Επιχειρησιακή Έρευνα: Στοχαστικά Μοντέλα 9/3/2021

Πιθανογεννήτριες - Μετασχηματισμοί L-S

① Ασκ. 1.8 / φυλ. 2

$$\tilde{F}_X(s) = \frac{10s + c}{3s^2 + 30s + 75}$$ ← παράμετρος

$c = ;$ $E[X] = ;$ $F_X(x) = ;$

Λύση:

$\tilde{F}_X(0) = 1 \Rightarrow c = 75$

$$\tilde{F}_X(s) = \frac{10s + 75}{3s^2 + 30s + 75}$$

$$E[X^n] = (-1)^n \tilde{F}_X^{(n)}(0)$$

$$E[X] = (-1)\tilde{F}_X'(0) = \cdots$$

$$\tilde{F}_X(s) = \frac{10s + 75}{3(s^2 + 10s + 25)} = \frac{10s + 75}{3(s+5)^2} = \frac{A}{(s+5)^2} + \frac{B}{s+5}$$

$$\tilde{F}_X(s) = \frac{10s+75}{3(s^2+10s+25)} = \frac{10s+75}{3(s+5)^2} = \frac{A}{(s+5)^2} + \frac{B}{s+5}$$

$$\xRightarrow{\times(s+5)^2} \frac{10s+75}{3} = A + B(s+5) \xRightarrow{s=-5} A = \frac{25}{3}$$

$$\xRightarrow{\times(s+5)} \frac{10s+\overset{50}{\cancel{75}}}{3(s+5)} - \frac{\cancel{25}}{3(s+5)} = B \Rightarrow B = \frac{10}{3}.$$

Άρα $\tilde{F}_X(s) = \frac{25}{3}\cdot\frac{1}{(s+5)^2} + \frac{10}{3}\cdot\frac{1}{s+5}$

$X \sim Exp(5) \to \tilde{F}_X(s) = 5/(s+5)$

$X \sim Erlang(2,5) \to \tilde{F}_X(s) = (5/(s+5))^2$

$$= \frac{1}{3}\cdot\left(\frac{5}{s+5}\right)^2 + \frac{2}{3}\cdot\frac{5}{s+5}$$

$$\Downarrow$$

$$F_X(x) = \frac{1}{3}\left(1-e^{-5x}-\lambda x e^{-5x}\right) + \frac{2}{3}\left(1-e^{-5x}\right), \qquad x \geq 0$$

Γενικά η σ.κ. της Erlang(n, λ) : $1 - \sum_{k=0}^{n-1} e^{-\lambda x}\frac{(\lambda x)^k}{k!}$

② Διευκρίνιση για μετασχ. L-S

Βασικό υπολ. αποτελ:

$$\int_a^b g(x)\, d\varphi(x) = \sum_{x:\ \text{σημ. ασυν της } \varphi} g(x)\left(\varphi(x)-\varphi(x^-)\right) + \int_a^b g(x)\, \varphi'(x)\, dx$$

g: συνεχής, φ: αύξουσα

③ Ασκ. 1.9 / Φυλ. 2

$$\tilde{F}_X(s) = e^{-2s} \Rightarrow X = 2 \Rightarrow F_X(x) = \begin{cases} 0, & x<2 \\ 1, & x \geq 2 \end{cases}$$

$$\tilde{F}_X(s) = \frac{4e^{-2s}+6e^{-6s}}{10} = \frac{2}{5}e^{-2s} + \frac{3}{5}e^{-6s}$$

$$\Rightarrow F_X(x) = \frac{2}{5}\mathbb{1}_{[2,\infty)}(x) + \frac{3}{5}\mathbb{1}_{[6,\infty)}(x)$$

$$= \begin{cases} 0, & 0 \leq x < 2 \\ 2/5, & 2 \leq x < 6 \\ 1, & 6 < x \end{cases}$$

$X = c$ με πιθ. 1 $\quad e^{-cs}$

$\Updownarrow$

$$F_X(x) = \begin{cases} 0, & x<c \\ 1, & x \geq c \end{cases}$$

$\downarrow$

$$\tilde{F}_X(s) = \underbrace{\int_0^\infty e^{-sx}\, dF_X(x) = e^{-sc}}$$

④ Ασκ. 1.10 / Φυλ. 2

$$\tilde{F}_X(s) = \frac{1}{3} e^{-2s} + \frac{c}{3s+15} \qquad c = ; \qquad E[X] = ; \qquad F_X(x) = ;$$

Λύση:

$$\tilde{F}_X(0) = 1 \Rightarrow \frac{1}{3} + \frac{c}{15} = 1 \Rightarrow \frac{c}{15} = \frac{2}{3} \Rightarrow c = 10$$

$$\tilde{F}_X(s) = \frac{1}{3} e^{-2s} + \frac{10}{3s+15} = \frac{1}{3} e^{-2s} + \frac{2}{3} \cdot \frac{5}{s+5} \Rightarrow X = \begin{cases} 2, & \text{με πιθ } 1/3 \\ \text{Exp}(5), & \text{με πιθ } 2/3 \end{cases}$$

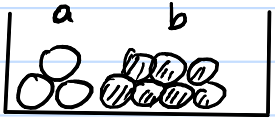
$\Downarrow$

$$F_X(x) = \frac{1}{3} 1_{[2,\infty)}(x) + \frac{2}{3}\left(1 - e^{-5x}\right), \quad x \geq 0$$

$$= \begin{cases} 2/3\,(1 - e^{-5x}) & 0 \leq x < 2 \\ 1/3 + 2/3\,(1 - e^{-5x}), & x \geq 2 \end{cases}$$

$$E[X] = \frac{1}{3} \cdot 2 + \frac{2}{3} \cdot \frac{1}{5} = \frac{12}{15} = \frac{4}{5}.$$

5 Άσκ. 1.2 / Φυλ. 1



Βήμα: Εξαγ. σφαιριδίου $\xrightarrow{Λ}$ Επανατοποθ.

$\xrightarrow{M}$ Αντικατ. με Λ

$X_n = \#$ Λ μετά από n βήματα.

$E[X_n] = ;$ Αναδρ. τύπος, Κλειστός τύπος.

$$E[X_{n+1}] = \sum_x \Pr[X_n = x]\, E[X_{n+1} \mid X_n = x]$$

$$X_{n+1} = \begin{cases} x, & \text{αν } X_n = x, \ Λ \\ x+1, & \text{αν } X_n = x, \ M \end{cases}$$

$$E[X_{n+1} \mid X_n = x] = x \cdot \frac{x}{a+b} + (x+1)\left(1 - \frac{x}{a+b}\right)$$

$$= x + 1 - \frac{x}{a+b}$$

$$= \left(1 - \frac{1}{a+b}\right)x + 1$$

$$\Rightarrow E[X_{n+1}] = \sum_x \Pr[X_n = x]\Big(\underbrace{\Big(1-\frac{1}{a+b}\Big)x+1}_{g(x)}\Big) = E[g(X_n)]$$

$$= E\Big[\Big(1-\frac{1}{a+b}\Big)X_n + 1\Big] = \Big(1-\frac{1}{a+b}\Big)E[X_n] + 1, \quad n \geq 0$$

Αναδρ. σχήμα: $E[X_0] = a$, $E[X_{n+1}] = \Big(1-\frac{1}{a+b}\Big)E[X_n]+1$, $n \geq 0$.

Εξισ. σταθ. σημ.: $x = \Big(1-\frac{1}{a+b}\Big)x + 1$ $(\rightsquigarrow x = a+b)$

$$E[X_{n+1}] - x = \Big(1-\frac{1}{a+b}\Big)(E[X_n] - x)$$

Άρα

$$E[X_n] - \overset{a+b}{\overset{\shortparallel}{x}} = \Big(1-\frac{1}{a+b}\Big)^n(\overset{a}{\overset{\shortparallel}{E[X_0]}} - \overset{a+b}{\overset{\shortparallel}{x}})$$

$$\Rightarrow E[X_n] = a+b - b\Big(1-\frac{1}{a+b}\Big)^n, \quad n \geq 0.$$