

Chapter 8

The Nested Epistemic Actions Model for Abstraction in Context: Theory as Methodological Tool and Methodological Tool as Theory

Tommy Dreyfus, Rina Hershkowitz, and Baruch Schwarz

Abstract Understanding how students construct abstract mathematical knowledge is a central concern of research in mathematics education. Abstraction in Context (AiC) is a theoretical framework for studying students' processes of constructing abstract mathematical knowledge as it occurs in a context that includes specific mathematical, curricular and social components as well as a particular learning environment. The emergence of constructs that are new to a student is described and analyzed, according to AiC, by means of a model with three observable epistemic actions: Recognizing, Building-with and Constructing—the RBC-model. While being part of the theoretical framework, the RBC-model also serves as the main methodological tool of AiC.

In the first section of this chapter, we give an outline of the theoretical aspects of AiC as background to the description of the elements of our methodology in the second section, and their application to a specific example in the third section. In the concluding section, we close the circle by exhibiting the strong relationship of theory and methodology in AiC as it is mediated by the RBC-model.

Keywords Abstraction in context • Epistemic actions model • RBC+C-model

T. Dreyfus (✉)

Department of MSTE, School of Education, Tel Aviv University,

P. O. Box 39040, Ramat Aviv, Tel Aviv 69978, Israel

e-mail: tommyd@post.tau.ac.il

R. Hershkowitz

Science Teaching Department, Weizmann Institute of Science, Rehovot 76100, Israel

e-mail: rina.hershkowitz@weizmann.ac.il

B. Schwarz

School of Education, The Hebrew University, Mount Scopus, 91905 Jerusalem, Israel

e-mail: baruch.schwarz@mail.huji.ac.il

© Springer Science+Business Media Dordrecht 2015

A. Bikner-Ahsbabs et al. (eds.), *Approaches to Qualitative Research in Mathematics Education*, Advances in Mathematics Education,

DOI 10.1007/978-94-017-9181-6_8

8.1 Theory

In this section, we give a summary of the theoretical aspects of AiC. We refer the interested reader to the literature for more extensive treatments (Hershkowitz et al. 2001; Schwarz et al. 2009). Our approach took shape in the course of research that accompanied innovative curriculum development, when questions arose such as “What did students learn and consolidate, and how? What mathematical concepts and strategies remain with them?” A more subtle issue has to do with the particularities of typical mathematical curricula in which ideas fostered in certain activities are solicited as elements in a reorganized structure: archetypical reorganizations are about the passage from the realm of numbers to the realm of algebra, and from algebra to functions and their representations. Although students may learn ideas that designers did not intend to foster, the structures of mathematical curricula impose constraints and open affordances for certain types of learning processes.

The salient characteristics of the mathematical curricula and classroom learning environments in which we study abstraction are as follows:

- Curricula are organized as *successions of activities* and themes proposed along these activities very often *transform* previous ones. Hence curricula express an *intention of continuous transformation*.
- Our theory of abstraction takes into account the particularities of contexts: The “C” in AiC implies that the role of mathematical, curricular, historical and social context is central to AiC. In particular, the social and interactional context may vary considerably according to the teacher’s decisions.
- Nevertheless, there is an underlying expectation of students’ responsibility for their own learning in an environment that encourages inquiry. Students have the responsibility to report and justify their work and their conclusions to their peers and their teacher, for example during whole class discussions.

In classrooms, abstraction often takes place in interacting small groups of two to four students. Hence, we focus on small groups as well as individuals and two dual issues are central in our approach: On the one hand, we look at what is shared—shared meanings or knowledge to consider what allows groups to continue further constructing knowledge in the same topic together (Hershkowitz et al. 2007). On the other hand, we try to discern agencies as we look at who initiates, how labor is divided and whether responsibilities are taken by different students.

This attention to a special kind of curriculum and to the emergence of learning processes within various contexts led us to theoretical forefathers that belong to different traditions, Freudenthal and Davydov. Freudenthal (1991) provided what many mathematicians have in mind when they think of abstraction. Freudenthal has brought forward some of the most important insights to mathematics education in general, and to mathematical abstraction in particular. These insights constitute a cultural legacy that led his collaborators to the idea of “vertical mathematization”. Vertical mathematization points to a process that typically consists of the reorganization of previous mathematical constructs within mathematics and by mathematical

means by which students construct a new abstract construct. As researchers in mathematics education, we preferred the expression “vertical reorganisation” to the expression “vertical mathematization” to discern between what is intended by the teacher—the mathematization, and what often happens—a reorganisation, which is a radical change which sometimes does not coincide with what is intended. In vertical reorganisation, previous constructs serve as building blocks in the process of constructing. Often these building blocks are not only reorganised but also integrated and interwoven, thus adding a layer of depth to the learner’s knowledge, and giving expression to the composite nature of the mathematics. Sequences of problem situations provide opportunities to capitalise on the new constructs repeatedly, and to turn them into building blocks for further constructing actions, where each construct includes ‘pockets’ of past constructs on one hand, and is itself a potential component for new constructs.

Davydov was one of the most prominent followers of the historical cultural theory of human development initiated by Vygotsky. For Davydov (1990), scientific knowledge is not a simple expansion of people’s everyday experience. It requires the cultivation of particular ways of thinking, which permit the internal connections of ideas and their essence to emerge, thus enriching rather than impoverishing reality. According to Davydov’s “method of ascent to the concrete”, abstraction starts from an initial, simple, undeveloped and vague first form, which often lacks consistency. The development of abstraction proceeds from analysis, at the initial stage of the abstraction, to synthesis. It is a dialectical process that ends with a more consistent and elaborated form. It does not proceed from concrete to abstract but from an undeveloped to a developed form. As such, abstraction clearly contrasts with *generalization*, which is an extension to less specific criteria—an extension, which is not more developed and therefore does not require reorganization.

The reference to both Freudenthal and Davydov’s theories of abstraction in AiC expresses an effort to bridge between the cognitive and socio-cultural dimensions: On the one hand, Freudenthal, an eminent mathematician influenced by Piagetian ideas, saw mathematization as a reflective effort done by the student toward a construct foreseen beforehand by the mathematician. Davydov’s approach is socio-cultural. His analyses of abstraction uncover dialectical processes often occurring in interactions between the adult and the student. The bridge between those two perspectives consists of the necessary design effort for providing opportunities for abstraction to occur. The design is accountable for what the mathematician and the mathematics educator intend to “instill”. This design hopefully opens the door for negotiations of meaning in interactions, possibly leading to abstractions as desirable constructs.

AiC adopts the views of vertical mathematization and ascent to the concrete and builds on them to define abstraction as a process of vertically reorganizing some of the learner’s previous mathematical constructs within mathematics and by mathematical means so as to lead to a construct that is new to the learner. Activity theory proposes an adequate framework to consider processes that are fundamentally cognitive while taking into account the mathematical, historical, social and learning contexts in which these processes occur. In this, AiC follows Giest (2005), who considers activity theory as a theoretical basis, which has an underlying constructivist philosophy.

According to activity theory, outcomes of previous activities naturally turn to artifacts in further ones, a feature that is crucial in tracing the genesis and the development of abstraction through a succession of activities. The kinds of action that are relevant to abstraction are epistemic actions – actions that pertain to the knowing of the participants and that are observable by participants and researchers.

We postulate that the genesis of an abstraction passes through a three-stage process, which includes the need for a new construct, the emergence of the new construct, and the consolidation of that construct. We will exhibit the crucial role of epistemic actions for the second and third stages below. The nature of the first stage is well expressed by Kidron and Monaghan (2009) when dealing with the need that pushes students to engage in abstraction, a need that emerges from a suitable design and from an initial vagueness for the learner:

The learners' need for new knowledge is inherent to the task design but this need is an important stage of the process of abstraction and must precede the constructing process, the vertical reorganization of prior existing constructs. This need for a new construct permits the link between the past knowledge and the future construction. Without the Davydovian analysis, this need, which must precede the constructing process, could be viewed naively and mechanically, but with Davydov's dialectic analysis the abstraction proceeds from an initial unrefined first form to a final coherent construct in a two-way relationship between the concrete and the abstract—the learner needs the knowledge to make sense of a situation. At the moment when a learner realizes the need for a new construct, the learner already has an initial vague form of the future construct as a result of prior knowledge. Realizing the need for the new construct, the learner enters a second stage in which s/he is ready to build with her/his prior knowledge in order to develop the initial form to a consistent and elaborate higher form, the new construct, which provides a scientific explanation of the reality. (pp. 86–87)

As mentioned above, a central component of AiC is a theoretical-methodological model, according to which the emergence of a new construct is described and analyzed by means of three observable epistemic actions: recognizing (R), building-with (B) and constructing (C). Recognizing refers to the learner seeing the relevance of a specific previous construct to the situation or problem at hand. Building-with comprises the use and combination of recognized constructs, in order to achieve a localized goal such as the actualization of a strategy, a justification or the solution of a problem. The model suggests constructing as the central epistemic action of mathematical abstraction. Constructing consists of assembling and integrating previous constructs by vertical mathematization to produce a new construct. It refers to the first time the new construct is expressed or used by the learner. Hence, while the term *constructing* refers to the epistemic action – a process – the term *construct* refers to an outcome of such action. This definition of constructing does not imply that the learner has acquired the new construct once and forever; the learner may not even be fully aware of his new construct, and the learner's construct is often fragile and context dependent. Constructing does not refer to the construct becoming freely and flexibly available to the learner. Becoming freely and flexibly available pertains to consolidation.

Consolidation is a never-ending process through which students become aware of their construct, the use of the construct becomes more immediate and self-evident,

the student's confidence in using the construct increases, and the student demonstrates more and more flexibility in using the construct (Dreyfus and Tsamir 2004). Consolidation of a construct is likely to occur whenever a construct that emerged in one activity is built-with in further activities, possibly in the course of a further constructing action. These further activities may lead to new constructs. Hence consolidation connects successive constructing actions and is closely related to the design of sequences of activities.

In processes of abstraction, the epistemic actions are nested. C-actions depend on R- and B-actions; the R- and B-actions are the building blocks of the C-action; at the same time, the C-action is more than the collection of all R- and B-actions that make up the C-action, in the same sense as the whole is more than the sum of its parts. The C-action draws its power from the mathematical connections, which link these building blocks and make them into a single whole unity. It is in this sense that we say that R- and B-actions are constitutive of and nested in the C-action. Similarly, R-actions are nested within B-actions since building-with a previous construct necessitates recognising this construct, at least implicitly. Moreover, a lower level C-action may be nested in a more global one, if the former is made for the sake of the latter. This nested character was observed in the classrooms and in interviews in which we studied abstraction and it substantiated our theoretical tenets according to which the curriculum was intended to afford a continuous transformation of constructs. Given these characteristics, we named the model the *dynamically nested epistemic actions model of abstraction in context*, more simply the RBC-model, or RBC + C model using the second C in order to point at the important role of consolidation. The RBC-model is the theoretical and micro-analytic lens, through which we observe and analyse the dynamics of abstraction in context. We will, in the concluding section, come back to the RBC-model in order to show how the model as a part of the theory interacts with the same model as methodological tool, and hence theory and methodology mutually depend on each other, influence each other and undergo successive refinements.

8.2 The AiC Methodology

In this section, we give a concise description of the methodology of AiC, its elements, its tools and its methods. We decided to keep this description concise since in our opinion, methodological tools and methods – those of AiC as well as those of any other theoretical framework—obtain their significance only through their application to specific cases; our main intention, in this chapter, is therefore to show the application of the methodology to a specific set of data; this application will be exemplified in the next section, accompanied by frequent reference back to the present section.

8.2.1 *Design for Abstraction*

The importance of design in our theory is considerable. It constrains the kinds of actions participants may carry out, and affords the ones that seem desirable to the designer and teacher. Of course, students' actions are never determined by the design, and there is always a gap between intended actions and actions enacted. But the diminution of this gap is of crucial importance for abstraction to occur. The design is a challenging task and it is often an integral part of the research project (Kouropatov and Dreyfus 2013), or the project is a design-research-redesign project (Hershkowitz et al. 2002). Often also, a particular design that has been successful in some condition is taken over from elsewhere. For example, in a recent study on the tension between the discrete and the continuous in a differential equations course, the design, and even the data, were taken as is from a research study in a different country in which only one member in the team had been involved in the past (Tabach et al. 2014). The difficulty involved in the design originates from the fact that it has to comply with an intention of continuous transformation of constructs. This requires the elaboration of sequences of activities that offer the students opportunities to learn well defined mathematical ideas (e.g., how a solution to a first order differential equation can be approximated by discrete steps and that the approximation improves as the steps become smaller; the notion of integral as an accumulating quantity; that algebra is a tool for justification, etc.). It also requires the elaboration of further activities to apply these ideas as tools in familiar contexts or as tools in contexts that necessitate the elaboration of new ideas. What is common to all these learning aims is that they include adding new connections between students' previous knowledge, hence adding depth to the students' understanding, integrating their knowledge in ways not available to them before. In brief, the design intends to create a didactical hierarchy aimed at vertical reorganization of students' knowledge.

Beyond these very general remarks, designs may considerably vary to afford intended abstractions. However, in several research studies using AiC to study abstraction, some design principles have been articulated: In addition to creating a collaborative situation, some of these principles are triggering a cognitive conflict, asking for hypotheses and providing tools for testing them, and reflective argumentation (Prusak et al. 2012). Also, the sequence of activities often alternates contexts in which small groups work or argue together with teacher-led (reflective) discussions.

8.2.2 *A Priori Analysis*

In a move pertaining to design but specialized on its epistemological aspects, an effort is then made to foresee trajectories of students' learning: an *a priori analysis* of the activities (e.g., Ron et al. 2010). Assumptions are first made about the previous knowledge of the students. Then possible paths to deal with the activities and

answer the questions they raise are examined. The most relevant question asked in the a priori analysis is what knowledge is helpful or even necessary to deal with the task and to complete it to the designer's or teacher's satisfaction. We are particularly interested in knowledge that has not been relevant in previous activities carried out by the students. With respect to knowledge that has not been relevant in previous activities and is helpful or necessary in the current one, we assume that the designer or teacher intended this knowledge to be constructed by the students in the current activity. The aim of the a priori analysis is to identify such elements of knowledge intended by the design, typically concepts or strategies thought of in terms of the content domain of mathematics. It is our working assumption that the new constructs, which emerge for the students when dealing with the task, are in close correspondence with the knowledge elements identified in the a priori analysis, if the design and the a priori analysis are adapted to the learner. Nevertheless, students' constructs are of course to be distinguished from the knowledge elements intended by the design.

The selection of knowledge elements in the a priori analysis has a considerable influence on the a posteriori RBC-analysis that is being carried out later. It can therefore not be left vague or undetermined. Therefore, two or three researchers in a team usually carry out a priori analyses independently, and then differences are ironed out until there is agreement among the researchers. For each knowledge element, we do not only give a definition in terms of the mathematical meaning of the element in the context in which it is being used, but also an operational definition. The operational definition is methodologically important: It fixes under what circumstances the researcher will say that a student is using or expressing a construct that represents this knowledge element. Hence, the operational definition will take into account the definition of constructing, namely that constructing refers to the first time a new construct is expressed or used by the learner. Examples for knowledge elements with their general as well as their operational definitions will be given in the next section.

The result of the a priori analysis is not or not only a list of knowledge elements but rather a structure of knowledge elements, with some elements being contained in (or part of) others, and some being prerequisite for others. Often, there is an overarching knowledge element, within which all others are included. An example for this is the Seals activity (Dreyfus et al. 2001). The Seals activity was intended for grade 7 students, whose previous learning experiences did not include tasks where algebra was used as a tool for justification; and while they had learned and gained experience with the simple distributive law $a(c+d)=ac+ad$, they had never had a need or occasion to similarly expand $(a+b)(c+d)$ (the extended distributive law). The Seals activity presented the students with numerical examples of "seals" of the form shown in Fig. 8.1, and asked them to find as many properties as possible of such seals. After some exploration, they were led to focus on the property that the difference between the product of the elements in the minor diagonal and the product of the elements in the major diagonal is always 12, and asked to justify this phenomenon. This was intended to (and did in many cases) lead student groups to use algebra as a tool for justification and develop the extended distributive law in the

Fig. 8.1 The “seal”

X	X+2
X+6	X+8

process. Hence the a priori analysis resulted in algebra as a tool for justification as the overarching knowledge element and, nested within it, the extended distributive law as a further knowledge element. By the way, this overarching knowledge element was clearly recognized by the researchers only after we analyzed several groups solving the activity and observed difficulties some of them experienced in solving the task.

One aim of the a priori analysis is to focus, at least initially, the researchers' attention on the intended constructs, on which the RBC micro-analysis of students' constructing of knowledge is then based. Sometimes the a priori analysis, as meticulous as it may be, may miss the necessity of certain knowledge elements for performing a task or may fail to predict some of the students' ways of going about a task. In such cases, the a posteriori analysis of the data according to the RBC-model will be used to provide improvements to the a priori analysis. Hence, the a priori analysis serves as a working hypothesis that may later be confirmed or modified by the micro-analysis of the data.

8.2.3 Data Collection and Preliminary Analysis

The setting of a specific research study occasionally influences the decision on which students to focus the data collection. In individual or small group interview studies, data from all students are collected. In studies that take place in classrooms, the potential volume of data is enormous, and it is mandatory to make a choice. We then select a few focus groups or focus students for close observation. Group work of the focus students, as well as whole classroom discussions, are videotaped and audio-taped. Tapes are transcribed, in some cases with intonation signs. The analysis uses all student productions (worksheets), the transcripts, and researchers' field notes. Since meaning making in mathematics is very often achieved in multi-modal channels (Radford 2009), the videotapes are important for example for gestures, facial expressions, as well as to identify in a group of students who wrote or drew what when (for which purpose we have recently started using *Livescribe* pens as well (Hickman and Monaghan 2013). It is very often helpful to complete the data collection with interviews of the teacher as well as of some focus students.

The analysis of whole classroom discussions according to the RBC-model is delicate because, in such a context, the interventions of individual students may be sporadic. However, whole classroom discussions can complete analyses in a succession of activities. (See also comments in the concluding section.)

We prepare our analysis by reading the entire transcript and often viewing the videos in order to get an overview of the entire learning process; usually, we first prune the transcript to leave relevant data, then split the pruned transcript into episodes (Chi 1997) the end of an episode being determined by either cognition, content or external factors that influence content or cognition. Cognition means that a change occurs in students' orientation, in questions asked or approached by the students, or in the methods they use to attempt making progress. A change in content is often indicated by the transition to a new task or subtask. External factors, like the teacher calling attention to a whole class discussion, frequently have a similar effect of introducing a break or change or orientation. Among all episodes, we then identify those, in which it appears that new knowledge may have emerged for one or several focus students. Later stages of our analysis will begin from these episodes.

8.2.4 *Need*

Although the RBC-model focuses by definition on epistemological aspects of mathematical activity, abstraction stems from a need and overt expressions of this need may sometimes be identified in the preliminary analysis: enthusiasm, uncertainty, surprise, curiosity or bafflement, for example as the consequence of a cognitive conflict, are emotional states whose appearance are good threads to begin with to identify and explain the emergence of a constructing action. Similarly, learners may express the need to justify for themselves or for others a mathematical fact or an idea that they just discovered.

As researchers, we also have the task of identifying needs whose manifestation is subliminal, and often these can be detected. There are many possible indicators for a need as a precursor of constructing: Student's questions, manifestations of uncertainty, of surprise or even just a request for time of reflection that is related to the situation at hand, are all indicators of a possible need for a specific construct, even though the student is at this stage unlikely to be aware of this. Of course, this manifestation may remain undeveloped. Also, one cannot establish whether the above behaviors really express a need or a whimsical emotional expression lacking clear volition. A brief analysis of the episode generally clarifies this issue. Besides these emotional expressions as indicators of a need, natural places at which to look are self-regulatory or self-monitoring manifestations: the learner's explicit orientation in the problem space (e.g., by attempting to establish what has been achieved and what still needs to be achieved).

However, in some cases, constructing actions can occur without the researchers having been able to pinpoint a specific need for a new construct. Kidron et al. (2010) have shown that in some situations constructing actions can occur on the basis of a

general epistemic need rather than on the basis of a specific need for a new construct, and that this general need is closely connected to Davydov's description of abstraction as a transition from the vague to the precise.

8.2.5 *Analysis According to the RBC-Model*

After the preliminary analysis and the possible identification of episodes where a need is detectable, one can turn to the heart of the method, the analysis of the episodes with R, B and C epistemic actions as building blocks of the process of abstraction. The goal in this RBC analysis is twofold: first, to unveil the processes by which the students' new constructs emerge as a vertical reorganization of previous constructs in the current context; second, to contribute to the refinement of AiC through the unfolding of the processes that occurred during the episodes (as will be shown further on).

The operational definitions generated in the a priori analysis are crucial for the RBC analysis. A construct's operational definition provides clear criteria for assessing whether a student's utterance or action provides evidence that a constructing action for the said construct has occurred. Sometimes, such an action or utterance by the students will occur during or immediately after the student's constructing action and sometimes only at a later stage. This may depend on how involved the student is in interaction with other students, with the teacher, or with the interviewer at the critical junctures. This does, of course, not mean that students less involved in the interaction do not construct; it only means that the researchers may not have immediate access to these students' constructs, and need to work on the basis of their later actions or utterances.

The analysis by the RBC+C model is a micro-analysis that proceeds utterance by utterance through the analyzed episodes. This often enables us to find the exact moment where a constructing action has been completed, sometimes by direct evidence and sometimes by concluding back from the evidence to earlier utterances. It also enables us to follow the constructing process and the way epistemic actions are nested in each other. This fits exactly our definition of abstraction: The three epistemic actions are the windows through which we can evaluate the main building blocks of the process of abstraction, and the micro-analysis enables us to trace these epistemic actions and the "mathematical glue" which ties them together.

In order to achieve this goal, researchers might begin by identifying a constructing action that has occurred, mark the relevant action/s or utterance/s in the transcript as the end of the constructing action of the said construct and begin to work backwards through the transcript to identify the recognizing and building-with actions that contribute to the said constructing action. Other researchers trace the epistemic actions in more flexible way: After reading and obtaining a general overview of the episode they go forward and backward, being led by their need to understand and interpret.

The researcher can usually identify recognizing actions quite easily on the basis of an action on or an explicit mention of a previous construct by the learner. The recognized previous construct may then become one of the building blocks being reorganized during a constructing action. Evidence for recognizing actions is usually rather obvious, and our experience shows that disagreements arise rarely with respect to the constructs being recognized by the learners and the evidence that shows that they are recognized. We stress explicitly that recognizing does not refer to a potential new construct that may emerge from the current activity but to previous constructs available to the learner.

Similarly, building-with actions refer to making use of previous constructs having been recognized by the learner as relevant to the problems situation at hand, and not a potential new construct that may emerge from the current activity; indeed, such a potential new construct, having not yet been expressed or used even once by the learner, cannot yet possibly be available to this learner. Building-with refers to explicit actions of computing, sketching, justifying, reasoning, etc. with previous constructs. For this very reason, building-with actions are similarly easy to identify as recognizing actions. Here also, our experience shows that disagreements are rare and novice researchers need only moderate training. Building-with actions also help the researcher to assess which recognizing actions are relevant, according to whether the recognized previous construct has been used in one of the building-with actions. A previous construct that is recognized but not used in any way does not play a relevant role in the constructing action.

If working backward through the transcript, the goal of the researcher is to identify and mark all actions and utterances that contribute to the emergence of the emerging construct. These utterances and actions need not be, and except in simple cases are not, contiguous and some may have occurred much earlier, long before the place where the end of the constructing action has been identified. Even so, this central part of the analysis frequently allows us to find the roots of the constructing action as well as its parts, and allows the researcher to produce a fine-grained description of how the construct emerged by vertically reorganizing previous constructs by means of recognizing and building-with actions. In some cases (e.g., Kidron et al. 2011), we have even been able to identify quite early seeds for a constructing action, for example the observation by students of phenomenological patterns that at the time of observation did not lead anywhere but later supported the initiation of a constructing action.

We have often found it convenient to represent a process of abstraction in a table such as Table 8.1. For purposes of demonstration, we assume that the activity under discussion relates to only two new constructs C_1 and C_2 , and one previous construct C_a . (In concrete examples, we tend to give meaningful indices to the constructs – see the next section for examples.) The table is an extension of the transcript. Its left-most column numbers the utterances; the second column shows the names of two students, A, B and the teacher T. The third column would contain the utterances (omitted here). A column with comments on gestures, timing etc. might be added. The fourth column in Table 8.1 exhibits the analysis of constructing action C_1 and the fifth one that of C_2 . In these two last columns, we enter R- and B-actions with

Table 8.1 A hypothetical table of epistemic actions

No.	Name	Utterance	C ₁	C ₂
1	A	...		
2	B			
3	A			
4	B		R _a	
5	T			
6	B			
7	A			
8	B		B _a	
9	A		R _a B _a	
10	B			R _a
11	A		B _a	B _a
12	A			R ₁
13	B			B ₁
14	A			B ₁
15	B			B _a B ₁

previous constructs, denoting clearly which previous construct the action refers to. In the C₁-column, only R_a and B_a actions (with C_a) are possible. In the C₂-column, R_a, B_a, R₁ and B₁ are possible. We again stress that in column C_k, actions R_k and B_k cannot possibly occur since the constructing action of C_k has not ended and hence C_k is not available to the student. In rows that contribute to the constructing action C_k, the corresponding cells of column C_k are shaded. An utterance may contribute to more than one constructing action. The end of a constructing action is marked by a bold horizontal separation below the relevant cell. (For a detailed example of using such a table for the analysis of a constructing action, see Dreyfus et al. 2001.)

The preceding description shows that a construct has to be recognized as relevant before being used to build-with. Hence an R_k action is usually nested within any B_k action. It similarly follows that the R- and B-actions with previous constructs are nested in C-actions, for example, R_a, B_a, R₁ and B₁ are all nested in C₂. Constructing actions can go on in parallel, as in the table, where constructing C₂ begins before constructing C₁ has been completed. Parallel constructing actions can interact, branch and combine in rather complex ways (Dreyfus and Kidron 2006).

Colleagues and research students alike often ask us how to distinguish constructing actions from building-with actions. This question is eminently justified since there is no analytic way to distinguish them by mechanical analysis. The decision whether a task requires building-with or constructing from a learner depends not only on the task but also on the learner's personal history: If the sequence of learner actions and utterances when dealing with the task expresses vertical reorganization and hence the emergence of a construct that is new to the learner, then the action is

a constructing action; otherwise, it is a building-with action. Someone might argue that this may introduce ambiguities into the RBC-analysis since different researchers may interpret a sequence of actions and utterances differently; we agree that there is a measure of interpretation involved, but our experience shows that differences of interpretation are rare and can be successfully resolved in interaction between researchers.

Students' constructs as revealed by the RBC-analysis can be expected ones that correspond closely to the knowledge elements identified in the a priori analysis, or they can be quite unexpected ones. This is fortunate, as good design for learning should not look like conveyor belt production in factories. Instead of considering unexpected knowledge elements as design failures, one should consider them as ways students handle opportunities for learning provided to them. An example for an unexpected construct will be presented below. In fact, students' constructs may, and often will correspond partially to the intended knowledge elements. Occasionally, this will lead to a modification of the a priori analysis, not in the sense of making it into an a posteriori analysis but rather in the sense of correcting what could or should have been expected. In other words, we as researchers have in quite a few cases used the data to discover that our analysis was not as fine-tuned or as reasonable as might be expected and to learn from the students' actions what might have been expected a priori. On other occasions, some students' constructs will be only partially correct from the mathematical point of view or sufficient to achieve only partially the tasks set by the activities. In this case, we speak of partially correct constructs – PaCCs (Ron et al. 2010). PaCCs are, by definition, partial with reference to an intended knowledge element and curriculum, and not in some absolute sense. PaCCs are examples of manifestations that were not considered in the first articulation of the RBC theory but that were incorporated in a refinement of the theory through further experiments.

8.2.6 Consolidation

Our theoretical approach to abstraction as based on activity theory lends itself naturally to the consideration of abstraction as including not only constructing processes but also the use of the resulting constructs further on. This implies that a student may be considered as having participated in a constructing action without fully being aware of it. Alternatively, the student may have fully taken responsibility for her constructing but may not have properly used the resulting construct further on. Examining student actions after the emergence of a construct is the focus of consolidation. In order to investigate consolidation, our analysis hence typically works forward from the moment the C-action has ended and identifies R- and B-actions with this construct. These are most frequently found either as B-actions carried out to deal with a task that does not lead to a new construct or, more

interestingly, as R- or B-actions serving as part of a constructing action of a further construct as R_1 and B_1 have served in the constructing of C_2 in Table 8.1.

A first step in the analysis of consolidation of a construct is therefore to identify all R- and B-actions with this construct, a step that is necessary anyway for the purpose of analyzing the constructing of further constructs. The second step in the analysis of consolidation is to determine whether and in what way these R- and B-actions provide evidence of the student's confidence in using the construct, of the immediacy with which the student uses the construct, of the self-evidence the construct has for the student, of the flexibility the student demonstrates in using the construct and the student's awareness of the construct. Each of these separately (and a fortiori some or all of them together) provides evidence for the continuing consolidation of the construct.

8.2.7 *Who Is Constructing?*

Just as choices have to be made which data to collect, choices sometimes have to be made, which data to analyze. While no such decision needs to be taken when analyzing an interview with a single student, this situation is prone to excessive researcher intervention, for example when the student gets stuck, or simply in order to make the student express her thoughts. We therefore tend to work with small groups of students. However, even when a pair of students is collaborating in an interview situation, one student may be leading, or the pair may be interacting in a symmetric manner. In each case, the researcher has to decide whether to analyze constructing actions of the pair as a whole or of each student separately; in some cases, it may be impossible to separate the epistemic actions of the students or information about one of the students may be sparse. The situation clearly gets more complex when groups have three or four participants; it gets even more complex, when the group works in a classroom. This is a big challenge because of the many variables that play a role in a whole class situation and the potentially messy data. Hence we had to deal with the question of what constitutes knowledge shared by a group of individual students as they construct and consolidate it in the classroom (Hershkowitz et al. 2007). In this setting, we emphasize the interactive flow of knowledge from one student to the others in the group, until they reach a shared knowledge—a common basis of knowledge, which allows them to continue constructing further knowledge in the same topic together.

8.3 A Focus Group in a Classroom as an Example

In this section, we provide an example whose purpose it is to illustrate the use of the methodology set forth in the previous section. The example is taken from a research project (Hershkowitz et al. 2007) whose main goal was to investigate aspects of

small groups' knowledge constructing in working classrooms. The topic chosen for instruction was elementary probability. Five grade 8 classes in different schools and their teachers participated in the project.

The probability unit deals with concepts and problem-solving aspects of empirical versus theoretical probability, and one- and two-dimensional sample spaces. This overall construct of probability includes three stages:

- A. Sample spaces in one dimension (1D), for example the probability of obtaining 3 when rolling a die; theoretical probability as the ratio of the number of relevant outcomes to the number of all possible outcomes; experience with the fact that empirical probability values tend to the theoretical value as the number of trials increases.
- B. Sample spaces in two dimensions (2D), where all simple events are equiprobable and can be counted and organized into a table, and the probabilities of the compound events can hence be found by inspection, for example, the probability of obtaining at least one 3 when two dice are rolled simultaneously.
- C. Sample spaces in 2D, where each dimension has two possibilities whose probabilities are not necessarily equal.

8.3.1 *Design for Abstraction*

The project team designed a ten-lesson probability unit with the specific aim of giving students opportunities for constructing knowledge related to stages A, B, and C above, and to cooperate in constructing knowledge. The underlying design principles include a didactical hierarchy with occasions for hypothesizing, for facing conflicts, and for arguing (Hadas et al. 2008).

The unit is structured hierarchically in such a way that each stage (and often each task) introduces new elements, which evolve from the previous stages (globally) and tasks (locally). The global aspect of the didactic hierarchy principle can be demonstrated by the appearance of the calculation of probability values as a relevant ratio at each stage of the unit. It is reflected in the tools used for presenting the probability values: In stage A, the students present the probability values of 1D events on a chance bar (a $[0,1]$ number segment). In Stage B, the students deal with situations involving the probability of 2D events such as two dice or a spinner and a die; a table model based on the equiprobability of the simple events represented by the table cells is introduced as a tool for finding such probabilities. In Stage C, the students deal with binomial but not necessarily equiprobable 2D events using an area model for probabilities in the resulting four-event sample space. Stage C is designed on the shoulders of Stages A and B, which are interwoven together with the two dimensions being represented by two orthogonal chance bars generating a square. The unit area of the square represents the probability of the sample space as a whole, while the area of each of the four rectangle-cells in the square represents the probabilities of the four events. This briefly illustrates the didactical hierarchy design principle.

Task 1 1a Joe and Ruth roll two white dice. They decide that Ruth wins if the numbers of dots on the two dice are equal and Joe wins if the numbers are different. Do you think the game is fair? Explain! 1b The rule of the game is changed: Joe wins if the dice show consecutive numbers. Do you think the game is fair? 1ci How many possible outcomes are there when rolling two dice? 1cii Reconsider your answers to Tasks 1a and 1b: Are the games fair? 1d Suppose Joe and Ruth play with one red die and one white die. Does this change the answers to 1a, 1b and 1c? Task 2 We again roll two white dice. This time, we observe the difference between the larger number of dots and the smaller number of dots on the two dice. (If the numbers of dots on the two dice are equal, the difference is 0.) 2a Make a hypothesis whether all differences have equal probability. Explain! 2b How many differences are there?

Fig. 8.2 Tasks 1 and 2 of Stage B

We demonstrate the other three design principles using the first two tasks of Stage B. These tasks are presented in Fig. 8.2. The hypothesizing principle is most explicit in Task 2a. Experience in teaching probability led the designers to expect that some students will hypothesize equal probabilities for all differences, and that many of the others are likely to hypothesize that the probability is decreasing and, in particular, that the probability of difference 0 is larger than the probability of difference 1. Making use of this expectation for creating a situation favorable for constructing one of the central knowledge elements in 2D probability, the design continued after Task 2b, in the following lesson, by means of a computer simulation that creates a histogram for the differences; this histogram is likely to generate a conflict between what students expect based on their hypotheses and what the computer simulation shows—hence realizing the conflict principle. This conflict caused by the refutation of their hypothesis by computer experiment is a highly fertile situation for constructing knowledge about the probabilities of events in context of the 2D sample space of two dice. This is a suitable moment to encourage students to argue their reasons for and against the hypothesis and to evaluate the cogency of different, sometimes contradictory arguments, thus engaging also the argumentation principle in support of constructing knowledge. It will be shown below that this specific case closely relates to the hierarchy of knowledge elements produced by the a priori analysis hence demonstrating also the local aspect of the hierarchy principle.

8.3.2 *A Priori Analysis*

The RBC+C analysis will be exemplified below using data from students' work on the tasks at the beginning of Stage B of the learning unit (see Fig. 8.2). A preliminary knowledge element from Stage A that may become relevant here is E_{PV1} (PV1 stands for *probability value in 1D situations*): When all possible outcomes are known, the theoretical probability value can be computed as the ratio of the number of relevant outcomes to the number of all possible outcomes. We note that this knowledge element is built on and includes knowledge elements such as *event* or *outcome*, *relevant outcome*, and *all possible outcomes*. Obviously, additional knowledge about dice having six faces numbered 1–6 and about differences and fractions are needed to deal with the activity but these are assumed to be generally known to students in grade 8 and we do not list them.

Four basic probability concepts are considered here as the main knowledge elements that underlie the learning design and may be expected to be constructed by the students when working on Tasks 1 and 2: *simple event*, *compound event*, *sample space* and *probability value*. The simple event element has two constituent elements, *pair* and *order*. Figure 8.3 presents the knowledge elements as well as the relationships between them. For example, the probability of a 2D compound event is calculated as the ratio between the number of simple events in the compound event and the number of simple events in the sample space. Hence, the probability value element contains the compound event element and the sample space element as constituents, while both the compound event element and the sample space element contain the simple event element as constituent. These knowledge elements will below be defined operationally in such a way that the researcher can tell from students' actions and utterances whether or not they are using these elements. We note that strictly speaking, students are never constructing a knowledge element E_X or building-with a knowledge element E_X ; rather, in the constructing action, constructs emerge for the students that (hopefully) correspond at least partly to knowledge elements intended by the designer. This is what we will, for simplicity, refer to as “a

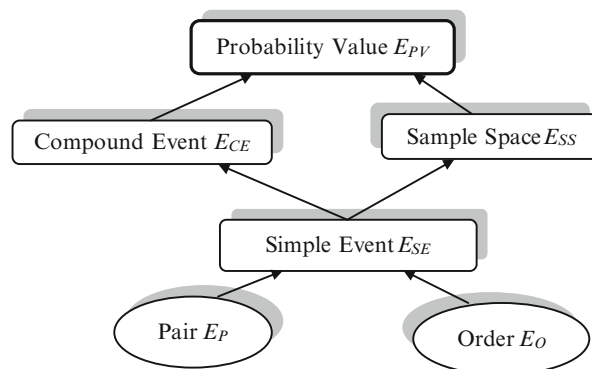


Fig. 8.3 The knowledge elements

student constructed or built-with E_X "; however, we always take this to mean that the student constructed or built-with his or her personal construct corresponding to E_X .

Simple event E_{SE} A 2D simple event E_{SE} is an ordered pair of 1D simple events, for example getting 3 on the first die and 4 on the second die. When tracing the process of constructing the simple event element we shall focus on two constituent elements, Pair E_P and Order E_O . We shall say that students have constructed the pair element when they, by their words or actions, relate to the occurrence of the two 1D simple events as to a single event. We shall say that students have constructed the order element if they relate to pairs like $(\square, \Delta)$ and $(\Delta, \square)$ as to two distinct events. The order element can be constructed only if the pair element has been constructed.

Compound event E_{CE} A compound event is a set of simple events, which share a common attribute, for example "the two dice show a difference of 3". We shall say that students have constructed the compound event element, if they organize or list all relevant simple events in a systematic way, with the aim of finding their number, depending on what they consider as a simple event. In particular, if students did not construct the order element, but count all the non-ordered relevant pairs, we shall say that they have constructed the compound event element. This is a reflection of our approach, which considers processes of constructing knowledge from the student's existing perspective rather than from an expert's perspective.

Sample space E_{SS} The sample space is the set of all possible simple events. We shall say that students have constructed the sample space element, if they carry out actions to organize or list all events of the sample space in a systematic way, with the aim of finding their number, depending as above on what they consider as a simple event.

Probability value E_{PV} The probability of a 2D compound event is calculated as the ratio between the number of simple events in the compound event and the number of simple events in the sample space. We shall say that students have constructed the probability value element if they calculate the probability of a compound event as the ratio between the number of simple events in the compound event and the number of simple events in the sample space.

Our list of knowledge elements is, of course, incomplete. On one hand, it is based on the didactical decisions for the design of the probability unit. On the other hand, the aim of the research to analyze knowledge constructing processes, dictates concentrating on knowledge elements that were observed in the students' work. Thus both, the design and research considerations have influenced the list of elements and its organization.

The designers' intention was that dealing with Tasks 1 and 2 may be expected to lead to a need for constructing of E_P and at least partially constructing E_{SS} (for example Task 1c) and E_{CE} . The intention of Task 1d is to motivate students to deal with the dilemma of E_O . Task 2 represents a more sophisticated situation since there are six different differences, each of which is a complex event with a different probability whose calculation requires E_{PV} .

8.3.3 *Data Collection and Preliminary Analysis*

In all lessons taught in the framework of the project, researchers were present and documented the lesson by means of two video cameras. One camera was directed at a focus group of students (the same group in each lesson) and the second camera at the teacher and the whole class activity. The researchers also took field notes and collected students' completed worksheets and written tests.

In the specific lesson serving as example here, the focus group consisted of three girls, Noam, Rachel and Yael. We consider the focus group while all students in the class were working in small groups as part of the regular class work. We present parts of the analysis of the work of the focus group on Tasks 1 and 2. The preliminary analysis of the data yielded a division into three episodes as used in the next subsection.

8.3.4 *Analysis According to the RBC-Model, Including Need and Consolidation*

The need for a new construct, its emergence and its consolidation are three inter-linked stages that form part of a process of abstraction for this target knowledge. Although these three stages have in the previous section been treated separately for added clarity, they are in practice closely interwoven and are therefore presented here in combination.

8.3.4.1 *Episode 1: Constructing E_p*

After a brief introduction during which she reminds the students of the notions of frequency and relative frequency, the teacher distributes the worksheets with Tasks 1 and 2. Rachel reads 1a aloud. Noam and Yael immediately agree that the game is not fair, that Ruth's chances are lower, with Noam even saying 1 in 6 and Yael, counting $1+2+3+4+5+6=21$ and adding another 6 arriving at 6 out of 27. This provokes Noam to ask for clarification.

[21] Noam What are you doing? Eh?

[22] Yael I did: What are the chances? One one, two two, three three, four four, five five, six six. And he has one two, one three, one four, and two one, two two. No two one and one two are the same thing... one two, one three, one four, five plus 4 is nine, nine plus three is twelve

[23] Rachel I don't understand what you are doing. What are you doing?

[24] Yael Because I have to know what is our whole. It's like, what results can be, what results might occur? ... It's either (1, 1) (2, 2) (3, 3) (4, 4) (5, 5) (6, 6) or ... (1, 2) (1, 3) (1, 4) (1, 5) (1, 6) and then 2.

[25] Noam Yael, I don't understand at all what you are doing.

[26] Rachel Yes, me neither.

Yael clearly and repeatedly refers to pairs of numbers [22] as simple events, is already busy enumerating, more or less systematically, all the relevant events for calculating Ruth's and Joe's chances to win [22, 24], and asks explicitly what are all possible simple events [24].

Using the operational definition for E_P given above, we interpret that Yael has constructed a suitable construct for E_P in the context of two dice before [21], and without providing any explicit evidence for the constructing action to the researcher. She is already recognizing and building-with E_P in [22, 24] and beyond. This is the reason why we have freely used the notation (a, b) for pairs from [24] onward, although the parentheses can hardly be heard in the students' utterances.

Using the operational definitions for E_{CE} and E_{SS} we conclude that Yael is in the process of constructing these elements: She is attempting to organize lists of all relevant simple events, with the aim of finding their number, but does not yet do this quite systematically.

We also note that Yael gives a brief indication of being aware of the issue of order (of the elements of a pair) in [22] and, in fact, negates order; she may be in the process of constructing what might be called an "anti-order" construct, which, of course, does not form part of our list of knowledge elements.

There is no parallel evidence for Rachel and Noam; quite the contrary, their questions seem to indicate that they may not yet have constructed a view of events as pairs of numbers (E_P), and they do not appear to understand what, how, and why Yael is counting [21, 23, 25, 26].

Yael seems to realize this and repeats

[27] Yael Listen, there are some possibilities that 1 will appear: (1, 2) (1, 3) (1, 4) (1, 5) (1, 6) and we finished with 1, now 2: (2, 3) (2, 4) (2, 5) (2, 6).

[28] Rachel O.K., O.K., we understood that, but why are you adding? I don't understand.

We observe that Yael's [27] is more explanatory and more systematic than [24], and interpret this as showing that she has now constructed E_{SS} . Although there is no explicit evidence from Rachel's [28] that she has now constructed E_P , the fact that her question has changed from a vague expression of unease [23, 26] to a precise question about Yael's counting system [28] may indicate this. And indeed, as will be shown in Episode 2 below, Rachel acts as if building-with E_P while she and Yael continue to discuss the numbers 6, 15 and 21 of simple events for Ruth, Joe and the total, respectively, in [29–57]. During this entire excerpt, events as pairs are mentioned only a single time, and even then by Yael and not by Rachel. Nevertheless, in [64] Rachel provides explicit evidence for having constructed E_P .

[58] Noam See 'cause you didn't ... you did like one side on the die is 3 and the other side is 4 and you did like 3 plus 4 and that's like ...

[59] Yael I didn't do 3 plus 4. I'll tell you exactly what I did.

[60] Noam No wait a second, second. This is what I understood from what you did

[61] Yael I'll tell you what I did.

- [62] Noam One minute! No! You have to do 3 and 4; it is one possibility, and 4 and 5 is a second possibility, so it is two [possibilities].
- [63] Yael What?
- [64] Rachel That is what she did; (3, 4) is one possibility and (5, 4) is one possibility.

Rachel [64] explains that this is exactly what Yael did, by repeating Noam's explanation. Here Rachel uses E_P , which she had presumably constructed earlier, to build-with it her explanation. She thus provides evidence that she is already consolidating E_P . This excerpt also allows us to follow in some detail Noam's constructing of E_P . Her transition from "3 plus 4" in [58] to "3 and 4; it is one possibility" in [62], may be interpreted as the moment she begins to see the pair (3, 4) as a unit and hence has constructed E_P . She still expresses this fact rather clumsily, and this illustrates well "the first time the new construct is expressed or used by the learner", which forms part of our definition of constructing and indicates that her construct is still fragile. This fragility has disappeared by 109 where Noam builds-with and thus consolidates E_P when explaining what is the meaning of the difference in Task 2.

- [109] Noam If we look on the difference of one die and the other die: (2, 2), then the difference is 0.

In summary, we conclude from the data that the three students constructed and began consolidating E_P . While Yael's process of constructing was quick to the point of being immediate (and hence not observable until after the effect) and Rachel's process of constructing was hidden because she did not express it, Noam's was explicit, even for this very simple construct.

The question arises, in what sense the three students shared the knowledge they have constructed. The knowledge flow started from Yael. The constructs of the two other students emerged first through interacting questions and explanations from Yael to the two others, and then through the explanations of Yael and Rachel to Noam. At this point in time, it seems that E_P has become a shared common basis of knowledge for the group, which they can use together for further constructing. From observations of later activities, we have evidence that E_P was consolidated and remained available as part of the group's shared knowledge. The article by Hershkowitz et al. (2007) presents a more detailed analysis of the sharing aspects of this focus group's constructing processes.

8.3.4.2 Episode 2: Partially Constructing E_{CE} and E_{SS}

The knowledge elements E_{SS} , E_{CE} and E_{PV} for the 2D case as needed in Stage B are all generalizations of the corresponding 1D knowledge elements, which the students met, and presumably constructed, in Stage A. While the generalizations might seem straightforward, they are not necessarily so for students, and may therefore require processes of abstraction. For the three students in our report, the generalization of E_{PV} was indeed automatic and never formed the topic of specific attention. However, the generalization of E_{SS} and E_{CE} was associated with considerable difficulty, at least

for Rachel and Noam. As in Episode 1, Noam lagged behind, and we will focus on Rachel and use her process of constructing E_{SS} (she constructed E_{CE} concurrently) and use it to illustrate additional methodological aspects of AiC.

Chronologically, Episode 2 develops partially in parallel to Episode 1. We differentiate between the two episodes in order to clearly separate the constructing of E_P , which is a prerequisite for the constructing of E_{CE} and E_{SS} , and because the constructing process of E_P is quite different in nature from that of E_{CE} and E_{SS} .

As described in Episode 1, while working on Task 1 Yael is counting events [22, 24] and explains that she is trying to find all the possible simple events in the sample space (in her words, “the whole” [24]) in order to calculate the required probabilities. Rachel, who has already constructed E_P , still does not follow why and how Yael is counting [28], and Yael explains again:

[32] Yael What is my whole? This is my whole ‘cause... O.K., look! There are 6 out of... 5 + 4 is 9, and 3 is 12, and 2 and 1 is 15, and the 6 is...

[33] Rachel 21

Rachel adds the numbers; however, she still wonders about the meaning of the number 21, as we conclude from

[41] Yael I know that I am right. His frequency is 15 out of 21...

[42] Rachel What?

and later

[48] Rachel Why 21?

and

[51] Rachel What is 21? I don’t understand what 21 is! [laughs with embarrassment]

[52] Yael O.K., there are 6 possibilities that we will get the same number on the dice.

[53] Rachel True!

[54] Yael Then 15 plus 6 is 21.

[55] Rachel Ah, O.K.

...

[69] Noam Why 21?

[70] Yael Ufff !!

[71] Rachel Because there are 15 possibilities for him and for her 6 possibilities and 15 plus 6 is 21.

[72] Yael It is the whole, all the possibilities that can occur on the dice, two dice.

[73] Noam Ah, you roll...

[74] Yael Then, 6 divided by 21, it’s her frequency, and 15 divided by 21 it’s his frequency.

[75] Rachel [dictating the answer to Question 1a] No, because the chance of two equal numbers is 6 divided by 21 and the chance of different numbers is 15 divided by 21.

We omitted the segment [56–68] from here since most if it was discussed in Episode 1; in this segment, Rachel speaks only in [64] explaining to Yael what Noam meant – and this has to do with E_P , rather than with the constructs of interest presently.

In [69]–[75], Rachel and Yael, together and in similarly lucid language, explain the meaning of the number 21 to Noam. Based on this, we infer that Rachel’s construct for E_{SS} emerged previously, and given what she asked in [51], this can only have happened in [52]–[55]. In other words, we interpret her “True!” in [53] and her “O.K.” in [55] as expressions of her constructing action. This interpretation is strengthened by the fact that Rachel does not shy away from admitting her lack of understanding ([23], [26], [28], [42], [48], [51]). The evidence of Rachel’s constructing E_{CE} is similar: It occurred no later than [52]–[55] for the same reasons as E_{SS} . However, it is more difficult to locate precisely in a specific segment of the transcript because Rachel asked no questions expressing its lack. We can therefore locate the end of Rachel’s process of constructing E_{SS} in [55] but find it difficult to locate its beginning. There are three distinct possibilities, between which we cannot decide on the basis of the empirical evidence. In other research studies, we have found that all three of these possibilities actually do occur. One possibility is that constructing E_{CE} precedes constructing E_{SS} , i.e. that Rachel’s process of constructing E_{CE} has ended before [51]. Another is that the two processes partially overlap, with constructing E_{SS} beginning after constructing E_{CE} begins but before it ends, and constructing E_{CE} ending before constructing E_{SS} ends. The third one, and the one that seems to us most likely, is that constructing E_{CE} occurs within constructing E_{SS} , in other words that one constructing action is nested in another one. In some cases, such nesting occurs by necessity because the construct nested inside the other one is unavailable to the students and necessary for constructing the other one (see e.g., Dreyfus et al. 2001). This is not the case here; it is conceivable and has happened in our research project that students construct E_{SS} before they construct E_{CE} , that they construct E_{CE} before they construct E_{SS} , or that the two constructing actions partially overlap.

When explaining E_{SS} [71, 75], Rachel refers to the compound events “Ruth wins” and “Joe wins”, which have 6 and 15 elements respectively, i.e. she recognizes these compound events as relevant for the sample space; she also builds-with them the sample space, which consists of the union of these two compound events. She thus recognizes and builds-with her construct for compound event. Similarly but less explicitly, she recognizes and builds-with her construct for simple event, without which the compound events do not exist. We stress again that the R- and B-actions nested in C_{SS} are not R- and B-actions with the SS construct currently being constructed but rather R- and B-actions with the previous constructs: R_P , B_P , R_{CE} , and B_{CE} .

We are, of course, aware that these actions are all implicit and hence interpreted into [52]–[55]; however, given the lucid explicit explanation Rachel gives to Noam in [71, 75], we feel justified to make this interpretation. Moreover, since we consider Rachel’s constructing actions of E_{CE} and E_{SS} as terminated in [55], her explanation to Noam is an opportunity for her to formulate the constructs in her own words,

linguistically correct and elaborate, which we consider as her beginning to consolidate the constructs E_{CE} and E_{SS} . Searching forward through Rachel's activity on further tasks in the next lessons, for example a task with two spinners, we find further evidence for consolidation of these constructs. We are also able to observe how the constructs for E_P , E_{CE} and E_{SS} become increasingly self-evident for Rachel and how she uses them with increasing flexibility, for example in other physical contexts. A brief example will be given in Episode 3. However, giving more detailed examples for this would require lots of additional space without adding much more to the explanation of our methodology.

If, on the other hand, we search backward through the transcript for expressions by Rachel of a need for the SS construct, we find a clear need for it in Rachel's insistent and repeated questions in [42], [48] and [51]. We note that the questions become more explicit and clearer as her thinking proceeds. Even in its clearest form, however, Rachel's question asks what is the meaning of 21 rather than what is a sample space. This is only natural since only in very rare cases can learners be expected to ask for something they do not yet know about. This is why we interpret questions of the type Rachel asked as a clear expression of need for a new construct.

Finally, we observe that from an expert's point of view, Rachel's (and Yael's) results are wrong. In fact, there are 36 possible events in the sample space for two cubes, not only 21. They do not take into account that (1, 2) and (2, 1) are different events. This is, of course, due to the fact that Yael and Rachel have constructed and work with a construct for simple event that includes a construct for Pair (E_P) but does not include a construct for Order (E_O). Their construct for simple event is therefore only partially correct and since the simple event construct is a crucial constituent of the compound event construct and of the sample space construct, Rachel's (and Yael's) construct for sample space is also only partially correct. Ron et al. (2010) give a detailed treatment of partially correct constructs, together with a methodology for identifying them.

8.3.4.3 Episode 3: The differences' Task (Task 2)

The students in our focus group quickly agree that the game in Question 1b (Fig. 8.2) is not fair either, because Ruth has six winning possibilities and Joe has only 5, that the answer to Question 1ci is 21, and that playing with coloured dice does not make a difference. Yael and Rachel give these answers. Noam reads the questions and occasionally asks for clarification. This changes when they start thinking about Question 2. From the start, Noam contributes. She reads the question [107] and is the first to build-with E_P the difference between a pair of numbers on the two dice [109, see above]. Possibly, the fact that she is now sharing the E_P construct with her peers gives her some self-confidence and motivation.

When Yael points out that "difference 1" is a compound event [115] and uses this to reformulate the question more concretely [119], Noam immediately produces an answer [120–126].

- [115] Yael If you get (1, 2), the difference is 1 and if you get (2, 3) the difference is also 1.
- [119] Yael Are the probabilities of 1 and 4 equal?
- [120] Noam I understand. No!
- [121] Yael Are the probabilities of 1, 2, 3, 4 equal?
- [122] Noam We understand! NO!
- [123] Yael How do you know? Did you calculate it?
- [124] Noam No! Because they are not equal.
- [125] Yael How do you know?
- [126] Noam If we say difference of 4 ... or we say difference of 5 there is only one.
- [127] Yael One moment! Let's take 1, then we have (1, 2) (2, 3) (3, 4) (4, 5) (5, 6) and the same number (1, 1) (2, 2) (3, 3) (4, 4) (5, 5) (6, 6).
- [128] Rachel What's the connection?
- [129] Yael Well, I check the probability of 0 and the probability of 1.

This is a case where the designer intended no new constructs but R- and B-actions were required to achieve a local goal. While Yael and Rachel [127–129] behave accordingly, building-with their existing constructs (E_{CE} , E_{SS}) to count how many pairs fit each difference in order to compute probabilities (E_{PV}), Noam uses only her compound event construct in order to construct a new strategy to achieve the same goal, namely comparing the frequencies of the compound events. She concludes that different frequencies mean different probabilities, without any need for calculating the probability value itself. We consider this strategy to be a new construct that has emerged for Noam. It is an unexpected and somewhat non-conventional construct, which the researchers did not include in the a priori analysis. We have met other such unconventional unexpected student constructs in other research studies (Ron et al. 2011).

In other cases, there are good reasons for the researchers to modify or refine the a priori analysis in order to include unexpected constructs; one example for this has been implicitly mentioned above: Yael (and many other students') construct for simple event did not take the order element into account. We therefore modified our original operational definition for the compound event element E_{CE} by adding, "if students did not construct the order element, but count all the non-ordered relevant pairs, we shall say that they have constructed the compound event element".

We note that when the three students summarize and write down their conclusions, which are identical, the difference in the approaches taken by Rachel and Noam leads to an argument between them that reaches far beyond the constructs under consideration here into constructs related to what it takes to justify a claim: For Noam, examples are sufficient to conclude that different differences have different probabilities. Rachel does not agree; she thinks that examples are not sufficient:

- [143] Noam No, let's give examples.
- [144] Rachel It decreases, you understand?

- [145] Noam Come on, let's give simple examples, O.K.? Let's give examples!
- [146] Rachel No! Because the difference between 0 and 1...
- [147] Noam It IS an example, you dupe!
- [148] Rachel [writes] Because as the difference grows, the probability becomes smaller.

In summary, we note that each student constructed her individual knowledge in her own way and time. While Yael and Rachel reached a stage where they shared their knowledge with respect to all constructs under discussion here, we have no evidence that Noam has constructs corresponding to E_{SS} and E_{PV} , and the strategy she constructed in Episode 3 may even point to the fact that she was lacking a construct for E_{SS} . However, this strategy constitutes an unexpected construct for Noam that could potentially be useful to build-with in other situation. All three students shared their lack of knowledge concerning the element E_O and hence all their other constructs were partial.

8.3.5 *Additional Methodological Comments*

There are a number of additional aspects of our methodology, which we were not able to demonstrate in this chapter because their presentation would require information about additional research studies and additional data and therefore be quite onerous in terms of additional space. We will therefore only comment on these aspects, referring the reader to other publications for more detailed information.

8.3.5.1 **Knowledge Construction and Social Interaction**

The study used for illustrative purposes above exhibits important aspects of the role of social context in knowledge constructing processes, including the role of the flow of knowledge between individuals during the interaction, which have been discussed in more depth by Hershkowitz et al. (2007). However, our previous study dealing with the relationship between peer interaction and the construction of knowledge (Dreyfus et al. 2001) already proposed a methodology using two parallel analyses of the protocols of the work of student pairs, an analysis of the epistemic actions of abstraction as well as an analysis of the peer interaction. The parallel analyses led to the identification of types of social interaction that support processes of abstraction. We also found an excellent fit between episodes defined for the epistemic actions analysis and episodes defined for the peer interaction analysis.

8.3.5.2 **Tools as Contextual Elements in Knowledge Construction**

Artifacts such as manipulative or computerized learnware are another element of the context that has been shown to possibly have a crucial role in the construction of knowledge. For example, following the students' work on Task 2 above and a

computer simulation of the situation in Task 2 that led to conflict, teachers were asked to present the sample space of two dice as a six by six table of all simple events. This helped many students construct E_O .

In a more substantial study on the influence of tools, Weiss (2011) has analyzed the role of an analogical model in knowledge construction during model-based tasks. The tasks were in the area of fraction comparison and the analogical model was the tower of bars. From the methodological point of view, the study by Weiss required not only an RBC analysis but also an analysis based on the Emergent Models approach of Gravemeijer (1999), which is rooted in the Realistic Mathematics Education philosophy of the Freudenthal tradition and makes use of vertical mathematization. This double analysis has led to the identification of deep theoretical relationships between AiC and RME.

In a different study, Kidron and Dreyfus (2010) describe how instrumentation led to cognitive constructing actions and how the roles of the learner and a computer algebra system (CAS) intertwine, giving the CAS a major influence on interactions between different parallel constructing actions.

8.3.5.3 Revision of the Instructional Design on the Basis of RBC Analysis

Finally, the micro-analytic nature of the research methodology described in this chapter allows the researcher to observe constructing actions very closely. One of the related practical considerations is that this allows educators to identify problems in the micro-design of an educational intervention. Kouropatov and Dreyfus (2014) briefly describe an example of a revision of the micro-design of an activity in the realm of the integral as accumulation.

8.4 The Relationship of Theory and Methodology in AiC

The flow of theoretical and methodological paradigms that determine the frames for research in science and mathematics learning recently became richer, more divergent and more sophisticated. In addition, it seems that more than in the past, researchers today do not feel obliged to and/or satisfied with sticking to one methodological paradigm. Research trends in our area are nowadays characterized by flexibility and creativity in combining research methods and methodological tools that fit the researchers' theoretical framework and meet their goals and needs to explain and answer some 'big questions' emerging from their explorations.

In this section we discuss issues concerning the contour lines between the theoretical framework of AiC and the methods and methodological tools within AiC research. Hershkowitz (2009) argues that "these boundaries (the above mentioned contour lines) are flexible and even a bit vague in the sense that the same scheme or model may serve as a theoretical framework in one piece of research, as a methodological tool in a second one, and as both of them in a third piece of research" (p. 273). There are many examples for this in AiC. One is the notion of nesting, which was originally meant for R-actions being nested in B-actions and B-actions

in C-actions, but empirical results showed that C-actions may be nested in higher level C-actions and this led us to expand both, our theoretical conception of nesting and our methodology for describing it (Dreyfus et al. 2001). A second example is a refined classification of B-actions into subcategories, distinguishing, for example, actions that provide an orientation in the student's problem space from actions that aim at solving a problem; this again led to an expansion of both, our theoretical conception of B-actions and our methodology for identifying them (Dreyfus and Kidron 2006). A third example is provided by the notion of PaCCs (Ron et al. 2010); introducing this notion led us to systematically carry out *a priori* analyses (a methodological tool), which in turn enriched the theoretical framework, as well as required a different method of analyzing the epistemic actions in order to identify PaCCs. Instead of providing more brief pointers to such local examples, we will in the remainder of this section discuss in some detail the most substantial and deepest case of intertwined growth of theory and methodology, which is related to the role of the social context in the construction of knowledge. This will also give us an opportunity to briefly review the history of AiC and conclude the paper with an outlook.

At the beginning of the AiC research work, we came up with a first hypothesis for the model using both theoretical considerations and the analysis of considerable amounts of data in parallel. In this undertaking, we were led by the need to give theoretical expression to the specific characteristics of our data, which pointed to constructing of knowledge by means of mathematical thinking. In the process, we took into account and incorporated elements of existing theories. As we described in the theory section, abstracting was taken as human activity of mathematization, specifically vertical reorganization of previous mathematical constructs, interweaving them into one process of mathematical thinking with the purpose of constructing a new mathematical construct—Abstraction in Context has emerged.

At that stage, the researchers found themselves in a circular situation where theory stemmed from the analysis of data, and the analyzed data and its interpretation served as evidence for validating the theory. We were quite aware of this situation and explained: “This definition (of abstraction) is a result of the dialectical bottom-up approach . . . a product of our oscillations between theoretical perspective on abstraction and experimental observations of students' actions, actions we judged to be evidence of abstraction” (Hershkowitz et al. 2001, p. 202). It is clear that for analyzing the above actions, the researchers had to use some basic methodologies, which fit protocol analyses of an individual and the more complicated analysis of cognitive and interactive work within dyads and groups. The three epistemic actions, recognizing, building-with and constructing, and the dynamically nested relationships between them were hypothesized as the main building blocks of the model. These building blocks and the dynamic relationships among them express the vertical reorganization within the theory, and at the same time are validated by being used as the lens and compass to describe and interpret the data analyses themselves. Such a situation held for the first steps towards the validation of the model as a theoretical framework and as a methodological tool as well.

In these first steps, the research context and the theory itself were expanded. As an example let's follow the development of the socio-interactive dimension: In our first AiC research we interviewed individual grade 9 students. Along the interview the student has an interaction with the interviewer only (Hershkowitz et al. 2001). In our second AiC research (Dreyfus et al. 2001) we interviewed dyads of students; the interaction between students turned out to be a main factor in the process of knowledge construction and therefore took its role in the theory, specifically as part of the context. The methodology changed as well: The interviewer became an observer, and a graphic tool for demonstrating the RBC flow together with the interaction flow was invented.

In these as well as the following stages of development, the RBC+C model has been validated, both as a theoretical framework and as a methodological tool, in various social settings and learning environments. The settings considered include (teacher-led) classroom discussion, small-group problem-solving processes (see the examples in Sect. 8.3), tutoring situations and individual activities (e.g., introspective self-reports of single learners; see Kidron 2008). The age range of the learners extends from elementary school to adult experts and the longitudinal dimension varied. And, indeed, research made it clear that the RBC+C model is an appropriate tool/theory/theoretical tool/methodology to describe abstraction and provide insight into processes of abstraction in a wide range of situations of abstraction and consolidation on a medium-term timescale, where consolidation is a process by which the construct becomes progressively more self-evident, the student's awareness of the construct increases and the use of the construct becomes more flexible (Dreyfus et al. 2006; Hershkowitz et al. 2007). More than 10 years of research and more than 40 research publications, contributed by more than a few people, separate the 'birth' of the AiC framework from today's RBC+C model, as an empirically based theoretical framework.

We give a small number of illustrative examples for the linked expansion of theory and methodology from more recent research. Tabach et al. (2006) present an example of knowledge constructing within the context of peer learning in a working classroom. It shows how the design of the tasks and the computerized tools available to the students afford the constructing of conceptual knowledge (the phenomenon of exponential growth and variation, as it is expressed in its numerical and graphical representations). The researchers traced the constructing of knowledge through a series of dyadic sessions for several months in a classroom environment, and analyzed three sessions with intervals of a few months between them. The analysis showed that knowledge is constructed cumulatively, each activity allowing for consolidating previous constructs. This pattern indicates the nature of the processes involved in creating a new abstract entity: knowledge constructing and consolidating are dialectical processes, developing over time, even over time intervals that seemed to break the continuity. Hence, the main function of the RBC+C model in this research was to serve as a methodological tool to illustrate construction and consolidation processes. In addition, the data of this specific study served, in turn, to let a new methodological characteristic emerge: The RBC+C model is an appropriate tool for analyzing construction of knowledge over long intervals of time.

Wood et al. (2006) examined “the relationship between the patterns of interaction that exist in the classroom and children’s expressed mathematical thinking” (p. 228) in classes from different cultures, and for this aim the RBC-model served as the “conceptual framework employed to examine the quality of students’ expressed thinking” (p. 225). We think that the term conceptual framework, when applied to a certain model, expresses the flexibility with which this model may be applied as a framework for both theory and methodology. And indeed in Wood et al.’s research, it seems that on the one hand the authors believe theoretically that the model with its three epistemic actions expresses quite accurately the level of mathematical thinking that children have; on the other hand they used the model as a methodological tool for analyzing the protocols of the class members and identifying the levels of thinking expressed by class members. They accumulated these data for the purpose of quantitative analyses of the levels of thinking expressed by the class members in discussions in the different classrooms. This research shows some maturation of the model as a theoretical framework and as a methodology. The authors needed two conceptual frameworks in their study and the model is one of them. The model is not any more the focus of the study. It allows the researchers to determine levels of thinking that are available for inspection in the classroom.

Finally, Dooley’s classroom research (2007) concerns what she calls collective abstraction processes, which emerged in one lesson, mostly during the last phase of the lesson, where a whole-class interaction took place. The researcher’s aim was to show how the class community, as one entity, reaches ‘sophisticated constructs’. She explained: “One pupil’s ‘recognizing’ led to ‘building with’ by another and to ‘constructing’ of new ideas and strategies by others” (p. 1658). Again, this is a situation where the researcher uses the RBC-model as a methodological tool to explore the existence and nature of collective abstraction, and by doing this she confirms the RBC-model as a conceptual framework for this type of collective abstraction as well.

The RBC+C model may exemplify the flexible contour lines between a model as theory and a model as methodological tool: the model aims to serve as a framework for describing, analyzing and interpreting a human mental activity. The model is appropriate for exploring individual student mental activity as well as for exploring collective mental activity that is distributed in a group or a classroom among different individuals.

The model with its three epistemic actions, has a very general nature, general in the sense that it can be used in many and varied contexts. The nested relationships among the epistemic actions of the RBC+C model are global and the three actions of the model are observable and can be identified. Therefore the model lends itself easily to be adapted and to contribute to research in many different contexts of constructing abstract knowledge.

The research by Dooley raises questions that are at the heart of classroom research more generally. For example, in a study that focuses on “the travel of ideas” in the classroom, Saxe et al. (2009) claim that “developing methods to understand the travel of ideas is foundational to understanding learning in classroom communities” (p. 221), where the individuals in the classroom have a main role in the travel of ideas. Researchers who plan to observe and analyze in detail micro-processes of

constructing knowledge, or argumentation, or any other learning processes which are related to the classroom activity in a given context, along a time segment that may range from minutes to weeks, and do not want to ignore the researched phenomena as it is expressed by the individuals, face great difficulties: The observation and documentation processes are complicated, data are messy and massive and there are no systematic clear-cut methodologies for analyzing them (Schoenfeld 1992).

It seems that there is no theoretical problem in conducting such classroom research, which analyses in parallel the knowledge-constructing processes in the different social settings, which are formed in a natural way in the classroom, including the paths of such processes within individuals and among them, and then interweaving the analyses together. This seems the optimal way for gaining insight into processes of constructing knowledge in the everyday classroom. But will the ‘heavy battery’ needed for documenting all of the above not affect the natural learning environment in the classroom? And what about creating the methodologies for analyzing the huge volume of the accumulated data needed for interweaving the findings together?

Looking back on our ‘research journey’ with AiC, we discern a trend from investigating an individual learner or dyad with an interviewer in a laboratory setting via investigating small focus groups in a working classroom, to investigating students’ processes of constructing knowledge, and shifts of the constructed knowledge in a working mathematical classroom. The first phase served to develop the AiC theory and the RBC+C model, whereby we used the RBC+C model in two parallel roles: as a methodological tool for analyzing the data and for validating the theory as explained in the concluding section of this chapter. In the second phase, we applied the RBC+C model for analyzing students’ abstraction processes as they worked in a group in a working classroom. In the third and current phase, we aim to develop a methodology to coordinate analyses of the individual, the group and the classroom collective in a working mathematical classroom (Tabach et al. 2014; Hershkowitz et al. 2014). For this purpose, we combine AiC with another theoretical framework, Documenting Collective Activity (Rasmussen and Stephan 2008), which serves to analyze whole class discussions. We hope that observing different kinds of interactions and learning settings in the classroom in parallel on one hand, and overcoming the methodological problems of analyzing, interpreting and interweaving the findings of the different research settings on the other, may give researchers a coherent and meaningful insight into natural processes of learning activities (in the widest sense) in the classroom.

References

- Chi, M. T. H. (1997). Quantifying qualitative analyses of verbal data: A practical guide. *The Journal of the Learning Sciences*, 6, 271–315.
- Davydov, V. V. (1990). *Types of generalisation in instruction: Logical and psychological problems in the structuring of school curricula* (Soviet studies in mathematics education, Vol. 2; J. Kilpatrick (Ed.), trans: Teller, J.). Reston: National Council of Teachers of Mathematics. (Original work published 1972)

- Dooley, T. (2007). Construction of knowledge by primary pupils: The role of whole-class interaction. In D. Pitta-Pantazi & G. Phillipou (Eds.), *Proceedings of the fifth congress of the European society for research in mathematics education* (pp. 1658–1667). Larnaca: University of Cyprus.
- Dreyfus, T., & Kidron, I. (2006). Interacting parallel constructions: A solitary learner and the bifurcation diagram. *Recherches en didactique des mathématiques*, 26, 295–336.
- Dreyfus, T., & Tsamir, P. (2004). Ben's consolidation of knowledge structures about infinite sets. *Journal of Mathematical Behavior*, 23, 271–300.
- Dreyfus, T., Hershkowitz, R., & Schwarz, B. B. (2001). Abstraction in context II: The case of peer interaction. *Cognitive Science Quarterly*, 1, 307–368.
- Dreyfus, T., Hadas, N., Hershkowitz, R., & Schwarz, B. B. (2006). Mechanisms for consolidating knowledge constructs. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th international conference for the psychology of mathematics education* (Vol. 2, pp. 465–472). Prague: Faculty of Education, Charles University in Prague.
- Freudenthal, H. (1991). *Revisiting mathematics education. China lectures*. Dordrecht: Kluwer.
- Giest, H. (2005). Zum Verhältnis von Konstruktivismus und Tätigkeitsansatz in der Pädagogik [On the relationship between constructivism and activity theory in education]. In F. Radis, M.-L. Braunsteiner, & K. Klement (Eds.), *Badener VorDrucke* [Baden preprints] (Schriftenreihe zur Bildungsforschung – Band 3, pp. 43–64). Baden: Kompetenzzentrum für Forschung und Entwicklung.
- Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning*, 1, 155–177.
- Hadas, N., Hershkowitz, R., & Ron, G. (2008). Instructional design and research-design principles in probability. In M. Kourkoulos & C. Tzanakis (Eds.), *Proceedings of the 5th international colloquium on the didactics of mathematics* (pp. 249–260). Rethymnon: The University of Crete.
- Hershkowitz, R. (2009). Contour lines between a model as a theoretical framework and the same model as methodological tool. In B. B. Schwarz, T. Dreyfus, & R. Hershkowitz (Eds.), *Transformation of knowledge through classroom interaction* (pp. 273–280). London: Routledge.
- Hershkowitz, R., Schwarz, B. B., & Dreyfus, T. (2001). Abstraction in context: Epistemic actions. *Journal for Research in Mathematics Education*, 32, 195–222.
- Hershkowitz, R., Dreyfus, T., Ben-Zvi, D., Friedlander, A., Hadas, N., Resnick, T., Tabach, M., & Schwarz, B. B. (2002). Mathematics curriculum development for computerized environments: A designer-researcher-learner-activity. In L. D. English (Ed.), *Handbook of international research in mathematics education* (pp. 657–694). Mahwah, NJ: Lawrence Erlbaum.
- Hershkowitz, R., Hadas, N., Dreyfus, T., & Schwarz, B. B. (2007). Processes of abstraction, from the diversity of individuals' constructing of knowledge to a group's 'shared knowledge'. *Mathematical Education Research Journal*, 19, 41–68.
- Hershkowitz, R., Tabach, M., Rasmussen, T., & Dreyfus, T. (2014). Knowledge shifts in a probability classroom – A case study coordinating two methodologies. *Zentralblatt für Didaktik der Mathematik – The International Journal on Mathematics Education*, 46, 363–387.
- Hickman, M., & Monaghan, J. (2013). Networking methodologies: Issues arising from a research study employing a multi-media artifact. Paper presented at the 8th conference for European research in mathematics education (CERME 8), Feb 2013. Antalya. Retrieved 28 April 2013 from http://cerme8.metu.edu.tr/wgpapers/WG16/WG16_Hickman_Monaghan.pdf.
- Kidron, I. (2008). Abstraction and consolidation of the limit concept by means of instrumented schemes: The complementary role of three different frameworks. *Educational Studies in Mathematics*, 69, 197–216.
- Kidron, I., & Dreyfus, T. (2010). Interacting parallel constructions of knowledge in a CAS context. *International Journal of Computers for Mathematical Learning*, 15(2), 129–149.
- Kidron, I., & Monaghan, J. (2009). Commentary on the chapters on the construction of knowledge. In B. B. Schwarz, T. Dreyfus, & R. Hershkowitz (Eds.), *Transformation of knowledge through classroom interaction* (pp. 81–90). London: Routledge.

- Kidron, I., Bikner-Ahsbahs, A., Cramer, J., Dreyfus, T., & Gilboa, N. (2010). Construction of knowledge: Need and interest. In M. M. F. Pinto, & T. F. Kawasaki (Eds.), *Proceedings of the 34th conference of the international group for the psychology of mathematics education* (Vol. 3, pp. 169–176). Belo Horizonte: PME.
- Kidron, I., Bikner-Ahsbahs, A., & Dreyfus, T. (2011). How a general epistemic need leads to a need for a new construct—a case of networking two theoretical approaches. In M. Pytlak, T. Rowland, E. Swoboda (Eds.), *Proceedings of the 7th conference for European Research in Mathematics Education (CERME 7)* (pp. 2451–2461). Rzeszow. [On CD].
- Kouropatov, A., & Dreyfus, T. (2013). Constructing the integral concept on the basis of the idea of accumulation: Suggestion for a high school curriculum. *International Journal of Mathematics Education in Science and Technology*, 44, 641–651.
- Kouropatov, A., & Dreyfus, T. (2014). Learning the integral concept by constructing knowledge about accumulation. *Zentralblatt für Didaktik der Mathematik – The International Journal on Mathematics Education*, 46, 533–548.
- Prusak, N., Hershkowitz, R., & Schwarz, B. B. (2012). From visual reasoning to logical necessity through argumentative design. *Educational Studies in Mathematics*, 79, 19–40.
- Radford, L. (2009). Why do gestures matter? Sensuous cognition and the palpability of mathematical meanings. *Educational Studies in Mathematics*, 70, 111–126.
- Rasmussen, C., & Stephan, M. (2008). A methodology for documenting collective activity. In A. E. Kelly, R. A. Lesh, & J. Y. Baek (Eds.), *Handbook of design research methods in education: Innovations in science, technology, engineering, and mathematics learning and teaching* (pp. 195–215). New York: Routledge.
- Ron, G., Dreyfus, T., & Hershkowitz, R. (2010). Partially correct constructs illuminate students' inconsistent answers. *Educational Studies in Mathematics*, 75, 65–87.
- Ron, G., Dreyfus, T., & Hershkowitz, R. (2011). *Looking back to the roots of partially correct constructs: The case of the area model in probability*. Tel Aviv University: Technical report available from the authors.
- Saxe, G. B., Gearhart, M., Shaughnessy, M., Earnest, D., Cremer, S., Sitabkhan, Y., Platas, L., & Young, E. (2009). A methodological framework and empirical techniques for studying the travel of ideas in classroom communities. In B. B. Schwarz, T. Dreyfus, & R. Hershkowitz (Eds.), *Transformation of knowledge through classroom interaction* (pp. 203–222). London: Routledge.
- Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 334–370). New York: Macmillan.
- Schwarz, B. B., Dreyfus, T., & Hershkowitz, R. (2009). The nested epistemic actions model for abstraction in context. In B. B. Schwarz, T. Dreyfus, & R. Hershkowitz (Eds.), *Transformation of knowledge through classroom interaction* (pp. 11–41). London: Routledge.
- Tabach, M., Hershkowitz, R., & Schwarz, B. B. (2006). Construction and consolidation of mathematical knowledge within dyadic processes: A case study. *Educational Studies in Mathematics*, 63, 235–258.
- Tabach, M., Hershkowitz, R., Rasmussen, C., & Dreyfus, T. (2014). Knowledge shifts in the classroom—a case study. *Journal of Mathematical Behavior*, 33, 192–208.
- Weiss, D. (2011). *Processes of mathematical knowledge construction with analogical models*. Unpublished Ph.D. thesis. Tel Aviv University.
- Wood, T., Williams, G., & McNeal, B. (2006). Children's mathematical thinking in different classroom cultures. *Journal for Research in Mathematics Education*, 37, 222–255.