

New representational infrastructures: broadening the focus on functions

JEAN-BAPTISTE LAGRANGE*

Laboratoire de Didactique André Revuz, Université Paris-Diderot, 75013 Paris, France

**Email: jean-baptiste.lagrange@univ-reims.fr*

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For more than 10 years, I had the honour and pleasure to work with Celia Hoyles and Richard Noss. We share a common concern for more learnable mathematics, especially in algebra, and for the need to build new representational infrastructures taking advantage of technology. Beyond this common concern, my choice to work in the French institutional context and paradigms for the Casyopee project led to a different approach. Roughly speaking, in these context and paradigms the stake is to help students access existing representations while Hoyles and Noss privilege building new more learnable representations. The aim of the article is to investigate empirically how, in spite of this gap, Hoyles and Noss' approach can be an inspiration for enlarging the focus on a given topic, freeing oneself of a too strict dependency to one's own context.

Some years ago, I had the honour and pleasure of working with Celia Hoyles and Richard Noss for a chapter in a book at the invitation of Jürgen Maasz and Wolfgang Schlöglmann (Hoyles *et al.*, 2006). Our friend and colleague, Jim Kaput, had been tragically killed shortly before and we realized that his work on representational infrastructures provided a crucial challenge for mathematics education. By situating current representational systems in their historical context, Jim showed the possibilities for designing alternatives. We realized that for Jim, and for us, a key concern was to open up, democratize and make more learnable the complex ideas of mathematics, ideas whose complexity often owes as much to the way they are represented as to the ideas themselves.

One focus of our chapter was algebra, and more specifically for me, functions. We stressed that paper/pencil algebraic representations make necessary for individuals to pay considerable attention to manipulation, and that key mathematical topics are only amenable to those who had already been inducted into fluent algebraic representations and calculations. Our goal was to demonstrate that digital technologies can radically change this scenario.

We based our reflection on two distinct projects. We thought that these projects were converging in their goal—the design of technological environments to enhance learnability in algebra—but adopted different orientations. While Celia Hoyles and Richard Noss based their project *WebLabs* on the idea of building *new* representations for more meaningful student mathematical activity, the project I was working on, Casyopée, was designed to help students to make sense of *existing* mathematical representations with the support of the computer.

In 2006, my work of designing mathematical software for the classroom was in its infancy and, since then, I have carried out further development while attempting to understand in more depth Celia and Richard's ways of offering new, more learnable, representations. Eight years later, in this article,

I try to take stock of how this attempt contributed to enlarge my view of a subject-like functions and on the potential of technology.

In a first short section, I briefly describe how I see representations in the rich work of Celia and Richard. Then I present an attempt to exploit a ‘classical’ approach of representations in algebra and calculus at the time of our 2006 chapter, and then I present further work aiming at enlarging my view with some inspiration from my interactions with them both. Finally, I discuss this work alongside the main ideas related to representational infrastructures.

1. From representations to new representational infrastructures

Noss & Hoyles (1996) consider the role of representations in problem solving (p. 39) insisting on the ‘structuring role of the different types of representation adopted’. They oppose ‘representations which are global in the sense that they are drawn-on’, especially paper/pencil algebraic representations, preferring representations that are ‘locally evoked . . . and constructed in response to the specificities of problem solving in time and space’. They note the importance of ‘global’ representations in that they influence what people do. However, they stress the necessity of exploring the reciprocal effect of people on the development of representations.

In the 2006 chapter, as well as in more recent work (Hoyles & Noss, 2008), they build on Kaput *et al.* (2002) idea of representational infrastructures to discuss the role of systems of representation in cognition and learning, and especially to address the challenge of designing learnable systems. They stress that ‘the difficulty of a mathematical idea often inheres in the system with which it is expressed’.

For Hoyles & Noss (2008), ‘the point is that some infrastructural changes are so powerful, so transformative in, not just the possibilities of new pedagogy, but of epistemology – they redefine what counts as possible, what is expressible’ (p. 90). They mention their work on designing new kinds of representational infrastructures for opening new trajectories for learners who cannot adequately deal with conventional representational forms.

In this article, I will explain on the one hand how my approach is different and the other hand, how the powerful ideas and concepts raised by Noss & Hoyles are helpful in this approach.

2. Helping students to make sense of mathematical symbolism with the support of the computer

In the 2006 chapter we stressed how, in the paper/pencil context, symbolism is a major obstacle for most students.

Paper/pencil algebraic infrastructures make it necessary for individuals to pay considerable attention to manipulation, and key mathematical topics are only amenable to those who had already been inducted into fluent algebraic representations and calculations. This means that many never engage with the mathematical topic at hand (p. 269).

At that time, my main concern was whether, knowing these difficulties and the potential of dynamic technologies like dynamic geometry, spreadsheet, etc. for the exploration of mathematical notions and problems, mathematical symbolism would survive outside pure professional mathematics. I noted that

Computer symbolic systems (CAS), have been presented as a means to overcome students’ difficulties in paper/pencil manipulations (. . .) Although promising, this approach has not been, in our view, sufficiently discussed from an epistemological point of view, and its ‘viability’, that is to say the conditions in which it could be effective in actual classrooms, remains problematic (p. 270).

I explained that the main aim of the French Casyopée project that I had begun with some colleagues and teachers, was to address the challenge of offering a CAS tool that could be effective in secondary classrooms. I added:

The focus in Casyopée is on functions, because functions are objects and tools in many algebraic activities at upper secondary level. This choice of functions had consequences on the objects that Casyopée handles:

- (a) letters identify the variable, functions and parameters,
- (b) graphs and tabular representations of functions complement symbolic expressions,
- (c) equations correspond to the search for a value of the variable for a given value of the function or they can be about the equality of two functions, or the search for a value of a parameter. (p. 276).

With reference to the topic of mathematical representation, this project was mainly inspired by Duval (1999)'s framework, which will be presented in the next section and exploited to discuss a task for students that had been motivated by this approach.

2.1 A 'register' approach to multirepresentation

For Duval (1999, p. 4), 'there is no other ways[sic] of gaining access to the **mathematical objects** but to produce **some semiotic representations** (. . .) On the other hand, the understanding of mathematics requires not confusing the mathematical objects with the used representations[sic]' (emphasis added). According to Duval, rather than assuming material and mental representations are in opposition, it is necessary to distinguish two types of implementation of representations by the subject, the automatic activation of an artefact (material or mental), that he names as an organic system, and the intentional exploitation of a semiotic system (Fig. 1).

Implicitly, Duval presupposes the existence of mathematical objects. In automatic activation, this object itself produces the representation, for instance 'proportionality' produces points in a line on a concrete graph, whereas in intentional use, a semiotic system, that is to say a set of symbols and rules for using them, represents the object (that is to say is used in place of the object). The graph and its associated features denotes 'proportionality'. Thus, representing 'something' can be thought of in two ways: the activation of a mental or physical device showing the outcomes of making this 'something' work or the conscious use of a semiotic system in place of the 'something'.

For Duval, each semiotic system or 'register of representation' has its own specific means of representations and processing. These systems are a product of history.

History shows that progress in mathematics has been linked to the development of several semiotic systems (. . .) For example symbolic notations stemmed from written language have led to the algebraic writing and, since the nineteenth century, to the creation of formal languages. (Duval, 1999, p. 4)

Automatic	Intentional
Activating an organic system or a physical device	Bringing a semiotic system into play
The representation is the outcome of a physical action of the represented	The representation denotes the represented object

FIG. 1. Représentations and their implementations (from Duval, 1999).

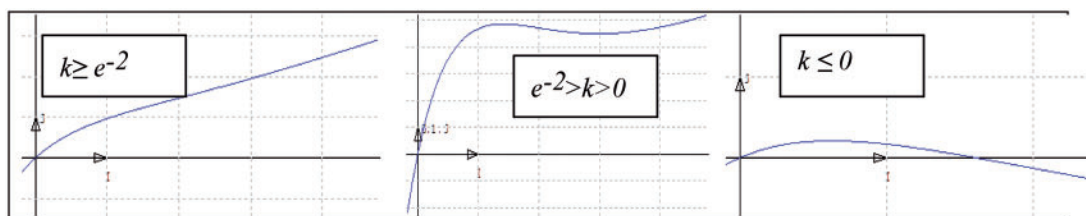


FIG. 2. Three behaviours of the function. This figure appears in colour in the online version of *Teaching Mathematics and its Applications*.

Duval stresses the need for a specific focus in teaching and learning on the processes of work inside the registers (treatments) and between the registers (conversion).

(...) repeated observations show that semiotic representations constitute an irreducible aspect of mathematical knowledge and that wanting to subordinate them to concepts leads to false issues in learning (...) Each semiotic register of representation has a specific way of working, of which students must become aware. (Duval, 1999, p. 8)

Relative to functions at upper secondary level, two registers are generally involved, the register of symbolic expressions and the graphic register of curves of functions. Many research studies show that in classroom practices the register of symbolic expressions dominates, and in this register rote treatments dominate over interpretation of expressions and understanding of the rules and purpose of transformations. Conversions are frequently from mathematical expressions to graphs, occurring in standardized tasks like finding the correct graph of a function from its algebraic expression.

In this section, I look at the conditions in which technology can help a more versatile use of each of these registers as well as providing balance and coordination between these registers.

This is a task for pre-calculus students, using Casyopée, studied by Minh (2011).

Consider the family of functions f_k over the set of real numbers $f_k(x) = x \cdot e^{-x} + k \cdot x$, k being a real parameter. Study the variations of f_k depending on k .

This task has been chosen because a purely symbolic solution is not directly accessible through standard techniques, since the root of the derivative cannot be calculated symbolically. An effective standard technique is to study the variations of the derivative, after calculating the second derivative, but this technique is not known by students. Another feature is that the function has three different behaviours depending on k (Fig. 2): the one on the left for k greater than a certain value (e^{-2}); the second between this value and zero; and the third for k negative.

The important thing is that this value is close to zero, and thus a pure graphical exploration might overlook the intermediate behaviour. It can be expected that most students will first animate the parameter for integer values and then actually skip this intermediate behaviour. However, a close inspection of the graph of the derivative for $k = 0$ and $k = 1$ (Fig. 3) will give a clue that ‘something happens’ between these values: passing from 0 to 1 the curve is translated vertically, and then an intermediate value would bring the curve to intersect the x -axis in two points. This idea of a translation is confirmed by the symbolic expression $f'(x) = (-x) \cdot e^{-x} + e^{-x} + k$ where k is an additive constant. Then, in order to carry out the task, the students have to perform non-routine processes inside the symbolic and the graphic registers, and to coordinate explorations and observations in the two registers.

Designed as a CAS system especially devoted to functions at this level, Casyopée allows students to perform automatically standard calculations like derivatives and solutions of equations, as well as to graph functions. The functionality to drive the values of k by means of a slider helps to animate the graph of the function, and also to instantiate this parameter inside expressions of the functions.

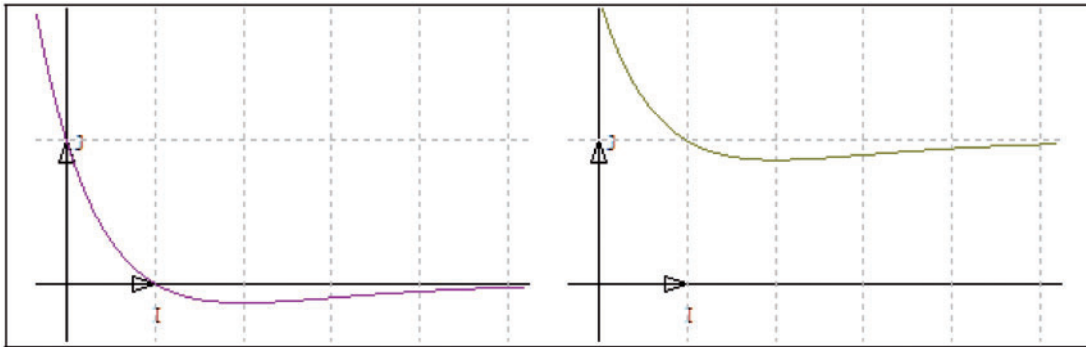


FIG. 3. The derivative for $k=0$ and $k=1$. This figure appears in colour in the online version of *Teaching Mathematics and its Applications*.

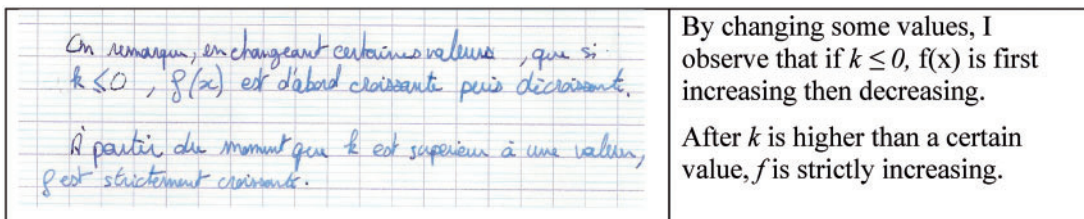


FIG. 4. A type 1 answer. This figure appears in colour in the online version of *Teaching Mathematics and its Applications*

In short, Casyopée is expected to give security for calculation purposes and to favour joint interpretation of graphs and symbolic expressions.

Investigating students' solutions at various stages, we observed three types of answers.

In a first type 1 (Fig. 4), the intermediate behaviour is not detected. The student keeps the default value 1 for the step in the animation of the parameter and does not closely look at the derivative. This type of answers is frequent from beginners not yet acquainted with the management of the parameters in Casyopée and who do have not a flexible use of the derivative.

In a second type of answer (Fig. 5), the student realizes that it is easier to observe the graph of the derivative than the graph of the function. Animating the graph with a smaller step of the parameter, she detects the intermediate behaviour, but she dwells in the graphic register, and then she cannot find the critical value (Fig. 5).

In a third type of answer (Fig. 6), the student finds the critical value e^{-2} by asking Casyopée for the solution of $f'(2)=0$ (Fig. 6). Analysing her approach, we identify four steps:

- she observes on the graph that the derivative is always maximum for $x=2$;
- coordinating the algebraic form of the derivative and its graphic representation, she notes that k is an additive constant and then the animation translates the graph vertically;
- she figures out that to find the critical value, she has to 'bring the graph' onto the x -axis, which is not possible by animating for decimal values of the parameter; and
- she solves symbolically by way of the equation in k $f'(2)=0$.

This third type of answer is observed from students with mature use of Casyopée (Minh, 2011). It is evidence of a very effective coordination between the symbolic and the graphic register. Casyopée

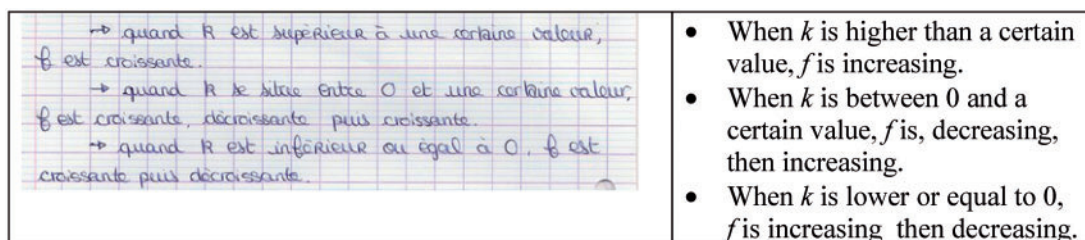


FIG. 5. A type 2 answer. This figure appears in colour in the online version of *Teaching Mathematics and its Applications*.

provides the means to make this coordination effective. It does not in itself give the students this capacity for coordinating registers. This capacity is the result of a process of students' growing awareness of Casyopée's functionalities *and* of the idea of functions as expressed in several registers. This process (defined by Guin & Trouche (1999) as 'instrumental genesis') was observed by Minh (2011) during 2 years of studying students' learning of pre-calculus.

2.2 New and alternative representations of functions

Duval's framework is very rich and many researchers draw on it, especially when dealing with digital representations. As said before, it presupposes that 'ideal' mathematical objects (notions, ideas, concepts...) exist. Duval's registers of representation were productive in the development of a tool for helping students deal with tasks about functions like the problem above. However, it was only a first stage and I regretted in the 2006 chapter that

Casyopée works on functions given by algebraic representations and till now provides no enactive representation for non-algebraic functional exploration and limitations in the students' experimental activity, especially when modelling phenomena. (p. 282)

I also pointed out the need to develop features allowing students to work with enactive non-algebraic representations. Shortly after 2006, I had the opportunity to work with colleagues on an extension of Casyopée in a project entitled 'Representing Mathematics with Digital Medias' (ReMath) (Kynigos & Lagrange, 2014). With regard to Casyopée, one goal was to explore new, non-standard representations of functions. We had also to carry out comparative or 'cross studies' of digital tools developed inside the project: each piece of software developed in the project had to be 'cross-experimented' by two teams and this cross experimentation was followed by a 'cross-case analysis' (Kynigos & Lagrange, 2014). In this section, I report first on new ways of representing functions introduced in this extension of Casyopée, and then on other views coming from the 'cross studies'.

2.3 Modelling dependencies in a physical system

One outcome of the work on Casyopée in ReMath and after was a broadened view of functions in the students' mathematical activity. The idea of modelling dependencies in the physical world was thought of as a way to allow students to work with enactive non-algebraic representations of phenomena and to pass fluidly between algebraic and non-algebraic representations. More precisely, the Casyopée team imagined a 'modelling cycle' organizing models of a dependency in specific settings. Figure 7 illustrates this cycle with the example of the 'amusement park ride' that I will use in this sub-section.

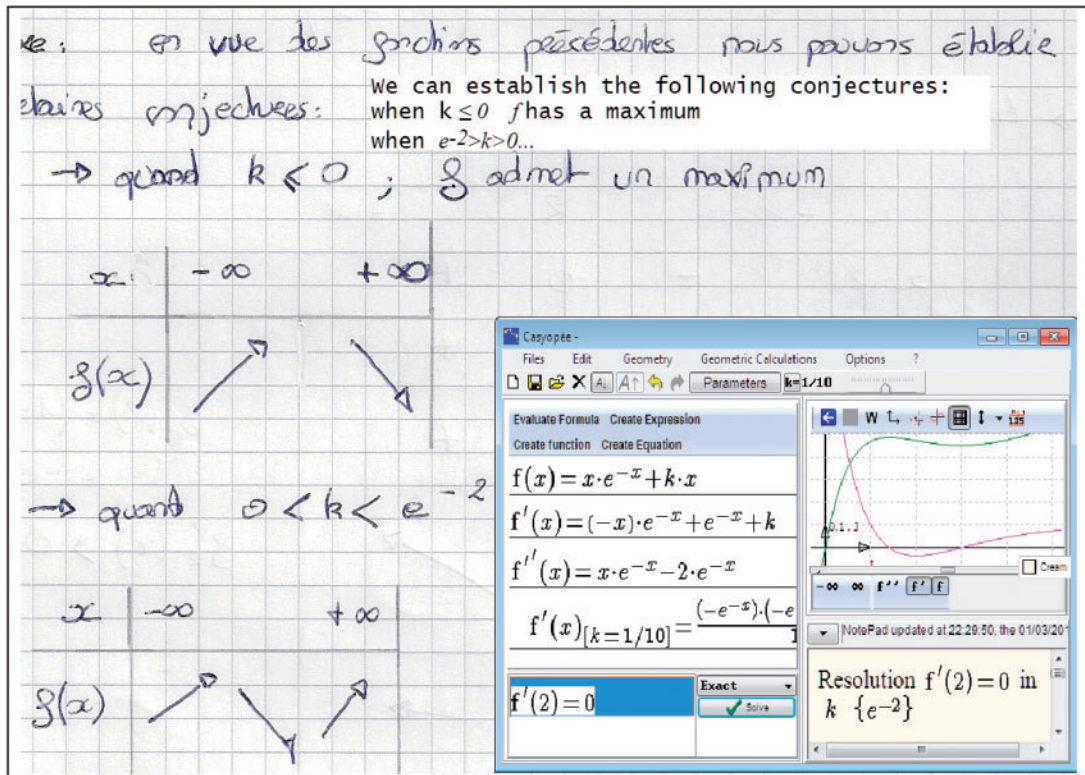


FIG. 6. A type 3 answer. This figure appears in colour in the online version of *Teaching Mathematics and its Applications*.

Basically, dependency is a human experience involving items in a *physical device*. In this setting, one item moves and another follows.

Human beings used drawings since the prehistoric times to ‘represent’ features of the physical world and they made them evolve to allow accurate work related to physical devices. We consider *dynamic geometry* as a product of this evolution, with the particular capacity of ‘representing’ dependencies: when the user drags an object on the screen, another object follows on the screen. Passing from a physical device to a dynamic geometry figure is in no way trivial, since it implies the identification of relevant features of the device and of their geometrical dependencies, modelling physical dependencies.

Quantities constitute another setting for working about dependencies. According to Thompson (2011), a *quantity* is the outcome of a process of *conceptualizing an object and an attribute of it so that the attribute has a unit of measure*. Thompson (ibid.) stresses the role played by quantitative reasoning in students’ operations of generalizing, and of the role that generalizing plays in students’ development of algebraic reasoning. He offers a number of ‘*dispositions*’ that can aid students’ construction of algebraic reasoning from quantitative reasoning. Among these, consistent with the school level for which the Casyopée project is intended, I retain reasoning with *magnitudes*, that is to say attributes of objects considered independently of the unit in which they are measured.

As Thompson (1994) says *one way to think of the evolution of today’s many ways to think of functions is as the current state of a long battle to conceptualize our world quantitatively and then this world of quantities or magnitudes, often overlooked in secondary curricula, is a key*

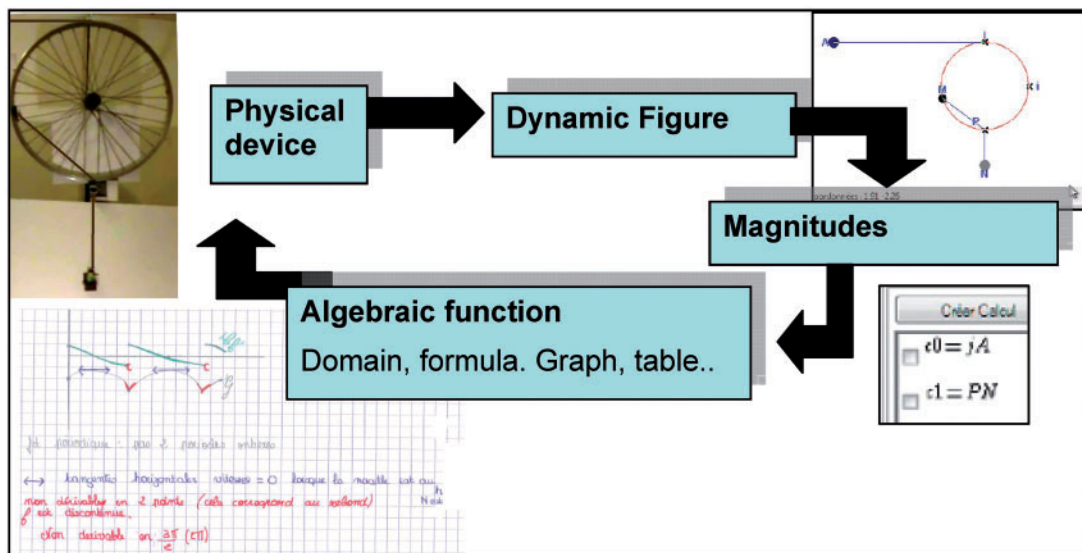


FIG. 7. An example of a modelling cycle. This figure appears in colour in the online version of *Teaching Mathematics and its Applications*

representational setting for students' access to functions. Especially, in this setting, distinguishing functional dependency from covariation, and identifying an independent variable, are crucial challenges.

As a fourth setting, *algebraic functions*, like those considered in the previous section, is the outcome of a historical process of emergence of functions as an individualized mathematical entity by mathematicians like Descartes, Newton and Leibniz as a tool for building infinitesimal calculus.

2.4 The extension of Casyopée

The extension of Casyopée carried out in the ReMath was intended to provide an environment supporting these various representations and their connections. It consisted in adding first a dynamic geometry window and second, representations of measures and of their covariation in a 'geometric calculation tab'. The existing part of Casyopée remained in an enhanced form, under the denomination of 'symbolic window'. The 'geometric calculation tab' is a unique feature, allowing covariations between pairs of magnitudes to be explored, to export those pairs that are in functional dependency into the symbolic window and then to define a function to model the dependency, likely to be treated with all the available tools. In order to help students in modelling dependencies, this can be done automatically (Lagrange, 2010). We will refer to this functionality as 'automatic modelling' below.¹

For the Casyopée team, the goal was that students meet functions through meaningful questions and through connected representations in these four settings. An example is the 'amusement park ride' situation.

A wheel rotates with uniform motion around its horizontal axis. A rope is attached at a point on the circumference and passes through a fixed guide. A car is hanging at the other end.

¹ For a detailed presentation of this functionality and Casyopée's other capabilities, please see the documentation downloadable from the website (<http://www.casyopee.eu/>).

The motion is chosen in order that a person placed in the car feels a smooth transition at the high point and an abrupt transition at the low point and then, in the classroom, the focus is on the identification of these different transitions on a physical model *and* on a mathematical function modeling the movement of the car. It is expected that students will associate the transition at the highest position of the car with a turning point of the function, and the transition at the lowest position with the non-differentiability of the function at the corresponding point. The dynamic figure and the expression of the movement as a dependency between magnitudes are essential parts of the modelling activity, helping to make sense of the relationship between the physical model and the mathematical function.

3. A classroom observation

The task was carried out with a 12th grade class (17–18 years old students) in a 90-min session. The problem was given in a ‘real life’ setting, the students being able to manipulate a scaled device. The students were first asked to describe what is happening at the lower point and whether it is different as compared to the high point. They were encouraged to refer at how a person in the car would ‘feel’ the difference and to draw a graph to explain their answer. Then they were told that, during the session, they have to build a mathematical model in order to better investigate the phenomenon.

The first step of modelling process was to build a dynamic geometry figure in Casyopée, replicating the device. The following indications were given to the students: the rope is attached to the wheel in a mobile point M and the guide is on the fixed point P. The car is in N (Fig. 7). The wheel is supposed to be put into rotation by pulling on a horizontal rope JA. This implies non-trivial constructions for the point M in order that the circular distance IM is equal to the linear distance Aj, and for the point N in order that MP + PN is constant. The students had to use Casyopée first to implement a dependency between magnitudes modelling the dependency in the figure, then to obtain an algebraic function by ‘automatic modelling’, to get the derivative. They should notice and finally identify precisely the points of non-differentiability.

I report on this situation in five steps: (1) the students’ spontaneous model of the physical situation; (2) how they built a dynamic geometry model; (3) how they chose the dependent and independent variables and how they interpreted this choice; (4) how they worked on the algebraic function obtained via Casyopée’s automatic modelling; and (5) how their understanding of the physical situation progressed after working on the algebraic function.

3.1 *Students’ spontaneous model*

At the beginning of the session, after presenting the situation and demonstrating by animating the scaled model, the teacher asked the students to describe what was happening at the lower point and whether it was different when compared to the high point. Figure 3 illustrates a typical answer. Students said that at the high point, the car stops and they had some difficulties to explain what was happening at the lower point. The more common expression, ‘drop shot’, is not accurate because it means that the car is arriving at a certain speed, stops and starts up again at a lower speed. Students illustrated by a graph of a piecewise linear function. Actually they thought that because the wheel rotates uniformly, the car’s movement should be piecewise uniform (Fig. 8).

3.2 *Building a dynamic geometry model*

This was a difficult part of the task for the students. Their poor practical knowledge of trigonometry explains why they needed help to define M in order that the circular distance IM equalled the linear

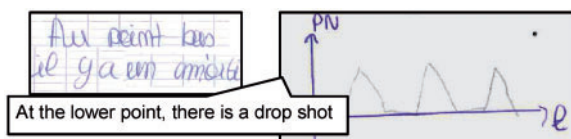


FIG. 8. Students' spontaneous model. This figure appears in colour in the online version of *Teaching Mathematics and its Applications*.

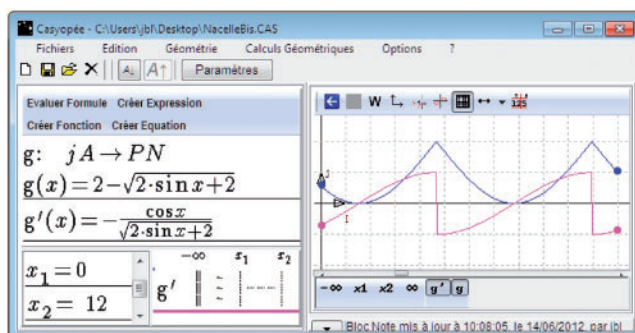


FIG. 9. A graph of the derivative with wrong vertical segments. This figure appears in colour in the online version of *Teaching Mathematics and its Applications*.

distance A_j . It seems more surprising that they found it difficult to define N in order to make $MP + PN = 2$ (the length of the rope). After the teacher indicated that PN is known when MP is known, some students used a circle centred in P with a radius of $2 - MP$ and defined N as an intersection point with the y -axis, and others directly defined N with the coordinates $(0; y_P - (2 - MP))$. Interestingly this work implies geometric construction as well as magnitudes.

3.3 Choosing the variables

Generally, the students had no difficulties in putting their decisions into action with the software. However, their expression was sometimes confused when explaining the decision. For instance, a pair of students wrote in the report: 'We choose distance A_j as the (independent) variable' and added ' A_j is a function of the coordinates of N '.

3.4 Working on the algebraic function obtained via Casyopée's automatic modelling

The students obtained the derivative by using Casyopée's automatic modelling under the form $x \rightarrow \frac{-\cos x}{\sqrt{2 \cdot \sin x + 2}}$. Casyopée issued warnings because this function is not defined everywhere. Students ignored the warnings and obtained a graph with wrong vertical segments (pink in Fig. 9).

The teacher drew students' attention on these segments and they recognized that there should be discontinuities of the derivative corresponding to the low points. The teacher asked them to compute the position of these discontinuities. No students did this from the formal definition of the derivative. Rather they came back to the physical device or to the dynamic geometry figure, looking for the value of jA corresponding to the lower point of the car. After they found these values and excluded them from the definition of the derivative, they got a correct graph (Fig. 10).

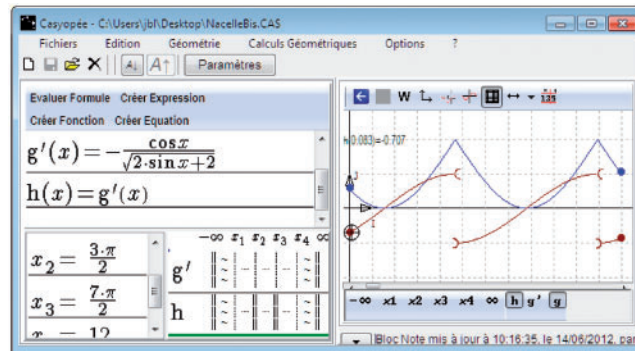


FIG. 10. A correct graph of the derivative. This figure appears in colour in the online version of *Teaching Mathematics and its Applications*.

3.5 Students' understanding of the physical situation after working on the algebraic function

Students' understanding was much better after working on the algebraic function. They identified the derivative and the car's speed, saying that the speed is null at the high point corresponding to a horizontal tangent on the graph of the movement. Implicitly, they recognized that at the lower point the car starts up again briskly at the same speed, speaking of 'rebound' corresponding to non-differentiable points, rather than of 'drop shot' implying softer stop and restart. This is evidence that the students connected features of the mathematical function with a phenomenon experienced in the physical word. Clearly students got a broad view of this function, coordinating many aspects or representations, without dominance of one representation. The algebraic formula was an aspect between others, the pole of the derivative giving a specific insight, complementary to the discontinuity on the graph, and the feeling of 'rebound' at the lower position of the car.

4. Other insight coming from a ReMath 'cross study'

The experimentation of several situations based on the above 'modelling cycle' comforted the team about the productivity of the enlarged view of functions on which Casyopée's extension is based. A 'cross study' carried out in ReMath provided additional unexpected insight. ReMath aimed at investigating contextual elements influencing research about mathematics education and technology, and to progress towards theoretical integration. The method in ReMath was to use 'cross studies'. Next I consider the Cruislet cross study that linked a Greek team and the team developing Casyopée.

The Greek team was influenced by the constructionist framework inside which Celia and Richard also work. They designed Cruislet as a navigation microworld in which users direct planes across Greek geography by issuing navigation instructions in either graphical/Cartesian or spherical/polar coordinate systems, in direct stepwise mode or by way of Logo programming. The modalities of employment exposed by the Cruislet team were consistent with their theoretical perspective:

Through the use of the Logo programming language the software becomes constructionist (...). Procedures constituting half-baked microworlds can be designed in the sense that students can observe the trips they define and then change the procedures to make changes to the trips. (Lagrange & Kynigos, 2014, p. 390)

For instance, the team implemented a 'guess my flight' situation based upon the use by students of a procedure that made one plane perform a flight to an arbitrary position while another reached a

dependent position, each of its coordinates being a linear function of the same coordinate of the first one. Using this procedure first as a black box and then decoding the procedure, students could make sense of the situation by investigating the co-variation of the planes and conceiving the first plane's position as an independent variable and the second plane's position as a dependent variable.

The Casyopée team encountered difficulties in implementing an experiment of Cruislet, mainly because the team perceived a distance between the software's design, capabilities and modalities of use, and the French curriculum. Analysing this distance in more depth, the Greek team pointed out that few of Cruislet's features had been used in the French teaching experiment although Cruislet, for instance by way of situation like 'guess my flight', had the potential to allow students to approach the idea of function, a central domain of interest for the Casyopée team. Again the Casyopée team's view of function was not broad enough: the domain of co-variation at stake in Casyopée is 2D geometry and the functions are polynomial, rational or transcendental real functions in one variable rather than multilinear functions. Although this reflection did not have direct implication on Casyopée's development, it helped to see that it would be pointless and counter-productive to consider *any* notion of function as transcending all possible representations. Together with the teachers that participated in the cross-experiment, we popularized Cruislet among French mathematics teachers in order to promote a wider view on functions.

5. Functions: a broad plurality of semiotic systems rather than a single concept

Duval's idea of considering several registers of representation, stressing the necessity of educating students both in the treatments inside registers and in the conversion between registers, was productive in the first phase of development of Casyopée, since it helped to build a tool and situations of use breaking with current narrow tasks and dominant conceptions of functions.

However, given the big variety of representations that we had to deal with in the extension of Casyopée and in the cross-studies, I came to think that it is unproductive to think of the idea of function as a single mathematical object, acting in different settings and denoted by all these dissimilar representations. Actually, the idea of analytic function, as invented by Descartes, Newton and Leibnitz dominates at secondary level, and teachers' and students' conceptions of functions are strictly dependent on an algebraic view of functions. It means that even when they consider situations where geometrical or physical dependencies are involved, these dependencies are used as a 'faire-valoir' (complementary or contrasting character in a play) rather than actual functions. There is then a danger that in the implementation of situations like the 'amusement park ride' above, too much emphasis is put on algebraic facts to the detriment of the other interpretations.

At other levels, especially by mathematicians, the 'set of ordered pairs' representation dominates. This representation is useful especially as a universal definition but it completely hides the idea of dependency that was at the foundation of functions. Non-standard functions like fractals go beyond analytic functions and are based on algorithmic representations.

It seems then more productive to think of functions as a constellation or a network of many representations, each rooted in a particular practice, no one dominating and with more or less strong connections between them rather than connections to a single object. This seems consistent with Radford's (1999) view that a *representation does not appear as a pointer to an abstract idea but as the contextual instantiation of social modes of knowing as expressed and contained in signs*. Especially considering the case of functions, Radford critiques a narrow view of functions and stresses that linking representations cannot be made without a common context.

The learning of functions is seen as the capability of a student to move from one representation to another (tables, graphics, algebraic formulas, etc.). These ‘external’ representations are seen as the expression of the same concept —that of function. In my view, each semiotic system (tables, graphics, etc.) leads to a particular concept.(...) What this means is that a contextualization among representations will be needed to link them, and this requires a different pedagogical action. (p. 149)

In the Casyopée team experience, contextualization can be carried out by students through meaningful broad questions, for instance the identification of different transitions as experienced in a movement like the car’s in the amusement park ride. We currently experiment with situations based on another question: why is the cable in a suspension bridge a curve? What kind of curve?

6. Conclusion

Before stressing the potential of technology to offer new representational infrastructures Kaput (1999) pointed the finger at a

deep, but largely tacit view of formalisms as separate and distinct from informal notations or utterances and from the phenomena that they are taken to represent intimately integrated with a Platonist and representationalist philosophical orientation that takes:

- Mathematical objects as pre-existing, to act as pre-given reference objects for mathematical notations,
- Language as an inert representational instrument which does not help create mathematics but only enables us (if we are sufficiently skilled in its use) to see and do mathematics, and
- Mathematics as separate from the material and social worlds we inhabit.(p. 157)

On the one hand, the Casyopée enterprise can be seen as an attempt to combat this dominant view, enlarging students’ representations of functions and connecting these with aspects of material and social life. On the other hand, the choice to focus on representations and modes of learning sustainable in the institution (in the sense of Douglas, 1989) which we consider part of our context, makes a difference: students are introduced to conventional representation systems, enhanced and connected to a broader network of meanings thanks to technology, rather than are free to create or privilege a kind of representation possibly far from conventional forms, as evocated in the first section of this article.

This is perhaps a big difference. However, looking at each other across a gap has proved fruitful. Recognizing the specificities of a context of research makes particularly important to ‘look outside’ to better appreciate the affordances of this context, beyond its limitations. I hope that this article bears witness to how working with Celia Hoyles and Richard Noss’ approaches of technology and representation in mind, was particularly productive.

REFERENCES

- DOUGLAS, M. (1989) *Comment pensent les institutions*. Paris: La découverte.
- DUVAL, R. (1999) Representation, vision and visualization: cognitive functions in mathematical thinking. Basic issues for learning. *Proceedings of the twenty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (F. Hitt & M. Santos eds), vol. 1. Cuernavaca, Mexico: Universidad Autonoma del Estado de Morelos, pp. 3–26.

- GUIN, D. & TROUCHE, L. (1999) The complex process of converting tools into mathematical instruments: the case of calculators. *Int. J. Computers Math. Learn.*, **3**, 195–227.
- HOYLES, C. & NOSS, R. (2008) Next steps in implementing Kaput's research programme. *Educ. Stud. Math.*, **68**, 85–97.
- HOYLES, C., LAGRANGE, J.-B. & NOSS, R. (2006) Developing and evaluating alternative technological infrastructures for learning mathematics. *New Mathematics Education Research and Practice* (J. Maasz & W. Schloeglmann eds). Rotterdam: Sense Publishers, pp. 263–312.
- KAPUT, J. J. (1999) Algebra & technology: new semiotic continuities & referential connectivity. *Proceedings of the Twenty-First Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (F. Hitt & M. Santos eds), vol. 1. Cuernavaca, Mexico: Universidad Autonoma del Estado de Morelos, pp. 156–158.
- KAPUT, J. J., NOSS, R. & HOYLES, C. (2002) Developing new notations for a learnable mathematics in the computational era. *Handbook of International Research in Mathematics Education* (L. English ed.). New Jersey: Lawrence Erlbaum, pp. 51–75.
- KYNIGOS, C. & LAGRANGE, J.-B. (2014) Cross-analysis as a tool to forge connections amongst theoretical frames in using digital technologies in mathematical learning. *Educ. Stud. Math.*, **85**, 321–328.
- LAGRANGE, J.-B. (2010) Teaching and learning about functions at upper secondary level: designing and experimenting the software environment Casyopée. *Int. J. Math. Educ. Sci. Technol.*, **41**, 243–255.
- LAGRANGE, J. B. & KYNIGOS, C. (2014) Digital technologies to teach and learn mathematics: context and re-contextualization. *Educ. Stud. Math.*, **85**, 381–404.
- MINH, T. K. (2011) *Apprentissage des fonctions au lycée avec un environnement logiciel: situations d'apprentissage et genèse instrumentale des élèves*. Thèse de Doctorat, Université Paris Diderot. Available at <http://tel.archives-ouvertes.fr/>.
- NOSS, R. & HOYLES, C. (1996) *Windows on Mathematical Meanings*. Dordrecht: Kluwer.
- THOMPSON, P. W. (1994) Students, functions, and the undergraduate curriculum. *Research in Collegiate Mathematics Education, I: Issues in Mathematics Education* (E. Dubinsky, A. H. Schoenfeld & J. J. Kaput eds), vol. 4. Providence, RI: American Mathematical Society, pp. 21–44.
- THOMPSON, P. W. (2011) Quantitative reasoning and mathematical modeling. *New Perspectives and Directions for Collaborative Research in Mathematics Education* (L. L. Hatfield, S. Chamberlain & S. Belbase eds), WISDOMe Monographs, vol. 1. Laramie, WY: University of Wyoming, pp. 33–57.

Jean-Baptiste Lagrange graduated from the Ecole Polytechnique of Paris in 1971. In 2000, he defended an “habilitation à diriger les recherches” in Didactics of Mathematics and became a Professor of Mathematics Education. In September 2012 he became Emeritus Professor of the University of Reims working as a senior researcher in the LDAR. Among a number of publications, he co-edited with Dame Celia Hoyles a book of reference: *Mathematical Education and Digital Technologies: Rethinking the terrain*. New York: Springer.