

CONNECTED WORKING SPACES: DESIGNING AND EVALUATING MODELLING-BASED TEACHING SITUATIONS

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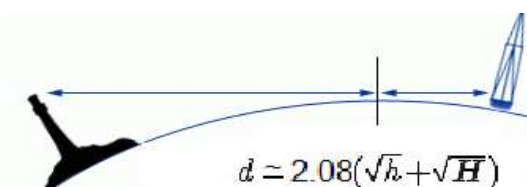
This contribution focuses on modelling at upper secondary level. Modelling is considered as a work on various models of a reality, belonging to different scientific fields, with varied mathematizations. The framework of Connected Working Spaces is chosen in order to describe the work on each model, and the connections made along the modelling process. The hypothesis is that these choices allow designing and evaluating situations that help students to understand comprehensively concepts taught at upper secondary level and enable them to appreciate how diverse fields contribute to a scientific perception of the sensible world. This hypothesis is tested by way of an experiment in realistic school settings.

MODELLING IN MATHEMATICS EDUCATION. WHAT FOR AND HOW?

Teaching/learning at upper secondary level should give meaning to mathematics as a tool for understanding the sensible world, and that is why modelling based teaching situations are often proposed. However, curricula and resources often reduce modelling to “translation” between reality and mathematics. Ideas brought by research like the “modelling cycle”, have the advantage of distinguishing models of different natures and of characterizing the type of activity involved in the transition from one to the other (English, Ärleback & Mousoulides 2016). This paper aims to build critically upon these ideas. A starting point is that the “modelling cycle” sharply separates “reality” and mathematics. I take an example from Blum & Ferri (2009, p. 48): it is asked to find the distance of a navigator to a lighthouse when the navigator perceives its light exactly on the horizon. The authors consider mainly two models: a “situation model” (*in reality*) and “the mathematical model”. Both models assume that the navigator is at sea level. I offer to consider here three alternative models (figure 1).

A “navigational science” model:

d the distance in nautical miles, h and H the heights respectively of the eye of the observer and of the light in meters.



A “geometrical-algebraic” model:

R the earth radius $d = \sqrt{h^2 + 2 \cdot R \cdot h} + \sqrt{H^2 + 2 \cdot R \cdot H}$

An “analytical model”: $d \simeq \sqrt{2R}(\sqrt{h} + \sqrt{H}) \simeq 1.93(\sqrt{h} + \sqrt{H})$

Figure 1: Three models for the distance to a lighthouse problem.

The first model considers actual navigation conditions, the navigator above sea level, and exploits empiric observations. Then, on the one hand, this model is more realistic than Blum & Ferri's "situation model". On the other hand, it is a scientific model since it belongs to "navigation", a science taught in naval schools. The two other models are mathematical models; they differ in the way they take into account the "preponderance" of the earth radius over the heights of the objects.

This simple example illustrates multifaceted links between reality and mathematics in modelling: for a given phenomenon, different scientific fields bring different models, each involving some aspects of reality and some mathematization. Each model also involves a specific type of work. For instance, the "geometrical-algebraic" model involves the identification of relevant geometrical relationship and calculations based on the Pythagorean Theorem, whereas the "analytical" model implies reasoning on approximation, using equivalence in a neighborhood. As for the "navigational science" model, the work implies considering and generalizing empirical data. Additionally, the example shows the possibility of multiple to-ings and fro-ings between models. These to-ings and fro-ings are necessary for confronting models: in the example, discussing the discrepancy between the "navigational science" and the "analytical" models and identifying refraction of the air as the origin. They also give meaning to the concepts involved in the different models by the connections to which they lead.

CONNECTED WORKING SPACES

Considering modelling as an activity involving models in different scientific fields and branches of mathematics, a theoretical framework is necessary to describe the work in each model. The framework of the Mathematical Working Spaces (MWS) allows characterizing the way the concepts make sense in a given work context. According to Kuzniak & Richard (2013) a MWS is an abstract space organized to ensure the mathematical work in an educational setting, based on the articulation of two fundamental levels: an epistemological level related to mathematical organization and tasks to perform, and a cognitive level related to individuals' activity and task enactment. The epistemological level associates a "representamen", that is to say a representation of the object on which one works (here a model), the artefacts relevant for this work and a theoretical framework of reference. The cognitive level associates three processes: visualization making use of material and mental representations, construction using "instruments" (i.e. internalized artefacts), and a discursive process consisting in reasoning and proof. This epistemological plan is connected to a cognitive plane by three types of genesis: (1) the semiotic genesis that gives the representamen its mathematical status of representation for visualization, (2) the instrumental genesis transforming artifacts into instruments for the construction process and (3) the discursive genesis that gives meaning to the frame of reference by mobilizing it for the discursive process.

Modelling, as we understand it, involves several models, and for each of these models, a working space. Minh & Lagrange (2016) built on the MWS framework to introduce

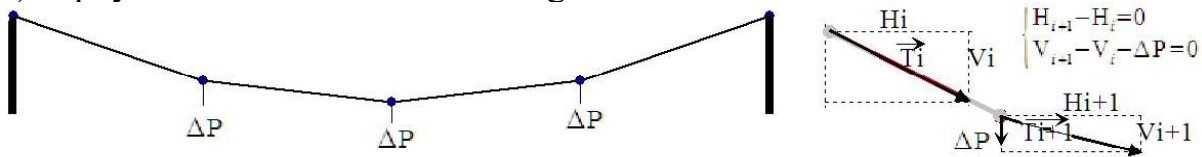
the idea of "connected working spaces" to give account of how connections between WMS bring meaning to the concepts involved. Here, this framework is expected to allow designing and evaluating teaching situations where students go back and forth between different models of a real world setting and make connections between working spaces, and help identifying geneses at work through these connections. In the following section, I will test this hypothesis by way of an experiment.

AN EXPERIMENT

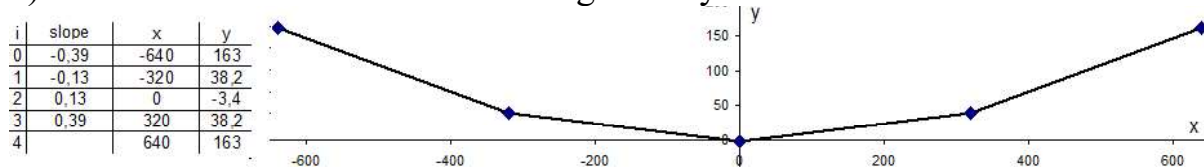
I report here about the design and experimentation of a situation for 12th grade students. It deals with the modelling of suspension bridges, aiming to make students discover a function modelling the main cable by way of a study of tensions. Inspired by the framework defined in the previous section this section presents first briefly four models and then the associated working spaces and a classroom implementation.

Suspension bridges: four models

a) A physical model of tensions along the cable.



b) A model of the cable in coordinate geometry.



c) An algorithmic model of the cable

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1://coordinates of the anchoring point
2: x ← -640
3: y ← 163
4://Tension on the anchoring point
5: V ← -5
6: Pour i allant de 1 à n
7:   x ← x + 1280/n
8:   y ← y + 1280/n * (V/H)
9:   V ← V + 10/(n-1)
10: Fin Pour

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d) A continuous model of the cable

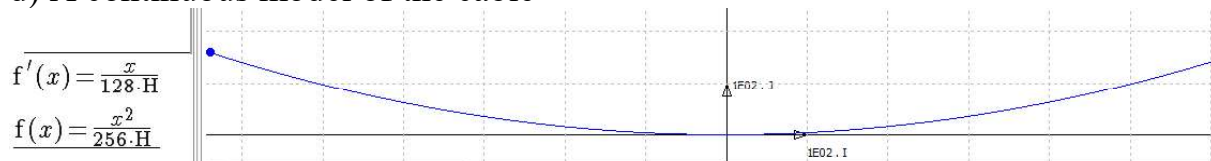


Figure 2: Models of a suspension bridge.

A suspension bridge is a type of bridge in which the deck (generally a roadway) is hung below main cables by vertical suspensors equally spaced. The weight of the deck ap-

plied via the suspensors results in a tension along the main cables. The central question is to find models of a main cable, allowing solving technical questions like the value of the tension for given data characterizing the bridge.

In a first model a main cable is represented by a physical mockup, i.e. weights equally spaced horizontally and suspended to a line. The static equilibrium law allows studying the sequence of tensions $\vec{T}_i$ in the line between the suspension points: the horizontal component H_i has the same value H in all segments and the sequence of values of the vertical component (V_i) is in arithmetic progression (figure 2a). The second model is a broken line in coordinate geometry. The slope of each segment is the ratio of the vertical by the horizontal components of the tension in this segment, and therefore is also in arithmetical progression. It is then possible to compute the sequence of the coordinates of the points (figure 2b). The third model systematizes the construction of the second model by way of an algorithm defining a piece wise linear function. Animating the global variables n (number of segments) and H (horizontal component of the tension) allows visualizing their influence (figure 2c). The fourth model is the curve of a mathematical function depending on a parameter H that can be also animated: the derivative is calculated as a limit of the slopes of segments in the broken line and the function is obtained by integration (figure 2d). The third and fourth models involve a software environment: functions are created by formula and domain, and also by an algorithm; graphs can be obtained and animated by way of sliders. The software Casyopée was used for the experiment (<http://casyopee.eu>).

Potentialities, constraints and general design

The brief presentation above shows that studying a suspension bridge implies considering data in the real world as well as a number of interrelated concepts in physics and calculus: tension, static equilibrium of forces, projection of vectors, slope of segments and gradient of curves, arithmetic progression and linear function, integration, discrete and continuous models, limits and integration... All these contents are taught in secondary curricula, thus the goal for students is not to "reinvent" each of them in isolation, but rather to recognize how modelling a real world situation involves understanding these concepts operationally and in interaction. In the French curriculum, the study of a suspension bridge can be carried out in the last year of the secondary scientific stream (12th grade, Terminale). The framework of connected working spaces was used to design a classroom situation exploiting the potential of the study of a suspension bridge. I consider four working spaces, each related to one of the four models. In each space, I characterize briefly the epistemological plane.

In the first working space, the representamen is the mockup with posts, line and weights, and the rules are the static equilibrium law and the properties of arithmetic progressions. Artefacts are concrete measurement devices used in physics and mathematics, dynamometers, protractor, and also more "abstract" tools like the decomposition of tensions in vertical and horizontal components. We name this space, the static systems working space, or shortly, the *statics working space*. In the second working

space the representamen is a broken line. The main rule is the analytical definition of a segment: students have to compute the coordinates of the end point, knowing the coordinates of the other point, the slope and the difference between abscissas. We name this space, *the coordinate geometry working space*. In the third working space the representamen is the graph of a continuous piecewise function and an important artefact is the programming module in the software environment that allows computing the series of points defining the function by way of a simple iterative treatment and animating global variables. We name this space, *the algorithmics working space*. Finally, the representamen in the fourth space is a function governed by classical rules in calculus. This is the *mathematical functions working space*. An important artefact is the software environment that can be used to get a curve of this model, compare to a picture of the bridge and to the curve of the algorithmic model's piecewise function, and animate the parameter H in order that the three models fit.

IMPLEMENTATION

In order to be feasible in the context of a French 12th grade class a few weeks before the baccalaureate, the implementation is limited to three and a half hours and organized in four phases. The first phase is one hour long and has been prepared with the physics teacher. It aims first to introduce students to questions related to bridges, particularly suspension bridges. They are invited to consult a dedicated website (<http://structurae.info/ouvrages/ponts-et-viaducs>), to select and sketch four bridges of different types, to look at a video illustrating the idea of tension along a horizontal rope and the fact that, whatever the tension, the rope is no more a straight line, as soon as force is applied vertically on a point, and to answer questions about suspension bridges. Also in this first phase, the students have to build apparatus combining dynamometers and weights, compute the horizontal and the vertical components of tensions and verify the static equilibrium of forces.

The second phase is 50 min long. At the beginning, the data related to the Golden Gate Bridge is presented to the whole class. Students also look at a physical mockup where the cable is a succession of dynamometers allowing observing the evolution of tensions along the cable. Then students are split into groups of four. Each group has one task, A or B, C or D. Task A is related to the *statics working space*: inspired by the work in the first phase, students have to consider the sequence of horizontal and vertical components of tensions at the connection points, recognize that the horizontal component is constant and compute a formula for the series of vertical components. Task B is related to the *coordinate geometry working space*. A formula for the value of the slope of each segment in a discrete model of the main cable is given to the students, depending on a parameter H , and on the number n of segments. Students have to compute the series of x and y -coordinates of the suspension points for a small value of n and a given value of H . Task C is related to the *algorithmics working space*. An algorithm like in figure 2c is given to students; they have to enter and execute the algorithm, interpret the parameter n , and adjust the parameter H in order that the model given by the algorithm conforms to the shape of the cable. Task D is related to the

mathematical functions working space. Students have to search for a function f whose curve models a main cable. They are informed that the horizontal component of the tension in the cable is a constant H and of a linear formula for the vertical component of the tension at a point of given x -coordinate. They have to find a formula for the derivative of f , taking into account that the tension is in the direction of the tangent to the curve. Then, using the software environment, they have to find a formula for f and adjust the parameter H in order that the curve of the function f conforms to the cable.

The third phase is also 50 min long. In this phase, the students are expected to develop connections between the working spaces of the second phase. They form new groups. Each of these new groups is made in order to bring together one or two students of each of the previous groups respectively doing task A, B, C and D. Students are invited to share their findings and to write a report emphasizing the important points of the study. The fourth phase (30 min long) is a collective synthesis led by the teacher.

Observation and evaluation

This implementation was observed in a class of 35 students by the end of March. The students were mostly average achievers. The contents at stake in physics and mathematics had been taught to students in previous lessons. The phases have been video recorded. In the first phase, most students sketched a suspension bridge without suspensors. They understood from the video that the weight of the deck "bends" the cable, but they did not link the shape of the cable with the uniform repartition of the weight, thanks to the suspensors. In the rest of the phase, the students correctly recorded angles and intensity of forces and recognized the static equilibrium law.

For the group work in phases 2 and 3, I report on a series of four groups observed doing each task in phase 2, and on one group in phase 3 bringing together students observed in phase 2. In phase 2, students doing task A mainly succeeded, while difficulties were observed for students doing other tasks. Students doing task B started by sketching a bridge with a lot of suspensors, not allowing to consider segments. They were prompted by the teacher to limit to 4 suspensors. They took time to find the coordinates of the anchoring point and had difficulties to use the formula given for the slope of the segments and the distance between suspensors, in order to calculate the coordinates of the next point. Students doing task C took time to enter the algorithm in the software. Nothing or a wrong display appeared on the screen, because of small mistakes. They corrected when the teacher helped them to analyze the algorithm. They identified the parameter n as related to the number of suspensors and proposed the value 83 (the number of suspensors in the Golden Gate Bridge). They considered that this value is "close to infinity" and that is why the curve did not appear as a broken line, in contrast to small values of n . Students doing task D found a formula for the vertical tension but had difficulty to interpret the fact that the tension is in the direction of the tangent.

In the group of phase 3, each student explained her task and her work in the preceding phase. Other students listened attentively and asked for further explanation. The parameter H was identified by students as playing a role in each task; for instance when a

student who did task C did not remember the effect of increasing H , confusing with the "height of the cable", the student who did task A corrected him, saying that it is a tension and then increasing should "straighten" rather than "slacken" the cable. The same student helped to overcome the difficulty met by the student who did task D to find the direction of the tangent to the curve and then the derivative of the function, saying "you just integrate the quotient of V and H ".

To further evaluate the connections students made between working spaces during the group work, I report on interviews with 3 students after phase 3. Each interview was 20 min long. The students were first questioned on their impression about the tasks and then they were invited to summarize their findings. They stressed that the situation was more complex than usually and that they were "not used to mix physic and mathematics". Commenting the first phase, they showed how their awareness of the structure of a bridge progressed, mentioning correctly the role of the suspensors. They still had difficulties in considering the slopes of the segments in task B in order to find the coordinates of the suspension points. They correctly interpreted the algorithm of task C, and were able to connect the evolution of H , and x and y respectively to task A and B. They did not show clear awareness that the function of task D was a limit of the continuous piecewise function of Task C. From graphical evidence they thought that it was more or less the same function for big values of n . The observer asked to explain why the gradient in a point of the curve is the quotient of V and H . The expected answer was that the tension has the direction of the tangent, but the students simply wrote $f'(x) = \Delta y / \Delta x = V(x) / H$. The first equality is common in the physics course, and the second derives from the definition of the components in task A. Thus, students made a connection between the statics space and the mathematical function space without explicit consideration of a limit.

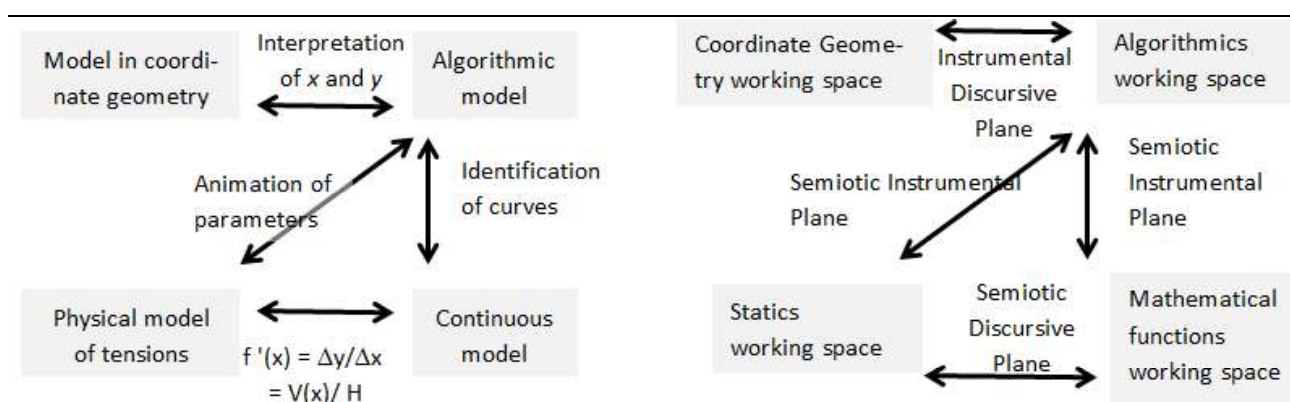


Figure 3: Connections made by students.

The figure 3 (left) summarizes the connections between models made by students. On the right, each connection is interpreted as connecting working spaces and participating in geneses. When students interpreted the evolution of the variables x and y in the algorithm, by connecting the body of the loop with the recurrence law of the coordinates in the geometrical model, it participates in a *discursive* validation of an *instrumental* object. The animation of parameters (an *instrumental* activity) within the

software environment supports visualization (a *semiotic* activity). The *discursive* explanation that students gave of the value of the gradient of the curve of the mathematical function is inspired by the *semiotic* notations of derivatives in physics. Each connection sheds light on concepts at stake: recurrence and iterative treatment, parameters in a function, tension, discrete and continuous models, gradient and derivative. Kuzniak, Tanguay, & Elia (2016, p. 728) name “vertical planes” the two by two combinations of geneses and depict them as “valuable tools for describing the evolution of the mathematical work in the solving process”. They especially warn that “activities with a strong constructivist feature” have to be designed in order to “avoid to be imprisoned in the Semiotic Instrumental plane”. Figure 3 shows that, in students’ activity, this plane is complemented by the two other planes, involving validation.

CONCLUSION

The experiment indicates benefits of a design inspired by the connected working spaces framework. The framework allows thinking of modelling as involving several models, each of them making reality and mathematics interact in different ways. This allows seeing models close to reality as belonging to scientific domains (physical or nautical science...) rather than to common sense and modelling as active appropriation of models of varied natures for a given reality rather than a “translation” between reality and mathematics. By considering several models and characterizing activity on each model as a work in a specific working space the framework helped to organize effective group work in realistic classroom settings. Developing modelling interdisciplinary activities for students is encouraged by curricula, but rarely achieved; the framework and the organization proposed here could be means to overcome implementation difficulties. In spite of the complexity of the situation, students made sense of the main aspects of the models and connections between them. These connections enabled them, through specific geneses including validation as well as visualization, to understand more comprehensively concepts taught at upper secondary level. Moreover, they could appreciate how mathematics and experimental sciences contribute together to understanding the sensible world.

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