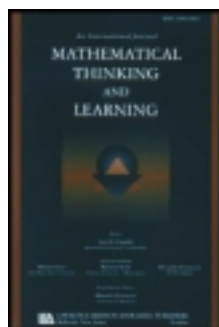


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How Emergent Models May Foster the Constitution of Formal Mathematics

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This article deals with the role that so-called emergent models can play in the process of constituting formal mathematics. The underlying philosophy is that formal mathematics is something that is, or should be, constituted by the students themselves. In the instructional design theory for realistic mathematics education, models always have been employed to foster a process in which formal mathematics is reinvented by the students themselves. This article describes how the use of models became more and more explicated over time and developed into the notion of emergent models. The design of an instructional sequence, which deals with flexible mental computation strategies for addition and subtraction up to 100, is taken as an instance for elaborating what is meant by *emergent models* and what role they play in fostering the constitution of formal mathematics. The analysis shows that there are 3 interrelated processes. First, at a more holistic level, there is a global transition in which “the model” initially emerges as a model of informal mathematical activity and then gradually develops into a model for more formal mathematical reasoning. Second, the transition from “model of” to “model for” involves the constitution of a new mathematical reality that can be denoted *formal* in relation to the original starting points of the students. Third, in the series of instructional activities, there is not 1 model, but the model actually is shaped as a series of signs, in which each new sign comes to signify activity with a previous sign in a chain of signification.

The difficulty of learning mathematics often is attributed to what is perceived as the gap between everyday life experience and what is colloquially called *formal mathematics*. In the approach discussed in this article, however, everyday-life experi-

ence and formal mathematics are not seen as two disjunct entities. Instead, what is aimed for is a process of gradual growth in which formal mathematics comes to the fore as a natural extension of the student's experiential reality. The focus in this article will be on the role so-called emergent models can play in such a process. The perspective taken here is that of instructional design. The design of an instructional sequence that aims at linear measurement and mental arithmetic will be used as an occasion to elaborate on the role of emergent models.

To provide the necessary background, I start by sketching the context of the instructional design approach, within which the use of emergent models can be depicted as a design heuristic. The instructional design under discussion is a design in the context of developmental or transformational research aiming at the development of a domain-specific instruction theory for realistic mathematics education (RME; Gravemeijer, 1994a; Streefland, 1991; Treffers, 1987). This RME instruction theory is not a fixed, ready-made theory but more a theory that is elaborated and refined continuously. Because of this, I think it is useful to describe the history of the way in which the role of models is conceived in RME. Further, before finally working out the intended exemplary instructional sequence, I sketch the history of that sequence.

I adopt a semihistorical approach in this article. As a consequence, I sometimes use descriptions that may seem unclear or questionable without elaboration, but I do so to do justice to the history.

DEVELOPMENTAL RESEARCH

The design heuristic, which is addressed here, is part of an instructional design approach, which, in turn, is part of a research approach that is known as *developmental research* (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Freudenthal, 1991; Gravemeijer, 1994b, 1998) or as *transformational research* (National Council of Teachers of Mathematics Research Advisory Committee, 1988). Developmental research is best described as an integration of design and research. That is, the design of instructional sequences serves as research on an instruction theory. This type of research poses the question: What does mathematics education that fulfills certain prechosen points of departure look like?

In the first phase of a developmental research project, a preliminary design of an instructional sequence is constituted. In the second phase, this preliminary design is elaborated and adjusted in a series of cyclic processes of deliberations on and trials of instructional activities. These cycles are to a large extent similar to Simon's (1995) mathematical teaching cycles.

In the last phase of the developmental research project, the knowledge assembled during the experiments with the various instructional activities is used to (re)construct an optimal instructional sequence. A central goal of the research work is to explicate the local instruction theory that underlies this reconstructed in-

structional sequence and to justify this theory in terms of theoretical deliberations and empirical assessments (see Gravemeijer, 1994a, 1994b, 1998).

This article is limited to the first part of such a developmental research activity. It focuses on the preliminary design of an instructional sequence. It should be noted that such a preliminary design does not necessarily have to be worked out in detailed descriptions of instructional activities because a core element of the developmental research approach is that instructional activities are designed and re-designed during the educational experiment. The experimental phase is not entered with a fixed sequence of instructional activities. Instead, the shape and form of the instructional activities that are actually used in the classroom are decided up on a day-to-day basis while taking into account what is learned from the preceding activities in the classroom.

The preliminary design may consist of a set of instructional activities that are supplemented with guidelines. The guidelines indicate both the order in which the instructional activities are to be enacted and the conjectured learning route in terms of the mental activities the students might engage in as they participate in the instructional activities. In relation to this, we speak of a *local instruction theory*. The concept of a conjectured local instruction theory is comparable with Simon's (1995) notion of a *hypothetical learning trajectory*. Such a notion encompasses the "consideration of the learning goal, the learning activities, and the thinking and learning in which the students might engage" (p. 133).

However, there are two differences: (a) The hypothetical learning trajectory deals with a small number of instructional activities, whereas the local instruction theory encompasses a whole instructional sequence; and (b) the hypothetical learning trajectory is tailored to the teacher's own classroom at a given moment in time, whereas the local instruction theory is more general. In my view, the connection between the two is that the local instruction theory entails a framework against which teachers can construe hypothetical learning trajectories that fit the actual situation in their classrooms.

If we return to the three phases of developmental research, we see that a conjectured local instruction theory is construed in the first phase. Subsequently, this conjectured local instruction theory is elaborated, refined, and adjusted in the second phase, the phase of the educational experiment. Then, in the third phase, the improved local instruction theory is reconstructed and justified. The issue I deal with in this article is the design of the conjectured local instruction theory in the first phase of a developmental research project. The focus is on the use of emergent models as a design heuristic, which can be seen as a constituting element of the RME theory.

RME

The RME theory is not a fixed a priori theory but is a theory that is always "under construction." RME is being developed in an ongoing process of designing,

experimenting, analyzing, and reflecting. Central to this development is the role of developmental research. There is a reflexive relation between RME theory and developmental research. On one hand, RME theory guides the design and research; on the other hand, developmental research serves the elaboration and refinement of the theory. The goal of the RME research program is to determine how mathematics education can be presented to students to facilitate their reinvention of mathematics.

The point of departure is Freudenthal's (1973, 1991) adage of "mathematics as a human activity." For him, mathematics was primarily an activity. Mathematics as a system of organized knowledge was secondary for him. According to Freudenthal, this is how it ought to be for students as well. In his view, students should learn mathematics by mathematizing: both subject matter from reality and their own mathematical activity. Via a process of progressive mathematization, the students should be given the opportunity to reinvent mathematics. In this manner, Freudenthal transcended the apparent dichotomy between mathematics as an activity and mathematics as a body of knowledge.

In fact, the *reinvention principle* is the first instructional design heuristic in RME. According to the reinvention principle, a learning route has to be mapped out along which the students can find the intended mathematics for themselves (Freudenthal, 1973). To do so, the curriculum developer starts with a thought experiment by imagining a route the students could have invented themselves. Knowledge of the history of mathematics can be used as a heuristic device in this process. Note that the students are not expected to reinvent everything on their own. In relation to this, Freudenthal (1991) noted *guided reinvention*. For him, the emphasis is on the character of the learning process rather than on inventing as such. The idea is to allow learners to come to regard the knowledge they acquire as their own private knowledge, knowledge for which they themselves are responsible. On the teaching side, students should be given the opportunity to build their own mathematical knowledge store on the basis of such a learning process. In relation to this, it may be noted that the RME theory is primarily a theory about knowledge construction; the idea is not to motivate students with everyday-life contexts but to look for contexts that are experientially real for the students and can be used as starting points for progressive mathematization.

At a closer look, the history of the reinvention heuristic reveals the dynamics of the RME theory. Initially, the reinvention principle was interpreted rather literally, referring to the history of mathematics as a source of inspiration for the design of instructional sequences. Gradually, however, students' informal solution methods have been recognized as a source of inspiration. Sometimes, the informal ways by which students solve applied problems "anticipate" the more formal procedures we are aiming for (Streefland, 1991). For instance, Streefland found that, when solving problems such as dividing three pizzas among four children, students ar-

rived at solution methods that contained the seeds of more formal procedures for adding and subtracting fractions with unequal denominators. For example, the students came up with solutions that corresponded with 3 divided by 4, 3 times one fourth, one half plus one fourth, or 1 minus one fourth. Written in a formal notation, one could conclude from this: $\frac{3}{4} = 3 \times \frac{1}{4}$, $\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$, and $\frac{3}{4} = 1 - \frac{1}{4}$. In this sense, the informal solution methods of the students anticipated conventional standard procedures (e.g., $\frac{1}{2} + \frac{1}{4} = \frac{2}{4} + \frac{1}{4} = \frac{3}{4}$), and one could think of ways to help students develop procedures similar to the conventional standard procedures.

RME started out as a vision, or as a philosophy of mathematics education that still had to be worked out. This is being done in developmental research projects, in which each time the research question is: What would mathematics education, which fulfills the initial points of departure, look like for a given topic? By reflecting on the yield of these projects, RME theory is developed, new theoretical insights in turn are used in subsequent research projects, and hence RME is developed in an iterative process of elaboration, adjustment, and refinement.

THE ROLE OF MODELS

In recent times, the RME community has expressed increasing interest in the role of models. From early on, it has been clear that models play a different role in RME than in mainstream information-processing approaches or in cultural-historical approaches, as elaborated by Gal'perin (1969).

In the latter approaches, models come to the fore as didactic models that embody the formal mathematics that is to be taught. That is, the formal abstract mathematics is "concretized" to be made accessible for the students. Actually, the students are to *discover* the mathematics that is concretized by the designer. In the RME approach, the models are not derived from the intended mathematics. Instead, the models are grounded in the contextual problems that are to be solved by the students. The models in RME are related to modeling; the starting point is in the contextual situation of the problem that has to be solved. This problem is modeled by the students who solve it with help of the construed model. The premise here is that students who work with these models will be encouraged to *(re)invent* the more formal mathematics.

Treffers (1987) discussed "bridging by vertical instruments" (p. 248). Here, vertical instruments also included schemes, diagrams, and symbols, although models were given a central position, and, although other roles of models were mentioned, it was emphasized that models primarily fulfill a bridging function between the phenomenological appearances of mathematics in reality on the one hand and the formal mathematics on the other. Still, models were treated as part of a broader category of instruments with a global function. In later publications, a more specific role of certain models was singled out:

It is especially by means of strong models that children are given the opportunity to bridge the gap between informal, context-bound work and the formal, standardized manner of operation through the constructive contribution of the children themselves. They must be “models of” context problems that will serve as “models for” the pure subject restricted and applied arithmetic in the pertaining area—multifaceted, in other words (Treffers, 1991, p. 33).

Despite the reference to the constructive contribution of the students, this description may evoke an image of two independent worlds: that of the student and that of the formal mathematics, which have to be brought together. Such an image, however, would not do justice to the RME adage that students have to construct their own mathematics. To put it differently, formal mathematics is not something “out there” with which the student has to connect. Instead, formal mathematics is seen as something that grows out of the students’ activity. The students are expected to develop formal mathematics by way of mathematizing their own informal mathematical activities. The notion of two disjunct worlds as it is taken from traditional mathematics education would not fit here.

The objective of the RME approach is that students experience formal mathematics no differently from informal mathematics. Freudenthal (1991) expressed this by speaking of mathematics as starting within common sense and staying within common sense. The premise is that common sense develops as the mathematics—that is, part of this common sense—develops. Within this perspective, we can distinguish between *formal* and *informal* by denoting *formal mathematical reasoning* as a form of reasoning that builds on arguments that are located in the newly formed mathematical reality. Seen this way, the distinction between informal and formal is a relative distinction—a distinction that is relative to a certain topic and that can be made especially from an observer perspective.

Streefland (1985) argued that models can support the process of mathematical growth: “Reality [is] attenuated into a model, a ‘model of,’ an *after* image” (p. 63). The model and the modeling facilitate reflection. Moreover, one may find that the model is also useful in other situations. The model can become exemplary, a “model *for* mathematical control of new realities” (p. 63). He elaborated: “In the mental shift from after-image to pre-image, realizing and level raising manifest themselves in the learning process. The aforementioned shift in perspective mirrors the new acquired insight in the wider applicability of the model (p. 63).”¹

In this sense, one can speak of a “model of” informal mathematical activity becoming a “model for” more formal mathematical reasoning. This shift can be exemplified with the fair-sharing situations that Streefland (1991) employed for instruction on fractions. In this setup, problems about sharing pizzas were modeled

¹In the translation to English, the Dutch word *voorbeeld*, literally meaning “example,” was translated as *pre-image*.

by the students by drawing partitioning of circles that signify pizzas (model of). Later, the students used similar drawings to support their reasoning about relations between fractions (model for).

THE EMPTY NUMBER LINE

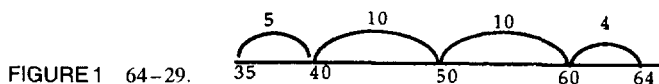
Thinking about the transition from model of to model for sparked a new impulse, with the research on the empty number line as a means of supporting flexible mental computation strategies for addition and subtraction up to 100. Inspired by Whitney (1988), Treffers (1991) introduced the use of an empty number line in RME.

An empty number line can be used to describe the solution of (two digit) addition and subtraction problems by marking the numbers involved and drawing the “jumps” that correspond with the partial calculation. Solving $64 - 29$ by subtracting 4, 10, 10, and 5, respectively, would result in the inscription shown in Figure 1.

Such a solution method, which could be seen as a form of curtailed counting down, belongs to the most common informal solution methods that students use. Accordingly, we may discern between collection-type solution methods and linear, counting-type solution methods. In the first type, the student partitions each number into a “10s part” and a “1s part” and then adds or subtracts those separately. In the second type of method, the student increments or decrements directly from one number, as in the previous example. The empty number line sequence is especially thought to fit with counting-type rather than collection-type methods. Moreover, the use of the empty number line not only fits these methods but also supports them. The number line supports the execution of counting-type methods by offering a way of scaffolding both partial calculations and partial results.

However, the number line has more advantages. One of these advantages is in the flexibility of the empty number line. The students can adapt the model to their thinking. We may contrast this with the common practice with base-10 blocks if we look at the way these blocks typically are used in instructional practices that aim at preparing students for the column algorithm (while acknowledging that these blocks can be used in a variety of other ways).

In this type of instruction, the students are expected to conform their thinking to the regulations that anticipate the written algorithms. When subtracting 37 from 84, the students are expected to start with the 1s to find that you cannot subtract 7 from 4, then exchange one 10 for ten 1s, which makes seven 10s and fourteen 1s.



Next, the seven 1s can be taken away, followed by the three 10s, which leaves four 10s and seven 1s, equaling 47.

In the approach of Whitney (1988) and Treffers (1991), the model is not employed to steer the student's thinking; instead, the model is meant to be adapted to fit the student's thinking. The student decides what steps to make and marks the number line accordingly. It may be noted that something similar can be done with base-10 blocks, but Whitney and Treffers both chose the number line because of their preference for counting-type solution methods.

Another reason for choosing the empty number line is that it also can be used to depict more sophisticated strategies such as compensating. For example, $95 - 19$ can be solved by first subtracting 20 from 95 and then adding 1. The empty number line can be used to explain and justify this strategy (see Figure 2).

Note that here the number line plays a different role than when used to model a context problem. This difference helps us to come to grips with the shift from model of to model for. At first, the focus is on the relation of the context problem to the number line; later, the numerical-mathematical relations become more important. At first, we encourage the student to try to mirror the problem situation by the number line representation; later, we encourage the student to use an incongruent representation if that would facilitate the calculation (see Figure 3).

Given the previous characteristics, the empty number line worked as a catalyst for the way the RME community was thinking about models. It helped the researchers realize that the shift from model of to model for concurs with a shift in the students' thinking, from thinking about the modeled context situation to a focus on mathematical relations. In the latter phase, number relations dominate the

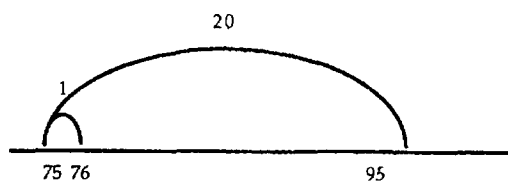
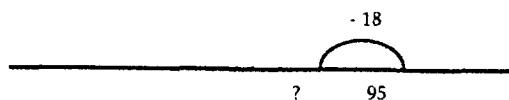


FIGURE 2 Compensating.

95-18 = ...



95-78 = ...



FIGURE 3 Solving subtraction as taking away or as a missing addend.

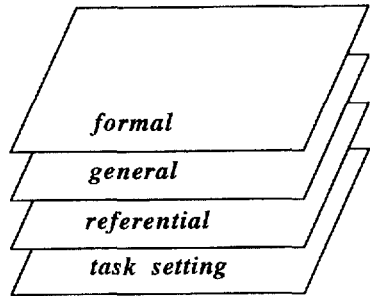


FIGURE 4 Levels of activity.

use of the number line. Therefore, we can discern two different types of activity (Gravemeijer, 1994a; Gravemeijer, Cobb, Bowers, & Whitenack, in press):

1. *Referential activity*, in which “models of” refer to activity in the problem setting described in instructional activities.
2. *General activity*, in which “models for” make possible a focus on interpretations and solutions independently of situation-specific imagery.

These general types of activity can be seen as different levels of activity, which can be completed with (a) a level of activity in the task setting itself and (b) a level of activity in which the student no longer needs a model. Together, these levels can elaborate the distinction between model of and model for by identifying four general types of activity, as shown in Figure 4.

Activity in the task setting involves interpretations and solutions that depend on understanding how to act in the problem setting (often out-of-school settings). *Referential activity* involves models of that refer to activity in the setting described in instructional activities. *General activity* involves models for that facilitate a focus on interpretations and solutions independent of situation-specific imagery. The *activity of formal mathematical reasoning* is no longer dependent on the support of models for to achieve mathematical activity.

Although the four types of activity clearly involve a developmental progression, this does not imply that they involve a strictly ordered hierarchy. In practice, discussions on the level of general activity and of formal mathematical reasoning frequently fold back to referential activity or even to activity in the setting (McClain & Cobb, 1998).

The goal of describing the four levels of activity is to illustrate that models are initially tied to activity in specific settings and involve situation-specific imagery. At the referential level, models are grounded in students’ understandings of paradigmatic, experientially real settings. In whole-class discussions, these models are integral to explanations in which students describe how they interpreted and solved tasks centering on the starting-point settings. General activity begins to

emerge as students' reasoning loses its dependency on situation-specific imagery, and the role of models gradually changes as they take on a life of their own. This transition can be seen as a process of reification, wherein the students begin to reflect collectively on their referential activity. In the process, the model becomes an entity in its own right and serves more as a means of mathematical reasoning than as a way to symbolize mathematical activity grounded in particular settings.

Here a connection can be made with the reification processes in mathematics as observed by Sfard (1991). Sfard noted that, in the history of mathematics, repeatedly processes develop into mathematical entities with an object-like character. Examples include the development from counting as a process into numbers as entities, as well as the shift from algebraic functions as recipes for processing an input to obtain an output, to functions as mathematical entities with certain characteristics (e.g., linear functions and quadratic functions).

The analogy with this process of objectivation, or *reification*, however, begs the question: What is it that is being reified? The analogy with the transition from process to object-like entity, observed by Sfard (1991), illuminates the point that it is not the inscription that is being reified but some activity. To identify the activity, it is necessary to look at the original instructional sequence in which the use of the empty number line is incorporated. In this instructional sequence, the empty number line is preceded by activities with a bead string (Treffers, 1991; Whitney, 1988). Initially, a bead string with 100 beads colored in groups of 10 is used for counting games. The beads are counted from left to right, and the number is indicated with a clothes peg that marks the total beads at its left. Students find out that the decimal structure can be used to count efficiently. Using the number line concept, the students solve tasks such as "Count 38 and add 24 more. Which number do you get?" In doing so, they might look for 38 on the bead string (possibly realizing that 38 is 2 less than 40); then, they might add 10 beads (knowing that the result will be 48) and another 10 beads (arriving at 58); and finally, they might add 4 (possibly as 2 and 2). In this manner, counting games on the bead string anticipate the use of the number line, which is introduced subsequently to model the activities on the bead string.

Initially, in the mode-of phase, the actions on the number line signify actions on the bead string (e.g., actions of counting beads, encompassing resultative counting, counting on, and counting down). It is this activity that is reified. In this manner, acting with the number line functions as an intermediary between counting beads and acting with numbers as elements of a network of number relations. In this sense, the empty number line changes from functioning as a model of acting with the bead string to functioning as a model for reasoning with number relations. Because the beads are colored in groups of 10, working with the bead string offers the opportunity to develop decuples-referenced solution methods. As a consequence, it may be expected that the students develop a framework of number relations that revolve around the decimal structure of our number system. In conclusion, we could argue

that working with the number line helps students to understand this decimal structure in such a manner that the numbers up to 100 start to derive their meaning from a system of number relations that become meaningful in their own right. In this sense, the goal of the number line sequence corresponds with what Greeno (1991) referred to with his environmental metaphor of number sense.

PROBLEMS OF THE ORIGINAL EMPTY NUMBER LINE SEQUENCE

Despite all its qualities, however, the empty number line sequence described previously also has some problematic aspects. One drawback concerns the artificial character of the bead string as a sort of tactile microworld. Another drawback is that there is no credible motive. There appears to be no reason for someone to carry out all sorts of counting activities with a bead string. Both points do not sit well with the RME approach, even though it could be argued that the bead string can be seen as a tactile representation of the number sequence that already is grasped by the students.

This critique was the catalyst for efforts to design an instructional sequence that capitalizes on linear measurement as a basis for the introduction of the empty number line (McClain, Cobb, Gravemeijer, & Estes, *in press*; Stephan, Cobb, Gravemeijer, & McClain, 1998).

To place this effort in the right context, however, I first elaborate on an earlier teaching experiment, in which we tried to circumvent the bead string. In that teaching experiment, we tried to build on activities known under the label "The Candy Shop" (see Cobb et al., 1997). To some extent, this choice was rather accidental. According to the preliminary design, the plan was to introduce the bead string following the candy shop activities. However, the state of affairs in the classroom seemed to justify the introduction of the empty number line as a means of representing the actions that were executed with the candies.

The instructional intent of the candy shop sequence was to support the students' initial coordination of units of 10 and 1. The story revolved around the production and sale of candies that were represented by unifix cubes. Within the scenario of the candy shop, the problem of how to pack the candies was made a topic for discussion. After having discussed the pros and cons of various options, the teacher revealed that the owner of the shop, Mrs. Wright, chose to make rolls of 10. When the students were familiar with packing candies into rolls of 10, other problems were posed within the candy shop scenario. Mathematically, this meant that the students were asked to generate different partitionings of a given collection of candies and to solve problems that involved incrementing and decrementing numbers of candies as required by the packing and unpacking process. The empty number line was introduced in a whole-class mental computation activity called "Target."

In this game-like activity, the task of the students was to determine how much one would have to change a given number to reach a fixed target number. During the Target activity, the ways in which the students solved these problems were symbolized by drawing number lines (see Figure 5).

The empty number line was subsequently extended to 100 during an activity in which it served as a record of simulated transactions in the candy shop. The hypothesis that the empty number line could be grounded in the candy shop activities initially appeared to be confirmed by the facility with which the children, working in pairs, created number lines to tell stories about what happened in the candy shop. However, later events in the classroom led us to revise our view. Problems arose when the teacher posed addition and subtraction tasks by drawing a horizontal empty number line and describing transactions in the candy shop but not acting them out. Here, it showed that the students interpreted the empty number line in a variety of different, yet personally meaningful, ways. The differences in interpretation are illustrated in Figure 6.

One of the tasks posed corresponded to $90 - 8$ (i.e., "Mrs. Wright has 90 pieces of candy, and she sells 8 of them"). As part of a counting-back strategy for solving $90 - 8$, some of the students interpreted this figure as showing that 3 candies had been subtracted: 89, 88, and 87. Other students argued that 2 candies had been taken away. It further became apparent that, for some students, the 9th decade when counting by 10 comprised 80, 81, 82, ..., 89. They appeared to reason that taking away 8 from the 9th decade would leave the candies signified by 80 and 81, and so there would be 81 candies left.

In retrospect, we concluded that the students adopted the empty number line primarily as a way of notating and not as a way of modeling a (mental) activity. We inferred that this calculational interpretation is stimulated by the fact that the students conceptualized the candy shop activities from a collection perspective and not from a linear-counting perspective. In the candy shop setting, the students were routinely structuring quantities in 10s and 1s. This split in 10s and 1s then guided their solution methods, whereas the empty number line representation did not lend itself to this type of solution method.

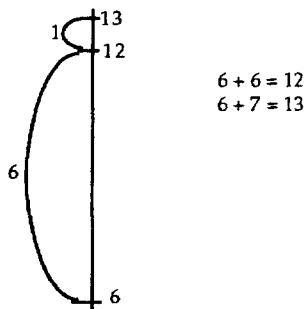


FIGURE 5 Symbolizing a solution for $6 + \dots = 13$ in the context of the Target activity.

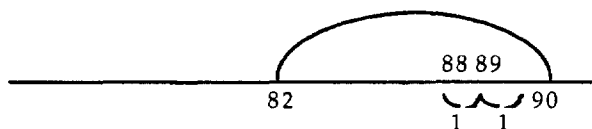


FIGURE 6 Counting down.

The original approach aimed to encourage students to model situation-specific solution methods that were situated in the activity of manipulating the bead string. In that situation, we expected the students to develop solution methods of a linear type, such as counting on or counting down in bigger or smaller jumps while using the decuples as reference points. Next, that activity was modeled by the students. Their past experience of working with the bead string provided the necessary imagery behind the increments on the number line.

In the aforementioned teaching experiment, this imagery was lacking. As a consequence, the students had to create a semantic grounding for the empty number line individually and privately by interpreting it in terms of counting activity. They did this in a variety of ways, which made it impossible for them to communicate effectively about what had previously been an unproblematic activity.

MEASUREMENT AS AN ALTERNATIVE FOR THE BEAD STRING

In the original instructional sequence, the activities with the bead string provided the necessary imagery. Note, however, that the way the bead string has to be interpreted is not self-evident. In this respect, we thought that measuring might offer a better basis for the imagery behind the number line. I explain this by comparing the bead string with measuring.

Just like the empty number line, the bead string can be interpreted in a cardinal or an ordinal manner. This may be illustrated with an incident during an informal empty number line teaching experiment that was carried out in West Lafayette, Indiana, in 1991. During this teaching experiment, we made a transition from working with a bead string to working with activity sheets on which bead strings were depicted. When the students were asked to mark a number (e.g., 53) they began to discuss the positioning of the mark. Should it go on the 53rd bead or next to it? (see Figure 7).

Eventually, this issue was resolved by the realization that the 53 refers to a cardinal number; thus, there were a total of 53 beads. Actually, this would be expressed best by circling all 53 beads or by putting one marker before the first bead and one behind the 53rd bead. Instead, the simpler convention was established that all beads to the left of the mark would count.

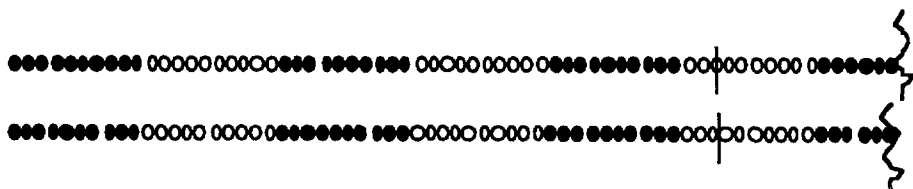


FIGURE 7 Different ways of marking 53 on the bead string.

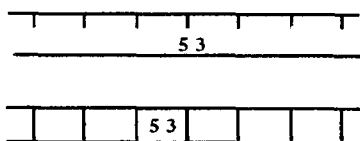


FIGURE 8 Different ways of marking on a ruler.

What is interesting is that the same issue arises in measurement. Applied to a measurement situation, the question might be: How do we mark the ruler? Do we mark the 53rd interval, or do we mark the boundary between the 53rd and the 54th interval? (see Figure 8).

Note that this relates to a conceptual understanding of measurement. Alternatively, from a mathematical point of view, it is not sufficient for the students to learn to measure accurately. Students will have to come to interpret the activity of measuring as the accumulation of distance (cf. Thompson & Thompson, 1994). The latter implies that each number word used in the activity of iterating signifies the total measure of the distance measured until that moment. Another aspect of a conceptual understanding is that the students conceive of the results of measuring as structured quantities. For example, a distance of 20 units can be thought of as being composed of 20 measurement units or as composed of two distances of 10 units or of distances of 5 units and 15 units.

DESIGN HEURISTIC

Given the similarities between the desired quantitative interpretation of the number line and measurement, it seemed sensible to investigate whether linear measurement might provide a sound basis for the introduction of the empty number line. This is where the emergent models design heuristic can show its usefulness. This heuristic can be used to think through a possible sequence. Starting with the hypothesis that such a sequence could be understood as a transition from model of to model for, guiding questions arise. For instance:

- What is the model?
- What activity is being modeled?
- What is the model a model for?
- What does the transition from model of to model for entail?

I now endeavor to deconstruct these questions. I start with the observation that there is a strong similarity between the empty number line and a ruler, and we can imagine a sequence in which activities with a number line build on activities with a ruler. However, from an instructional design perspective, the question arises: What do we take as the model? We could construe a conjectured local instruction theory, encompassing a sequence of two models that each pass through a model-of or model-for transition. Reasoning this way, we might find even more model-of or model-for shifts. One could discern various local shifts. The result of such an approach would be that the emergent models notion loses much of its meaning. Instead, I prefer to preserve the model-of or model-for language for the description of a more encompassing process that embraces a substantial instructional sequence. To do so, the idea of a model has to be interpreted rather broadly. This is done by interpreting the empty number line as a schematized ruler. In this manner, both the ruler and the empty number line can be seen as two manifestations of the same model.

If we take the ruler as the model, we must identify the activity in which the ruler is the model. Speculating on the genesis of the ruler in history, it makes sense to take the view that the ruler came about as a curtailment of iterating a measurement unit. So the ruler can be seen as a model of iterating some measurement unit. From our earlier reflections on the empty number line, we already know that the empty number line can be seen as a model for mathematical reasoning in the context of mental computation strategies for reasoning with numbers up to 100.

The connection between the two could be in the relation between measurement as accumulating distance and measurement as a cardinal interpretation of distance on the number line. In the context of measurement, distances on a ruler or number line would have the status of magnitudes. As the context of measuring moves becomes less dominant, the interpretation shifts toward numerical quantities in general. Finally, even the direct reference to specified countable objects becomes less prominent as the numbers on the empty number line start to derive their meaning from a framework of number relations.

In conclusion, we can envisage something like the following trajectory that may be embedded in a story line or in a series of stories:

1. The students start with measuring various lengths by iterating some measurement unit. This activity of iterating is modeled with a ruler.
2. The activity of measuring is extended to incrementing, decrementing, and comparing.

3. These activities are represented with the help of arcs on a schematized ruler.
4. This schematized representation is used as a way of scaffolding and as a way of communicating solution methods for all sorts of addition and subtraction problems. (Note that this trajectory is also in agreement with the reinvention heuristic.)

In such a sequence, special attention should be given to the decimal structure of our number system. In the context of measurement, this can be done by curtailing the iterating activity by means of a larger measurement unit, that is, a unit of 10. Measuring with 10s and 1s can form the basis for the development of a ruler that is structured by 10s and 1s. At the same time, it may support the development of strategies that use decuples as reference points.

A feature that may cause problems in such a setup is the rigid character of a ruler on one hand and the more global character of the empty number line on the other. Positions on a ruler have to correspond to exact distances. However, on the number line the distances between the numbers do not have to correspond exactly with their numerical values. The latter is essential because, otherwise, it would be impossible to represent unknown numbers or unknown differences. On reflection, it makes sense to differentiate between measuring and representing solution methods. Incrementing, decrementing, and comparing originate in measuring. However, when solving this type of problem, the opportunity arises to generate solutions by reasoning. For instance, consider the problem: "What is the result of incrementing 64 by 28?" A possible line of reasoning could be as follows:

1. You first measure 64.
2. Then add six 1s, which would get you to 70.
3. Then measure 2 times 10, which would get you to 80 and 90, respectively, and finally, add two 1s, which gets you to 92.

When describing this method on an empty number line, it would be sufficient to show that you start at 64, add 6, arrive at 70, and then add 10, arriving at 80, add another 10, arriving at 90, and add 2, arriving at 92. To try to strive for an exact proportional representation of all the jumps would severely hamper a flexible use of the number line. Therefore, we will have to distinguish between the ruler as a *measurement instrument* and the *empty number line* as a *means of expressing solution methods*. This distinction should be drawn both at the preliminary design level and in the enactment in the classroom. In a global sense, we are speaking of one model; in practice, however, this model manifests itself in various symbolic representations.

In an earlier publication on the empty number line (Cobb et al., 1997), we already acknowledged that the model may take various shapes. We differentiated between the model as an overarching concept and the symbolic representations that are used in the instructional sequence. To describe this counterpart, we used

FIGURE 9 A sign constituted of a signifier and a signified.

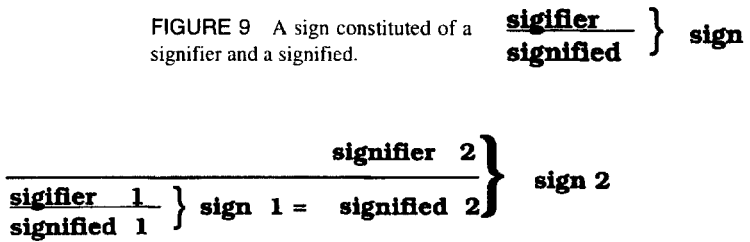


FIGURE 10 Sign 1 as a signified for Sign 2.

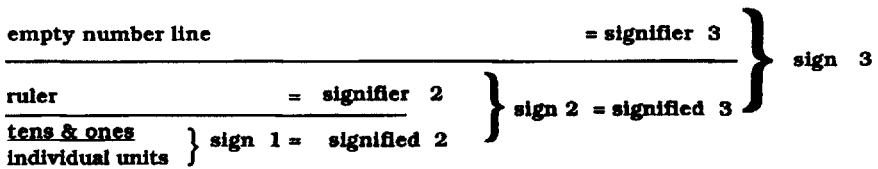


FIGURE 11 Preliminary chain of signification.

the notion of a chain of signification (cf. Walkerdine, 1988; Whitson, 1997). The concept of a chain of signification is grounded in the idea that a sign comprises a signified–signifier pair (Lacan, 1977; see Figure 9). A chain of signification then is constituted when the activity with one sign comes to signify activity with a previous sign. As the previous sign starts to serve as a signified for a new signifier, a signified–signifier pair is formed. This pair constitutes a link in a more global chain of signification (see Figure 10).

In the measurement–mental–arithmetic sequence in question, the chain of signification basically would look like the one shown in Figure 11. Iterating individual units along the lengths of items that have to be measured would be the starting point. This would come to be curtailed to iterating tens and ones. In this way, sign 1 is formed, of which “iterating individual units” is the signified, and “iterating 10s and 1s” is the signifier. The whole sign 1 (consisting of signified 1 and signifier 1) then becomes the signified (signified 2) for the next signifier: a ruler (signifier 2). These two together shape sign 2, which in turn becomes the signified (signified 3) for the empty number line (signifier 3), and sign 3 is formed.

Based on this preliminary chain of signification, we designed an instructional sequence for a first grade classroom. In elaborating the sequence, the actual chain of signification became more complicated than the one sketched previously (see Stephan et al., 1998). This was because (a) we introduced two different basic measurement units and (b) we inserted the so-called measurement stick to facilitate the transition from the ruler (which we called the *measurement strip*) to the empty number line.

MEASUREMENT SEQUENCE

The measurement sequence started with measuring by pacing "heel to toe." The idea was that measuring by pacing with your own feet offers a natural situation for iterating. Measuring with centimeters or inches would probably relate to familiarity with the centimeter as a sort of measurement strip and would therefore interfere with the intended buildup. Moreover, measuring with one's feet creates the opportunity to raise awareness for the need of a standardized measurement unit. The initial activities were situated in a scenario in which the king's foot was used as the standard measure.

The instructional intent was that the students came to interpret their pacing activity as an accumulation of distances. This imagery formed the basis for the next step in the sequence: measuring with a larger measurement unit. The scenario of measuring the king was employed to make the students think of more efficient ways of measuring. After a whole-class discussion of alternatives, the teacher told the students how this problem was solved in the kingdom. The solution was to make five footprints of the king's foot and to paste these together. This solution was discussed by the students before they began making their own footstrips. With the help of these footstrips, the students curtailed their counting of individual paces to counting by five paces. The intent was that the footstrips were seen as records of pacing, whereas the placement of one footstrip signified five paced feet.

Next, a shift was made to another measurement unit in the context of a story about Smurfs™. The teacher explained that Smurfs are small, dwarf-like people and that one particular Smurf was called on to measure a wall. Because of a lack of measurement tools, he used the food cans he had with him. As a consequence, more Smurfs started measuring with food cans. According to the story, these cans happened to have the size of a unifix cube. This made it possible for the students to act out the measurement activities of the Smurfs. Measuring with unifix cubes was curtailed in the same way as the measurement with the footprints. That is, a "Smurf bar" of 10 unifix cubes was constructed to represent 10 cans. Again, the story provided a need for this curtailment, and the students were provided with ample opportunity to invent and discuss solutions before the teacher explained how the problem was solved by the Smurfs. This pattern was repeated in all subsequent steps of the sequence. Let me note, however, that during the teaching experiment, more often than not, at least some of the students came up with suggestions that were similar to the solutions chosen in the story.

Measuring with the cubes and the Smurf bar became a matter of coordinating units of 10 and 1. Measuring started by iterating units of 10, which were then adjusted by adding or subtracting 1s. In this manner, measuring with 10s and 1s helped the students in structuring the number sequence up to 100. At the same time, a basis was being laid for the construction of a measurement strip that comprised 10 units of 10, each subdivided into 10 units of 1 cube. The idea is that mea-

suring with the measurement strip is grounded in the imagery of measuring units of 10 and 1. Thus, for the students, measuring with the strip signified iterating with the Smurf bar and individual cubes. The ultimate aim here was that, in measuring the length of an item and reading off a number, the students would signify the space that extends from the beginning of the measurement strip to the line signifying the end of the item. The students were acting in a spatial environment in which space was already partitioned.

In the following activities, a shift was made from actually measuring items to reasoning about lengths when solving tasks involving incrementing, decrementing, and comparing lengths of objects that were not physically present (e.g., comparing the measures of the heights of sunflowers in the context of a sunflower contest). These tasks offered opportunities for developing solution methods based on curtailed counting (using the decimal structure as a framework of reference). The shift away from measuring (as an activity) to reasoning about magnitudes was carried on with the introduction of a measurement stick. The measurement stick was introduced as a yardstick that was attached to a quay wall. The measurement stick had to be used to read the height of the water in the canal. (The context of the canal gives rise to all sorts of problems involving the rise and fall of the water level, the drafts of boats, and shallows in the river.) After discussing the students' suggestions about what a measurement stick would have to look like to make it easy to read, the measurement stick (see Figure 12) was introduced as the solution of the person responsible.

The appearance of the measurement stick was to some extent similar to the original bead string. An important feature was that, although there were no numbers, values were easy to read by counting 10s and 1s. The conjecture was that this feature created opportunities for developing strategies that used the decimal structure. Note that the objective was not that students should acquire particular calculational methods. Instead, the focus was on the students' construction of numerical relations that might give rise to these methods. In this respect, attention was focused on activities that involved reading off numbers, indicating numbers, and analyzing number patterns. Numbers close to a decuple, for instance, could be identified by using that decuple as a referent (e.g., $53 = 50 + 3$; $38 = 40 - 2$). These relations could be exploited when analyzing patterns that correspond to jumps of 10.

Problems in the context of water heights could be solved in two ways: by counting on the measurement stick or by arithmetical reasoning. The empty number line was introduced as a way of symbolizing the measurement strip arithmetical solu-



FIGURE 12 Measurement stick. Note that here the stick is shown in a horizontal position; in the context of the water heights, the stick has a vertical position.

tion methods that were grounded in reasoning with “10s and 1s.” (A jump on the number line described a move on the measurement stick that, in turn, was seen as signifying iterating Smurf bars, unifix cubes, or both.) The empty number line was introduced as a way of communicating solution methods but also functioned as a way of scaffolding. Finally, a generalization was made from magnitudes to numerical quantities in general; the students were asked to solve various addition and subtraction context problems while using the empty number line as a means to record their thinking.

On reflection, it can be argued that the new sequence based on measuring fits better with the RME philosophy than does the original bead-string-based sequence. Actually, the measuring-based sequence can be taken as an exemplary case to illustrate the contrast, mentioned previously, between the use of models that are construed as concretized formal mathematics and models that emerge in the process of modeling. Treffers (1987) elaborated on a distinction of *embodiment* versus *embeddedness* when he compared Dienes’s (1963) teaching–learning theory with the RME instruction theory (which relates to the work of Van Hiele, 1973). Treffers wrote:

Whereas in the first phases of Dienes’ learning cycles the children come in touch with concretized mathematical structures, with embodiments of games in artificially created environments, the Van Hiele basic levels entail an exploration of those aspects of reality for which the mathematical concepts and structures (can) serve as means of organization: “embeddedness” versus “embodiment”; the naturally organizable versus the artificially organized matter. (p. 275)

CONCLUSIONS

As our example of the measurement–mental-arithmetic sequence shows, the model in the model-of or model-for metaphor has to be understood as a global overarching concept. In our example, the ruler is the emerging model. The ruler emerges as a model of iterating a measurement unit (or measurement units). In this sense, the ruler is grounded in the activity of measuring. Gradually, however, the ruler changes character as the attention shifts from measuring to reasoning about the results of measuring. Finally, a schematized ruler becomes a model for reasoning about flexible mental-arithmetic strategies for numbers up to 100.

In the concrete elaboration in the sequence, the overarching model assumes various manifestations. The series of manifestations can be described in terms of a chain of signification. The chain of signification is in a sense the counterpart, on a more specific level, of what the model is on a more general level. A chain of signification is constituted as a sign that “slides under” a succeeding signifier (Walkerdine, 1988). That is, on one hand, the new signified encompasses the previous sign, whereas, on the other hand, the meaning of the previous sign changes.

The latter relates to the constitution of a new sign, in which the signifier and the signified are reflexively related; the signified gives meaning to the signifier, whereas the signifier shapes the signified.

From a mathematics education perspective, the notion of a chain of signification is helpful in that it shows how formal mathematical signs can be rooted in concrete activities of the students. The dynamic character of the chain of signification as such justifies the term *emergent models*. However, the meaning of the label *emergent* is broader. It refers both to the process by which models emerge within RME and to the process by which these models support the emergence of formal mathematical knowledge.

A crucial aspect of the emergent models heuristic is that the shift from model of to model for is reflexively related to the creation of a new mathematical reality. This can be elucidated by the role of the ruler as a model. What was expected was that, in the course of the sequence, a shift would take place in the student's view of numbers as referents of distances to numbers as mathematical entities. This shift involves a transition from viewing numbers as tied to identifiable objects or units (i.e., numbers as constituents of magnitudes—"37 feet") to viewing numbers as entities on their own ("37"). For the student, a number viewed as a mathematical entity still has quantitative meaning, but this meaning is no longer dependent on its connection with identifiable distances or with specified countable objects. In the students' experienced world, numbers viewed as mathematical entities derive their meaning from their place in a network of number relations. Such a network might include relations such as $37 = 30 + 7$, $37 = 3 \times 10 + 7$, $37 = 20 + 17$, and $37 = 40 - 3$. The critical aspect of this network is that the students' understanding of these relations transcend individual cases. That is, when students form notions of mathematical entities, they come to view relations like those just noted as holding for any quantity of 37 objects (including a magnitude of 37 units).

This shift from numbers as referents to numbers as mathematical entities is reflexively related to the transition from model of to model for. The students' actions with the model foster the constitution of a framework of number relations. However, through the students' development of this framework of number relations, the model can take its role as a model for mathematical reasoning.

In conclusion, this analysis shows that the emergent models heuristic refers to three interrelated processes. The first process consists of the global transition, in which the model first emerges as a model of informal mathematical activity and then gradually develops into a model for more formal mathematical reasoning. The second process involves the constitution of a new mathematical reality, which emerges as reflexively related to the model-of or model-for transition. The third process concerns the emergence of a chain of signification that can be seen as the concrete manifestation of the model in a series of signs, in which each new sign comes to signify an activity with a previous sign.

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REFERENCES

- Cobb, P., Gravemeijer, K., Yackel, E., McClain, K., & Whitenack, J. (1997). Mathematizing and symbolizing: The emergence of chains of signification in one first-grade classroom. In D. Kirschner & J. A. Whitson (Eds.), *Situated cognition theory: Social, semiotic, and neurological perspectives* (pp. 151–234). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.
- Dienes, Z. P. (1963). *An experimental study of mathematics-learning*. London: Hutchinson.
- Freudenthal, H. (1973). *Mathematics as an educational task*. Dordrecht, The Netherlands: Reidel.
- Freudenthal, H. (1991). *Revisiting mathematics education*. Dordrecht, The Netherlands: Kluwer.
- Gal'perin, P. Y. (1969). Stages in the development of mental acts. *Tijdschrift voor Nascholing en Onderzoek van het Reken-Wiskundeonderwijs*, 7(3/4), 22–24.
- Gravemeijer, K. P. E. (1994a). *Developing realistic mathematics education*. Utrecht, The Netherlands: Cd-β Press.
- Gravemeijer, K. (1994b). Educational development and developmental research in mathematics education. *Journal for Research in Mathematics Education*, 25, 443–471.
- Gravemeijer, K. (1998). Developmental research as a research method. In A. Sierpiska & J. Kilpatrick (Eds.), *Mathematics education as a research domain: A search for identity* (An ICMI Study Publication, Book 2, pp. 277–295). Dordrecht, The Netherlands: Kluwer.
- Gravemeijer, K., Cobb, P., Bowers, J., & Whitenack, J. (in press). Symbolizing, modeling, and instructional design. In P. Cobb, E. Yackel, & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms: Perspectives on discourse, tools, and instructional design*. Mahwah, NJ: Lawrence Erlbaum Associates, Inc.
- Greeno, J. G. (1991). Number sense as situated knowing in a conceptual domain. *Journal for Research in Mathematics Education*, 22, 170–218.
- Lacan, J. (1977). *Ecrits: A selection*. London: Tavistock.
- McClain, K., & Cobb, P. (1998). The role of imagery and discourse in supporting students' mathematical development. In M. Lampert & M. Blunk (Eds.), *Mathematical talk and school learning: What, why, and how* (pp. 56–81). Cambridge, England: Cambridge University Press.
- McClain, K., Cobb, P., Gravemeijer, K., & Estes, B. (in press). Developing mathematical reasoning within the context of measurement. In L. Stiff (Ed.), *Developing mathematical reasoning*. Reston, VA: National Council of Teachers of Mathematics.
- National Council of Teachers of Mathematics Research Advisory Committee. (1988). NCTM curriculum and evaluation standards for school mathematics: Responses from the research community. *Journal for Research in Mathematics Education*, 19, 338–344.
- Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22, 1–36.
- Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, 26, 114–145.

- Stephan, M., Cobb, P., Gravemeijer, K., & McClain, K. (1998, April). *Supporting students' learning in social context*. Paper presented at the annual meeting of the American Educational Research Association, San Diego, CA.
- Streefland, L. (1985). Wiskunde als activiteit en de realiteit als bron [Mathematics as an activity, and the reality as a source]. *Tijdschrift voor Nederlands Wiskundeonderwijs (Nieuwe Wiskrant)*, 5(1), 60–67.
- Streefland, L. (1991). *Fractions in realistic mathematics education: A paradigm of developmental research*. Dordrecht, The Netherlands: Kluwer.
- Thompson, A., & Thompson, P. (1994). Talking about rates conceptually: Pt. 2. Mathematical knowledge for teaching. *Journal for Research in Mathematics Education*, 27, 2–24.
- Treffers, A. (1987). *Three dimensions: A model of goal and theory description in mathematics education: The Wiskobas Project*. Dordrecht, The Netherlands: Reidel.
- Treffers, A. (1991). Didactical background of a mathematics program for primary education. In L. Streefland (Ed.), *Realistic mathematics education in primary school* (pp. 21–57). Utrecht, The Netherlands: Cd-β Press.
- Van Hiele, P. M. (1973). *Begrip en Inzicht [Understanding and insight]*. Purmerend, The Netherlands: Muusses.
- Walkerdine, V. (1988). *The mastery of reason*. London: Routledge.
- Whitney, H. (1988). *Mathematical reasoning, early grades*. Unpublished manuscript, Princeton University, Princeton, NJ.
- Whitson, J. A. (1997). Cognition as a semiotic process: From situated mediation to critical reflective transcendence. In D. Kirschner & J. A. Whitson (Eds.), *Situated cognition theory: Social, semiotic, and neurological perspectives* (pp. 97–150). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.