

Mathematisation – Vertical and Horizontal

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The idea of mathematisation which was espoused among others by Freudenthal was further developed by reflecting on the design practice by Treffers. He first made the distinction in 1976 between horizontal and vertical mathematisation to indicate the different aspects. By 1993 this differentiation was further clarified to distinguish the different roles that contexts can play in mathematics education. The process of level-raising by organising, symbolising and model building by starting from a paradigmatic situation in which children can orient themselves is termed vertical mathematisation, while the organising of contexts to make them applicable to mathematical treatment using the existing tools is termed horizontal mathematisation. NCERT textbooks are analysed through this lens and it is argued that while there are possibilities for supporting horizontal mathematisation, opportunities for vertical mathematisation are missing.

The National Curriculum Framework 2005 (NCF, 2005), which sought to introduce new mathematical practices into classrooms considered ‘mathematisation of the child’s thought processes’ as the ‘one main goal’ of mathematics education in schools, arguing, echoing Wheeler, that it is ‘more useful to know how to mathematise than to know a lot of mathematics’. (NCERT, 2006, p 1).

Mathematisation is a word that strikes a chord among many, especially those who have been pained by the mechanical manner in which mathematics has been taught and has been used by many to capture what they would consider as the heart of the mathematical enterprise – the thinking and the reasoning. In this paper I present the idea of mathematisation as formulated originally by Freudenthal and later modified and developed through practice by Treffers and others in the RME (Realistic Mathematics Education) tradition.

Freudenthal in his 1973 volume *Mathematics as an Educational Task*, made a case for mathematisation. His appeal for mathematisation was strongly related to his conception of mathematics as a human activity and his view of learning as reinvention. Writing as he was in the period that structuralist ‘New Math’ (that focused on set theory and the abstract deductive structures of mathematics) was sweeping across the world, Freudenthal’s obvious concern was to present an analysis of mathematics – not as a ready-made product but as ‘acted out mathematics’. Taking the example of complete induction, he argued against the ‘deductive course (which) is to derive from Peano’s axioms the principle of complete induction as a theorem and to apply it afterwards on various examples’. This, he called as an example of **antididactic inversion** and that ‘the analysis of the learning process shows that the didactic course is precisely the opposite.’ (Freudenthal, 1973, p122).

The mathematisation process that Freudenthal advocated was also related to his understanding of mathematics as an activity of organising fields of experience, in which the means of organizing the lower level becomes the subject matter on the higher level. (Freudenthal, 1973, p 123). His passionate appeal to see mathematics as a human activity and not as ready-made, made him argue that ‘there is no mathematics without mathematizing, in particular, no axiomatic without axiomatizing, and no formalism without formalising.’ He added that ‘the problem should grow out of the situation, and the child should learn to recognize the problem in the situation. Raising a problem is mathematics too.’ (Freudenthal, 1991, p 134-35)

Mathematisation And Didactisation

The understanding of what mathematisation entails developed with the Wiskobas project in which from 1968 onwards curriculum development work started in the Netherlands, involving classroom intervention (Treffers, 1993). Reflecting on a decade of the Wiskobas experience, Treffers in 1978 formulated an understanding of the levels in the process of mathematisation and concomitant process of didactisation (Treffers 1978), which has since become the common basis for understanding within the RME research community (van den Heuvel Panhuizen, 2001, Gravemeijer, 2000).

The idea of level-raising that developed in this period is shown by Treffers by taking example of a set of 11 questions dealing with ordered counting (Treffers, 1978, 43-81). He chose the example given below to detail out a path of mathematisation in the classroom (Treffers, 1978, p 58-66).

1) Someone rides every day the route from a via b to c following the shortest way. He decides to take every day a different ‘shortest’ route abc. How long can he continue to do this?

Treffers presents the landmarks in the process by which two children who had not been exposed to combinatorics solved the above problem.

Step by step children solve this problem, first recognizing what ‘route’ means and then what ‘different’ means, discover to their surprise that there can be an ‘outside’ route that is as short as the inner route. First they colour the different routes and then decide to draw them separately to make it clear. While they are busy sketching separately the different routes the teacher asks them

‘how many pieces are there from a to b, followed by how many steps are there to the right and how many below?’ When the teacher asks them whether they can write the different routes in terms of right and below, a new stage in the trajectory opens up for the children. They go through further schematisation of this representation by making tables of the different routes using the new characterisation and conclude that there are 13 routes. The teacher then makes a tree diagram to take the conversation further to see how many possibilities are there to go further once having reached b and children go on to notice the relationship and ‘see’ that there are 30 routes. (Treffers, 1978, p 60-64).

Adri Treffers, reflected on such classroom practices to develop an understanding about the different processes of organisation involved in mathematisation. Specifically reflecting on this lesson he showed how the process of organising, involving ‘recognising commonalities and differences’, ‘systematic approach’ ‘symbolisation’ ‘generalisation of a particular solution’ and so forth (Treffers, 1978, p 58) was not a process that ‘came up by itself’ and needed to be steered and supported through specific didactical methods. ((Treffers, 1978, p 59)¹. He identified eight core practices and characterised them through the term didactisation and argued that ‘just as mathematisation indicates the core mathematical activities, didactisation points to the core of the didactical practices’ (Treffers, 1978, p 65).

Learning principles (mathematisation)	Instructional Principles (didactisation)
Learning as construction	Laying a concrete orientation basis
Level- character of learning	Provision of models, schemes and symbols
Reflecting aspect of learning	Assigning of special tasks with free production and conflict problems
Learning as a social activity	Interactive instruction
Structuring and schematising character	Intertwining of learning strands

Table 1. Learning and Instruction principles (modified from Treffers, 1991, p 26)

Later he went on to identify five learning principles and connected five instruction (didactic) principles, in a process where mathematisation is expected to take place, with each of these aspects being connected to all the others (Treffers, 1991, 24-27).

Horizontal And Vertical Mathematisation

As a part of the process of reflecting on the experiences of classroom interactions Treffers not only laid out the relationships between mathematisation and didactisation, but already by 1976 made the distinction between two different forms of mathematisation - horizontal and vertical (1978, p 78-79).

The earlier cited example, in which the form of representation achieved becomes a tool for dealing with new situations, was seen as a case involving vertical mathematisation. In this various forms of mathematical treatments such as generalisation and symbolisation played important roles. At the same time he also stressed the importance of horizontal mathematisation which involves ‘effort to schematise a problem till a problem statement is created that can be solved using mathematical methods’ (p79). He pointed out that one of the 11 questions considered had a question which could not be solved merely by using the grid route problem that

¹ I have relied on the 1978 thesis and translated it while writing. The English translation of the thesis of Treffers was later published (Treffers, 1987), but I have not had access to it as of now.

children were familiar with. This involved finding out the ‘composition of boys and girls in the family’, in a survey of 160,000 families that have 4 children (Treffers, 1978, p 57-58). Treffers concluded that the question was posed at too abstract a level and that children should have had enough opportunity to understand chance and probability before such a problem could have been solved using combinatoric methods (p 78). He said, since there did not seem to be any mathematical method available for these children for solving the problem, empirical methods such as tossing a coin had to be first used to make the partitions and only then slowly would they come to see that the problem can be solved using grid routes with which they were already familiar. In such cases he said that the problem had to be first transformed using an empirical approach of observation, experiment and inductive reasoning before the problem can be dealt with using strictly mathematical methods. Therefore he characterised this as a case of horizontal mathematization.

Later Freudenthal also accepted this distinction as valid (Freudenthal, 1991, 41-44) and his characterization that ‘horizontal mathematization leads from the world of life to the world of symbols: In the world of life one lives, acts (and suffers); in the other one symbols are shaped, reshaped, and manipulated, mechanically, comprehendingly, reflectingly: this is vertical mathematization’ has become a leitmotif for this distinction (Gravemeijer & Terwel 2000; van den Heuvel-Panhuizen, 2001). Freudenthal mentioned that for a long time he hesitated to accept this distinction: ‘horizontal mathematizing which makes a problem field accessible to mathematical treatment ...versus vertical mathematizing which effects the more or less sophisticated mathematical processing.’ According to him, he declined to do so, because he felt that the equal status in practice of both these activities would be endangered by this distinction. He had experienced that many mathematicians interested in education narrowed mathematizing to the vertical component while many educationists turning to mathematics restricted to the horizontal one (Freudenthal, 1991, p 41).

Yet as the experience and reflection further progressed, the formulation of the distinction further evolved and got clarified and we can say that the understanding developed that both forms of mathematization deal with forming a model of reality but in different ways. In *horizontal mathematization*, the process of mathematization focuses mainly on ordering and schematising and building a model of the reality so that it becomes amenable to be dealt by mathematical means. As for example in the case of the discussion a horticultural image of Van Gogh’s Sunflowers. Treffers says that such ‘integrative themes have a specific horizontal mathematizing purpose; they are primarily directed at the application, practice and relating of learned knowledge and skills in context situations.’ (Treffers, 1993, p 94). Even though these are ‘application’ contexts, they need to be distinguished from what are normally called ‘word problems’ that are disguised bare number problems in which there is no mathematization to be done.

Vertical mathematization is on the other hand intimately related to the idea of ‘learning strands’ or trajectories of learning. In the case of the well known case of long division there is a continuing process of progressive mathematization or schematisation starting from a context. This aspect of a ‘model of’ a context that can function as a ‘model for’ for problem solving in other contexts was brought forward by Streefland in 1985 (Van den Heuvel-Panhuizen, 2001, p 4) and Treffers (Treffers, 1991, p 33-34). In fact this was already indicated by Treffers when remarked in the earlier discussions of problem solving with routes and grids that ‘through this the first step is set on the long path of model-forming. What serves at first as a problem can later be used as a tool to solve other problems’ (Treffers, 1978, p 47). But to function as such a ground for vertical mathematization, the context has to fulfil stringent criteria. Gravemeijer in elaborating on this aspect makes the very important point that the model of/model for heuristic emerged as a ‘result of an effort to come to grips with an effective *design practice*’ and not just theoretically. He brought out that in this case what is central is the paradigmatic situation with which children can engage with and detailed out the different levels involved in this process of vertical mathematization (Gravemeijer, 1997, p 29-32). This vertical aspect of mathematization that got clarified through practice also meant reassessment of the role that contexts play in mathematics education.

Anchoring Context

For the last ten years or so in our practice we have experienced the power of this form of vertical mathematization starting from a real context which functions as a paradigm. One has called such contexts as anchoring contexts which has now become a common shared term of the community of teachers with whom we work. When Freudenthal talked of real contexts he meant contexts that are experientially real to the children, taking care to note that even mathematical contexts can function as real to children provided they are that comfortable with them. Such contexts have been designed and used over the years from Grade I to Grade 6 for different topics and a few of them are presented to illustrate some characteristics of the level raisings involved when using such model contexts.

In designing an anchoring context to support vertical mathematisation two important aspects are taken into consideration. First is that children should be able to understand and get engaged with the problem posed and at the same time be able to solve it by some means, whether using concrete material, by counting or by calculating or whatsoever. We have found that stories can play a very important role in designing effective anchoring contexts. Recent experience of teaching children of Grade VI has convinced me that the impact of stories is not confined to the lower primary classes (Menon, 2012).

The second aspect is the adequacy of the situation in bringing forward the key mathematical idea that the teacher or the designer has in mind. Thus for example it was decided to try out a fraction trajectory with unit fractions as the basis. To develop the meaning of denominator a context was designed in which a boy had to choose where he would sit, given a choice of 4 different sharing situations. This was inspired by the pancake sharing context that Streefland had suggested but with a major modification since unit fractions were chosen as the first landmark. The fraction cut-outs play an important role in this trajectory, but over the five years that one has seen the trajectory in use, one has been surprised by the very flexible ways in which these cut-outs could be used. One of the processes of vertical mathematisation that one noticed in this trajectory was the change in the language of the children. To begin with for children, $1/6$ meant that 6 *children* were sharing a cake, later they would say that one *cake* is divided into 6 (equal) *parts* and later it becomes one is divided into 6 (equal) parts and later it became a number they could place on the number line.

These anchoring contexts whether they are that of cake sharing for fractions or birds flying out of a tree to find food to introduce addition and subtraction or biscuit packing to introduce factors, one key aspect of the anchoring context is that it anchors the discussions for some time. This is important to induce the focus to shift to numbers from the other aspects of the context. When the context is kept constant then numbers and number relationships start emerging more starkly for the children and they start noticing patterns. In fact we go to other type of contexts usually only after children are able to relate to the symbolic representation. Once this symbolisation is made real to children, then they are able to use it as a tool to deal with other situations.

It is seen that such contexts truly function as ‘anchoring contexts’, a haven where you return whenever there is a need. Even after children become comfortable with the fraction notation and are able to see it in abstract terms, when new challenges come some of them go back to visualisation of fraction cut-outs to think about the problem. Here we can see that what was earlier only a representation of a situation involving sharing of cakes has now become a generalised tool to think about other fraction operations such as multiplication and division, with the introduction of new (anchoring) contexts. One sees that there are different processes involved in a long-term trajectory involving anchoring contexts and one sees the need for further detailing out of the processes to get a clearer picture of the process of vertical mathematisation.

Currently in mathematics education, one can distinguish two broadly different approaches in the use of contexts – one can be termed a paradigmatic approach and the other an inductive approach in the use of contexts. The first one recognizes the important role that paradigms play in mathematics. Freudenthal in his perceptive statement said, “More often than one tends to believe, generality is achieved by the *aha* experience of one single paradigm only to be reinforced by a few (albeit not necessarily many) more of them” (Freudenthal, 1991, p 35). The other approach of exposing children to many contexts with the implicit understanding that children would generalise across these different contexts is what Treffers had referred to as the empiricistic approach which in the seventies was very dominant in the U.K. (Treffers, 1978, 1991). These two approaches can be diagrammatically represented as shown in Figure 1.

NCERT Textbooks

Since NCF 2005 emphasised mathematisation, it would be useful to consider how this approach is reflected in the textbooks produced since then. The textbooks produced since then form a refreshing set in which contexts play a key role. Textbooks can form a key input into the didactisation process and therefore an analysis of the textbooks can indicate the affordances for mathematisation in the classroom facilitated by the textbook. The NCERT textbook books from Grades 2 to 5 are considered here both in terms of horizontal and vertical mathematisation. A general overview is attempted given the space constraint. In the case of vertical mathematisation division is taken up for consideration.

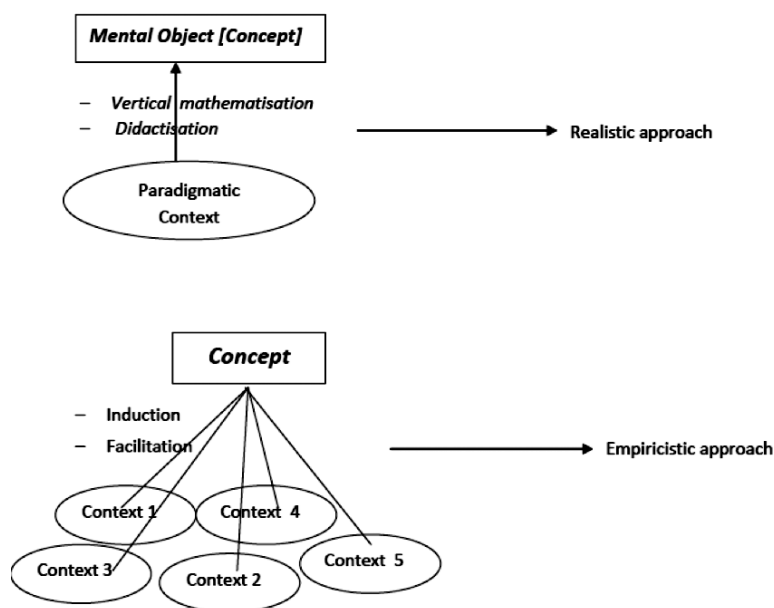


Figure 1. Contexts and Approaches to Mathematics Education

Horizontal Mathematisation

There are quite a few chapters in these books which can be used by teachers to support mathematisation. The chapter ‘A Trip to Bhopal’ (NCERT 2007, 23-34) of Grade 4 (Book 4) is a good example. The opening of the chapter conveys the excitement of starting on the trip and the question of the number of buses needed is posed very naturally and with all the urgency- provided of course the teacher enacts this in the class, the first page also gives hints about how the teacher could lead the conversation in the class about the ‘extra’ children and is a good start for prompting children to find their own means to find an answer. Here finding out the number of buses needed has a real purpose. This context is an adaptation of one of the classic RME contexts for division with buses for transporting football supporters – a very nice adaptation appropriate for the Indian context. The introduction of change in the capacity of buses and therefore the need to calculate the number of buses again happens in a very natural way. The calculation of the time needed to reach Bhopal, the impact of stopping over, on the reaching time, and especially the estimation of the width of Narmada and getting a sense of it using the length of the bus provide opportunities for mathematisation. Teachers would also get a good idea of how to give a sense of what 10,000 years mean with the example of using great grandmother’s age as a unit.

The strength of the chapter can be considered to lie in the fact that there is a purpose which can make sense to a child embedded in this chapter – there are actually problems to be solved, which is often not the case with many ‘integrated’ chapters elsewhere where there is only a description and a lot of questions to answer. But the effectivity of any such chapter depends on the manner in which the teacher transacts it in the class.

Yet there are some question marks that arise even with this chapter. It assumes that children are already comfortable with calculating the duration of time and with a sense of what one meter means and preferably also of multiples of 10 and 100.

Vertical Mathematization

I would discuss the issue of vertical mathematisation by focusing on two critical aspects related to the didactisation role of the teacher and therefore also of the textbook – that of providing a concrete orientation and of providing examples and models to support level-raising. The role of ‘concrete orientation’ is to provide a context and a problem with which children can engage and which they can solve using their informal strategies (See Table 1). Such a concrete orientation context would be one where the purpose for the child in problem solving is not obviously related to the target knowledge. This would be an adidactical situation as defined by Brousseau (2002).

Grade 3 textbook has the first step towards division and here the division symbol is also introduced. It also appears to be based on a different philosophy as compared to the textbooks of Grades 4 and 5. The chapter does

not present any situation which calls for problem solving. The ‘problems’ presented are what can be called as exercises (Schoenfeld, 1992, p 12) and introduces division by following the traditional pattern of worked out example (involving groups of butterflies, caterpillars and laddoos) followed by similar exercises. The only problem solving question is that of rearranging *jalebis* in plates to have equal sharing. The division concept is presented in this chapter through a partitive model. This is then followed by a situation of different animals jumping and then going on to bare number division problems with the hint to use the multiplication table.

Division gets focused in Chapter ‘Tables and Shares’ (NCERT, 2007, p124-132) in Grade 4. Here four different contexts are taken and each time a different solution method is suggested. In ‘Jumping Animals’, it is jumping on the number strip to decide how many jumps to reach any particular number. In making necklaces with ‘Sea Shells’ repeated subtraction is suggested. In ‘Gangu’s Sweets’, a situation is given which could in fact have provided a good orientation basis for division, but it is not presented as such. In ‘Children and their Grandfather’ the partial quotients method is introduced within a format. In this case also there is not sufficient concretisation of the problem. Thus for example, when grandfather asks ‘how to share the money equally among all of you’, the discussion starts by one child saying, “we know how to do $70 \div 5$ ” and not by saying, you can first give ten rupees to each of us and then....’ (p 128). This problem of insufficient concretisation is also followed in Grade 5 where returning the loan taken for a scooter is taken as the context, although it is a context that has a lot of potential for such treatment (p 179). In fact what one finds is a collection of contexts and no paradigmatic context that can support further abbreviation of the strategies as children reflect on their informal strategies.

The textbook suggests to the teachers in a note to allow children to solve these questions using any method of their choice including addition and multiplication. But the textbook has given a privileged position to repeated subtraction and addition is not presented. This can be understood in terms of the connection to the partial quotients method that is aimed at. But our experience with children over the last few years shows that repeated addition rather than subtraction is the main informal strategy that develops. Unfortunately research literature does not seem to distinguish between these two strategies and often club them into the same type (Anghileri et. al. 2002, p 162) and within literature there is some ambiguity about this aspect. In any case, in quotative situations there is evidence for the overwhelming presence of addition and of doubling and multiples of ten in the informal strategies of children.

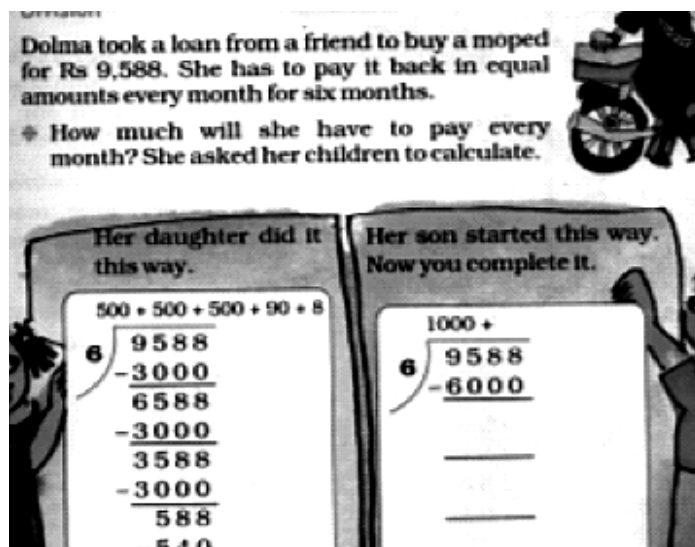


Figure 2. An example from grade 5 involving $9588 \div 6$

Supporting level-raising of children’s methods by steering the discussion and providing examples and models at the appropriate time is another key aspect of supporting vertical mathematisation. We would discuss this aspect with the example of the partial quotients method used in the textbook. The partial quotients method in fact connects to the multiplying-on methods that children tend to use once they are comfortable with multiplication. Otherwise they would most likely use repeated addition. In fact with smaller numbers, multiplying-on is quite an efficient strategy and in case children need to be prompted to go for partial quotients method they need to be prompted by the use of appropriately large numbers. It is true that ‘the beauty of this method’ is that children in a class can operate at their level of comfort by chunking differently (NCERT, 2007, p 128). But supporting vertical mathematization also means that children’s efforts at chunking needs to be supported with well chosen numbers and the development of appropriate subskills, keeping in mind the trajectories for the evolution of the division method.

For example in Grade 5, in the example involving $9588 \div 6$, (p 179) the two method examples discussed cannot be considered to be aligned to the possible path of development of this trajectory.

Multiplying by 500 is not an expected reaction – it would be either 1000 or 100. A bifurcation can happen after the subtraction of 6000. In this case some children would go for 500 times straight away (by halving 6000) while others who are not so confident would go on subtracting 100 times. An understanding of the actual processes supported by the textbook examples could help the teacher to steer the classroom discussion to support schematisation.

The other aspects of vertical mathematization by providing conflict problems to support reflection and the ensuing generalising is taken care of by some problems in the Grades 4 and 5.

This short review indicates that while there is room for supporting horizontal mathematisation in the new NCERT textbooks, vertical mathematization is not in fact supported. Further analysis of the other topics could throw more light on this.

To Conclude

The distinction made between horizontal and vertical mathematisation is a useful tool that can support curriculum development. Micro-didactical and macro-didactical approaches need to be combined to support longer term development of trajectories and intertwining of learning strands (van den Heuvel-Panhuizen, 2001, p 2). This might also mean a change in the approach to educational reforms, as recently Paul Cobb mentioned about his own experience, “The heuristics on which we had relied were relatively global in nature because we had derived them from a general background theory. In contrast, the specific design heuristics that Treffers outlined had emerged from and yet remained grounded in the activities of designing and experimenting in classrooms.” (Cobb et. al, 2008, p 6).

By focusing on contexts, the early reform movements had taken the first step in inverting the ‘antididactic inversion’ of starting from the product of mathematisation. The review of the reform process needs to orient us towards its further evolution. In a review of the Wiskobas programme Freudenthal himself concluded that in the seventies the programme has spent too much energy on the design of themes and too little on model contexts that could be vertically mathematised. Treffers mentions that integrative themes are important, ‘but an overemphasis leads to empiricistic mathematics with too little vertical mathematising (Treffers, 1993, p 94).

The process of connecting with the world, everyday and mathematical, that is real to the children of our country and designing longer-term trajectories based on them would require active and grounded involvement and collaboration of tens if not hundreds of people. The development of a range of trajectories for each topic so as to be able to connect to the lived experiences of different groups of teachers in the country would also be yet another task to be designed.

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