

New opportunities in geometry education at the primary school

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Abstract This paper outlines the new opportunities that that will be changing the landscape of geometry education at the primary school level. These include: the research on spatial reasoning and its connection to school mathematics in general and school geometry in particular; the function of drawing in the construction of geometric meaning; the role of digital technologies; the importance of transformational geometry in the curriculum (including symmetry as well as the isometries); and, the possibility of extending primary school geometry from its typical emphasis on vocabulary (naming and sorting shapes by properties) to working on the composing/decomposing, classifying, comparing and mentally manipulating both two- and three-dimensional figures. We discuss these opportunities in the context of historical developments in the nature and relevance of school geometry. The aim is to motivate and connect the set of papers in this special issue.

Keywords Geometry · Primary school · Technology · Drawing · Transformations

While geometry constituted the whole of mathematics in the Ancient Greek times, it is now seen as just one of many fields of mathematics. And in schools, as many researchers have observed, it receives relatively little attention, with much of the focus being placed on arithmetic (in primary

school) and algebra (in high school). In fact, studies in North America have shown that geometry receives the least attention of the mathematical strands (Clements and Sarama, 2011). This, despite the fact that study after study shows that students perform quite poorly on a wide range of geometry tasks. One of the reasons we proposed this special issue for ZDM is that we are seeing an emergence of attention to geometry at the primary school, not only in terms of the number of studies promoting the importance of geometry at this young age, but also in terms of the new directions being pursued. The papers in this issue concern, for the most part, children aged 4–8 years old, which includes primary as well as pre-primary school children in some countries, but also extends to slightly younger and older children, depending on the context of study. Given the diverse opportunities now available for geometry education at the primary school level, the central questions we are addressing in this paper are: why is it important to engage primary school aged children in more geometry? And what might be worth doing in geometry with these children?

1 A brief history of geometry in school mathematics

The teaching and learning of geometry has varied tremendously over the past hundred years, when it was initially reserved for high school students who took as their textbook, Euclid's *Elements* (at least in North America, see Sinclair, 2008). In the 1960s geometry first became an explicit topic of study in primary schools. This late emergence of geometry in the primary school may explain why there was relatively little research for Handbook authors Clements and Battista (1992) and then Battista (2007) to report on in terms of the teaching and learning of geometry

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in primary school—this despite a tradition of educators such as Froebel, Montessori, Pestalozzi, Steiner, Boole and Somervell, who constructed programmes featuring spatial awareness of tangible objects. While these educators pursued many different forms of geometry (including three-dimensional geometry and transformational geometry), the 1998 ICME study on geometry made clear that almost every country (including Britain, China, France, Germany, Italy) based their primary school geometry curriculum on the study of two-dimensional shapes, such as circles and polygons (see Chapter 6 in Mammana and Villani, 1998). The aim, presumably, was to help prepare these young learners for Euclidean-style geometry.

One's understanding of the nature of geometry determines one's sense of the aims of geometry in school mathematics. Gonz  les and Herbst (2006) identify the four following aims as dominating the twentieth century discourse of Euclidean-based school geometry: (a) the mathematical position that students should experience the making and proving of conjectures; (b) the logical position that students should develop logical reasoning; (c) the empirical position that students should learn a language that allow them to model the world; and, (d) the utilitarian position that students should acquire tools for future non-mathematical work. It would be fair to say that the current aims of geometry education in many countries combine these four aims. In addition to often working at cross-purposes, these aims ultimately divorce geometry from algebra and other fields of mathematics. As a result of this Euclidean perspective and approach, there may be more of a focus on two-dimensional geometry at primary school mathematics, sometimes at the expense of three-dimensional geometry.

In contrast, consider the kinds of questions that Freudenthal (1971) offered as relevant to geometry: “Why does a tied paper ribbon show a regular pentagon?”, “What is the intersection of a plane and a sphere, or two spheres?” and “Why does a door need two hinges, and how can we add a third?” (pp. 418–419). Interestingly, some of these questions form the basis of the learning-teaching trajectories for geometry proposed by van den Heuvel-Panhuizen and Buys (2008) for the k-2 grades. They also echo some of the questions advanced by Herbert Spencer in his *Inventional Geometry* (1876): “Place a cube with one face flat on a table, and with another face toward you, and say which dimension you consider to be the thickness, which the breadth, and which the length” (p. 16); “Can two lines meet together without being in the same plane?” (p. 18); “Make a linear figure having the fewest boundaries possible, and in it write its name, and say why such figure claims that name”, “Can you invent a method of dividing a circle into four equal and similar parts, having other boundaries rather than the radii?” (p. 45). They also have more in common with the approach to geometry taken by Henderson

and Taimina (2005) in their book *Experiencing Geometry*, where the first task is to explore why folding a piece of paper produces a straight line, using concepts of symmetry. In these points of inquiry, we are asked to step out of the bounds of fixed two-dimensional space to explore multi-dimensional dynamic spaces. And, as exemplified in the papers of this issue, it is these forms of inquiry that some mathematics researchers are now turning to with primary school children.

2 Current research trends in primary school geometry

This special issue contains articles that focus on the teaching and learning of geometry at the primary school level. Several themes emerge from this group of articles that highlight current trends. These are: (1) the role of spatial reasoning and its connection to school mathematics in general and school geometry in particular; (2) the function of drawing in the construction of geometric meaning; (3) the affordances of digital technologies in geometric and spatial reasoning; (4) the importance of transformational geometry in the curriculum (including symmetry as well as the isometries); and, (5) extending primary school geometry from its typical passive emphasis on vocabulary (naming and sorting shapes by properties) to a more active meaning-making orientation to geometry (including composing/decomposing, classifying, mapping and orienting, comparing and mentally manipulating two- and three-dimensional figures). Since each of the articles will provide a relevant overview of the research related to these themes, we focus here on providing the broader background of the research on the teaching and learning of geometry in primary school. This will provide a relief against which the recent trends featured in this special issue can be better appreciated.

2.1 A growing appreciation for the importance of spatial reasoning

There is an extensive body of research that has accumulated over the past 20 years that consistently shows the strong correlation between spatial abilities and success in mathematics and science. Already in the late 1970s, research had shown a positive correlation between spatial ability and mathematics achievement at all grade levels (Fennema and Tartre, 1985; Guay and McDaniel 1977). More recently, converging evidence from the psychology research literature has revealed that people who perform well on measures of spatial ability also tend to perform well on measures of mathematics (Delgado and Prieto 2004; Farmer et al. 2013) and are more likely to enter and succeed in STEM (science, technology, engineering and math)

disciplines (Wai, Lubinski, and Benbow 2009). Studies connecting music and other arts disciplines with spatial-temporal reasoning development and overall mathematics performance (Brochard, Dufour, and Despres, 2004; Cupchik, Phillips, and Hill, 2001) as well as creativity across disciplines (Kell, Lubinski, Benbow, and Steiger, 2013), also signal the growing appreciation for the importance of spatial reasoning.

The correlations between mathematics performance and spatial reasoning are particularly important because some individuals are harmed in their progression in mathematics due to lack of attention to spatial skills (Clements and Sarama, 2011). Stated more bluntly, Fuys et al. (1988) write that current curricular emphases have produced “geometry deprived” students. Indeed, Clements and Sarama (2011) argue that geometry should be of the highest priority because it too—as a vehicle for developing spatial reasoning—predicts later school achievement: “Empirical evidence indicates that spatial imagery reflects not just general intelligence but also a specific ability that is highly related to (the) ability to solve mathematical problems, especially non-routine problems” (p. 134).

These findings would not be surprising to those mathematicians who have argued that geometry provides mathematics with its basic meanings (through representations, models, visualizations, analogies and physical materials). To paraphrase René Thom: geometry is long on meaning and short on syntax; algebra has a lot of syntax but less meaning. The plethora of new English-language textbooks offering a visual approach to various mathematical topics (such as Nathan Carter’s *Visual Group Theory* and Tristram Needham’s *Visual Complex Analysis*) attest to Thom’s position. The recent ZDM issue (see Rivera, Steinbring and Arcavi, 2014) also highlights the fundamental role of visualisation across the curriculum. Presaging the work of Lakoff and Núñez (2000), Tahta (1980) writes that even the most abstract mathematical objects are described using geometric metaphors, citing the example of Dedekind for whom a set was a bag and Cantor, for whom it was an abyss. Turning to school mathematics, he writes: “At a more important elementary level, the failure of so many to handle numbers confidently may be due to the fact that they do not have any mental picture corresponding to the numerals on which they are required to operate” (p. 3).

While the role of the visual in other fields of mathematics has gained attention, it has been argued that few mathematicians actually *do* geometry anymore. Indeed, as Whiteley (1999) has argued in his history of the decline of geometry in school mathematics, many mathematics departments at the tertiary level in North America were emptied of geometers, as other fields of mathematics gained more traction. This has had a ripple effect on the mathematics courses available to undergraduate students,

including pre-service teachers who then enter their profession with very little interest in or experience with geometry. However, against the charge emerging from the Bourkaki-era of anti-imagery, Jean Dieudonné (1981) argued that “if anybody speaks of ‘the death of Geometry’ he merely testifies to the fact that he is utterly unaware of 90 % of what mathematicians are doing today” (p. 231). What mathematicians do today is not Euclidean geometry, to be sure. Indeed, as Craine’s (2009) book *Understanding geometry for a changing world* explains, the applications of geometry range widely, and increasingly so given developments in computer-based visualisation and modelling. While these issues seem more relevant to the secondary and post-secondary levels of mathematics education, they should also inform the curricular and pedagogical choices that are made at the primary school level. Across the papers in this special issue, compelling arguments, grounded in research evidence, are made for increasing our focus on geometry and spatial reasoning in the early years. The papers featuring the use of technology and learning tools in this issue may be forward-looking in the way that they unveil, illustrate an invoke sophisticated geometry ideas for young children, thus anticipating the kinds of geometric working environments and curricula that older students should encounter.

2.2 Recent conceptions of geometry

With geometry being shaken from its Euclidean roots, the question of what geometry *is* deserves some attention. Freudenthal (1971) praises a geometry that starts “as a science of physical space, of the space in which the child lives and moves, as an organization of the learner’s spatial experiences” (p. 418). Perhaps in an effort to underscore the relation of geometry to other fields of mathematics as well as its roots as a deductive system, Usiskin (1987) describes geometry in terms of the following four dimensions: (a) visualising, drawing and constructing figures; (b) studying the spatial aspects of the physical world; (c) representing nonvisual mathematical concepts and relationships; and (d) standing as a formal mathematical system. It is interesting to note that Usiskin’s dimensions (b) and (c) imply that geometry is based on observations of the physical world while also providing a way of construing that world. Tahta (1980) is more interested in working out how geometry fits into mathematics more broadly and, in particular, how it relates to algebra. He argues that defining geometry in terms of the study of shape and space does very little to explain its nature. He cites Caleb Gattegno, who proposes that “geometry is an awareness of imagery” while algebra is the formalisation of such awareness (p. 6). As Tahta argues, the premature shift to algebra, which is prevalent in school mathematics, can have devastating consequences for

learners by robbing them of something to act upon. Tahta's definition places imagery in a central role, suggesting that geometry is about working with—summoning, drawing, transforming, imagining, becoming aware of—imagery, which highlights its important connection to spatial reasoning. Thus for Tahta, doing geometry is part and parcel of doing mathematics: if there is no imagery, there is not mathematics.

It seems that the current uptake of research on spatial reasoning, which will refuse to reduce it to sterile written exercises or to vague empirical activity, may enable the pursuit of geometry on its own right. However, following Tahta, it will be important for researchers and educators to adequately attend to the interplay and complementarity of geometrical and algebraic activity, lest the pendulum swing too far in the other direction.

While constructs such as “imagery” and “visualization” (as well as others) have long been studied in the psychology literature by Levine, Goldin-Meadow, Lefevre and many others, recent research in mathematics education is now focused on gaining a better understanding of the various aspects of seeing, feeling and thinking that might be involved in these broad notions, and on how they fit with primary school learning contexts. In their paper, Moss, Hawes, Naqvi and Caswell (this issue) argue for the centrality of *spatial reasoning* and geometry in the early years and describe a dynamic curriculum that has been developed through four adaptations of Lesson Study activity in the Math for Young Children (M4YC) program. In particular, they offer vivid descriptions of playful tasks generated through exploratory lessons that promote geometry and spatial reasoning in the classroom and simultaneously support teachers in developing an interest in, and knowledge of, geometry as a critical feature of mathematics curricula in the first years of schooling. Van den Heuven and Elia (this issue) investigate imaginary perspective taking, which is often described as one of the central components of spatial reasoning for kindergarten children. They discuss how mathematics abilities predict imaginary perspective taking and confirm through their research, that the ability to deduce which objects are visible and which are not, from another point of view, is more accessible for young children than imagining the actual appearance of objects from another viewpoint. These authors also showcase a new measure for perspective taking based on their work on the PICO project.

The paper by Hallowell, Okamoto, Romo & LaJoy (2015) as well as that of Bruce and Hawes (2015) describe efforts to identify some key and previously neglected components of spatial reasoning to focus on, and then present empirical data from implementation of geometry interventions for primary school children that emphasize these components. Hallowell et al. report on their study of first

grade children's intuitive understandings of 2D and 3D figures, providing new insights into how children perceive 3D figures and their properties. Bruce and Hawes report on the malleability of 2D and 3D mental rotation abilities, and their findings of the gains in mental rotation skills for children ages 4 through 7 of all ability levels, during a geometry-rich mathematics program.

2.3 Building on children's strengths and predilections

The possible movement and expansion of the geometry curriculum is being studied on multiple levels. On one level, researchers are identifying geometric topics and ideas that might be relevant to students both at school and in the workplace, that go beyond those typically associated with Euclidean geometry. On another level, there is a growing amount of evidence both from psychology and mathematics education showing that children come to school with a great deal of informal geometric understanding (see Bryant & Watson, 2009), which is often not formally supported until much too late in the curriculum, once numerical and algebraic ways of thinking have become dominant. Some of these new findings are at odds with the traditional Piagetian assumptions about what and when children develop geometric reasoning. In his work with Inhelder, Piaget proposed a theory in which children first attended to topological features of shape such as being closed or having holes. Only later did they attend to distinctions such as rectilinear vs. curvilinear and then finally, they might attend to differences between rectilinear shapes (such as square and rhombus).

Many criticisms of these theories have been advanced, including that instead of focusing on topological vs Euclidean properties, it may be that visually salient features or “markers” such as holes and corners are more explanatory. Further, the role of familiarity, which has a historical and cultural component, seems relevant. There are also questions about how students' motor skills might be interacting with their supposed geometric understanding. Empirical evidence demonstrates that younger children (aged 2–3) can in fact distinguish between rectilinear and curvilinear shapes (Lovell, 1959; Page, 1959) and that 4-year-olds' drawings do not reflect primarily topological features (Martin, 1976). It is worth noting that in much of the research on children's drawing, there is an epistemological assumption that these drawings reflect children's internal representations of shapes. However, more contemporary theoretical perspectives attempt to avoid the implied dichotomy, preferring instead to view diagrams (and speech, gestures and other actions) as the active space for thinking itself, rather than infer any mental structures or schemas (see Châtelet, 2000; de Freitas & Sinclair, 2012; Thom & McGarvey, this issue).

Many papers in this special issue critically explore and examine relatively under-developed geometry topics that have not previously been associated with the skills, abilities and interests of primary school aged children. In the sections below we briefly outline the research context of these papers and also briefly summarise their main contributions.

2.3.1 *Diagrams and drawings*

Two papers in this special issue offer new ways of thinking about the role of diagrams in children's geometric thinking, including that of Thom and McGarvey (this issue), who examine whether the act of drawing serves as a means by which children become aware of geometric concepts and relationships. They highlight especially the importance, for both teachers and researchers, of examining the temporal way in which diagrams are made and their relation to spoken words, gestures and context. Kotsopoulos, Cordy & Langemeyer (this issue) analyse drawings that children produce in large-scale mapping tasks, with a particular interest in the way that children communicate motion through maps, gestures and verbalizations. Together these papers capitalize on the use of diagrams and drawings as a means to assess and attend to children's understanding and temporal sense of space.

While Piaget and Inhelder's work shapes a great deal of early research on children's understanding of geometry, focusing as it did on the types of features children attend to as well as on the diagrams they produce, the van Hiele model (1985) has had an even more lasting influence on mathematics education research at all levels, including the primary years. In short, and given our focus on the primary school, it is worth noting that the main levels in the van Hiele model described as relevant for this age span are the first two ("visual" and "descriptive/analytical"). Indeed, studies that use the van Hiele levels claim that most students show 'level 1 thinking' until they are in grade 5. There have been many refinements, criticisms and alternatives to the van Hiele levels over the past few decades, with researchers questioning the ideas that: (1) children "jump" discretely from one level to the next; (2) the levels are sequential and hierarchical; and that (3) the "snapshot" approach in which students are described as being "at" a particular level (see Clements and Battista, 1992). Further, the van Hiele levels seem to neglect the known complexities and malleability of spatial reasoning. It is important to underline that, because of the context in which these levels were identified, they have been used primarily to analyse the way children identify, describe and classify two-dimensional shapes. The persuasiveness of the van Hiele theory (in terms of being neatly amenable to providing numerical assessments of children's geometric thinking) may have driven research—and perhaps also curriculum—to focus on

the topic of two-dimensional shape identification, description and classification. For example, until very recently, the Piagetian interest on children's drawings has all but vanished in the mathematics education literature. Even at the middle school level, researchers have shown that students tend to rely on prototypical images when identifying shapes, and have difficulty coordinating these images with verbal or written descriptions and definitions (Clements and Battista 1992). The paper by Kaur (this issue) extends this work, with a particular emphasis on how children might use inclusive relations to identify different types of triangles with the support of dynamic geometry software, pushing the typical boundaries of young children's almost exclusive exposure to regular triangles.

2.3.2 *Angles and symmetry*

Another geometric concept that has received some attention at the primary school level is that of angle—again, perhaps influenced by Piaget and Inhelder (1956), who wrote that "It is the analysis of the angle which marks the transition from topological relationships to the perception of Euclidean ones" (p. 30). For somewhat different reasons, Clements and Battista (1992) argue that the concept of angle is fundamental to the development of geometric understanding. Further, it would seem that children show sensitivity to the concept of angle from very early years (Spelke, Gilmore and McCarthy, 2011). However, the concept of angle is shown to be multi-faceted and difficult to define—not just for students, but also historically for mathematicians. As, Sinclair, Pimm and Skelin (2012) write, the definitions currently used in textbooks produce many different answers to the question 'How many angles in a triangle?', none of them being three! Henderson and Taimina (2005) list the following different conceptions of angle: angle as a geometric shape, union of two rays with a common end point (static); angle as movement; angle as rotation (dynamic); angle as measure; and, amount of turning (also dynamic). Much of the research conducted on the development of the concept of angles has focused on children of age 9 and higher. Researchers have reported that young children have difficulty understanding angle as turn as well as difficulty connecting static angles to turns (Mitchelmore 1998; Clements, Battista, Sarama and Swaminathan, 1996). Further, many children tend to think that the length of the arms is related to the size of the angle (Gibson, Congdon & Levine, 2012) or that one arm must be horizontal and the direction always counterclockwise (Mitchelmore 1998). Another obstacle that has been identified is the salience of the right angle, which operates as a prototype of angle for many young learners, making it difficult for them to create or imagine angles that are larger or smaller than the right angle. Devichi and Munier (2013)

report that in a comparative study in which an experimental group of children ages 9 and 10 were introduced to angles as the space between infinitely long lines, they made less errors in terms of thinking that the length of the sides that form an angle affect the size of the angle, and in terms of the salience of the prototypical right angle.

Recent studies have gone beyond the confines of past curricula (Euclidean geometry) and past theories (especially Piaget) in pointing to the range of geometric concepts that primary school children might usefully engage with. For example, children come to school with informal awareness of parallel relations (Bryant & Watson 2009), and although such relations are highly relevant to their work in two-dimensional shape identification and description, they are not formally studied until middle school. While such a concept might strike some educators as overly ‘abstract’ or ‘formal’, researchers have shown that given the appropriate learning environments, children ages 4 through 7 have very robust understandings of parallel lines (see Sinclair, de Freitas and Ferrara 2013). Similarly, the concept of symmetry has been gaining attention as being relevant and suitable for younger children when presented thoughtfully. Previous research both by van Hiele and also Jaime and Gutiérrez (1995) had focused on the teaching of isometries in higher grades, but the paper by Ng and Sinclair (this issue) focus on symmetry in grades 1–3. In this paper, important ideas regarding the dynamic nature of symmetry are illustrated and made relevant, engaging and understandable for very young children. This research not only forces the reader to reconsider how we interpret typical geometry curricula in today’s classroom, but also requires us to re-evaluate what young children are capable of doing and understanding.

2.3.3 Three-dimensional geometry and volume

Another example of reconsidering what young children can do and understand mathematically, concerns three-dimensional geometry. Despite the prevalence of three-dimensional figures in many primary school classrooms, and the historical importance accorded to these figures—as in Froebel’s “gifts”—little research has focused on children’s geometric thinking in relation to these figures. This may be in part due to the influence of the *Elements* text on the geometry curriculum (and research on geometry) since three dimensional figures are only considered in the final book—while extensive work with two-dimensional shapes form the bulk of Book 1. Related to this, much of the van Hiele-driven research has also focused on geometric thinking with two-dimensional shapes—though Gutiérrez (1992) extended van Hiele to three dimensions based on a teaching experiment with 12 year olds. In fact, most of the research involving three dimensions has been aimed at older children (see also Pittalis and Christou 2010). Recently, however,

Ambrose and Kenehan (2009) conducted a teaching experiment involving nineteen eight- and nine-year olds in which children built and described polyhedra over the course of several lessons. The lessons were designed to enable the children to examine a range of examples and non-examples, thereby directing their awareness to component parts of the polyhedra. The researchers found that the children progressed from “visual-informal componential reasoning” to “informal and insufficient-formal componential reasoning” (using Battista’s 2007 refinement into sub-levels of van Hiele level 2). This suggests that children at this age can indeed begin to work geometrically with polyhedra. Moreover, while it seems reasonable that changes in reasoning of this kind might affect or be affected by similar activities involving planar shapes, almost no research has focused on this possible complementarity. Three papers in this special issue concern three dimensional geometry directly. The Bruce and Hawes (2015) paper and the Hallowell et al. (2015) paper examine how young children consider properties and orientations of two and three-dimensional shapes. In contrast, Tirosh, Tsamir, Levenson, Barkai and Tabach (2015) examine how preschool teachers demonstrate knowledge of 2D polygons with little difficulty, but struggle more with defining and identifying examples of cylinders. This paper calls into question the current limited emphasis on geometry and mathematics overall during training of preschool and elementary teachers (Stipek, 2013), particularly given the evidence presented in this special issue of young children’s capabilities in geometry and spatial reasoning.

In the spirit of investigating the extent to which young children can make sense of concepts usually reserved for upper secondary school, Ruttenberg, Mamalo and Whiteley (this issue) study how grade five children use spatial reasoning to work on optimizing volumes using rates of change. While much research has been devoted to children’s developing understanding of various measurement concepts, this paper focuses more on the concept of optimization, which can be seen, along with invariance and symmetry, for example, as a key idea in mathematics.

2.4 Maximizing the affordances of digital technologies

Another key area of exploration in this special issue concerns the increase of appropriate digital technologies for young geometry learners. In contrast to older software, such as Logo-based programming, which require numerical and/or symbolic input, or older mouse- and keyboard-drive hardware input, which can present motor dexterity challenges, newer touchscreen and multi-touch environments can greatly facilitate mathematical expression. Research has recently shown how new digital technologies that promote visual and kinetic interactions can help support the teaching and learning of geometry (Battista 2008; Bruce et al. 2011; Clements

and Sarama 2011; Highfield and Mulligan 2007; Sinclair, de Freitas and Ferrara 2013; Sinclair and Moss 2012). These new technologies are already challenging assumptions about what geometry can be learned at the early primary school level; they are also showing that long-assumed learning trajectories might change drastically if geometry becomes a more central component of the curriculum.

One important type of digital technology is the “virtual manipulative” (VM), which adapts existing concrete manipulatives such as Miras and geoboards to the screen. For example, Moyer, Niezgoda and Stanley (2005) studied the use of a Pattern Block VM with kindergarten children and found that the patterns they created were more creative, complex and prolific when using the VM than when using concrete materials. Further, the children were able to create designs that are more precisely assembled than if they were working with physical objects since, as the shapes can be “snapped” into position, they stay fixed. Similarly, Highfield and Mulligan (2007) report that preschool children using a Pattern Block VM as well as a drawing tool called *Kidpix* experimented with more patterns, created more precise patterns and made more use of transformations than children who worked only with physical materials. The authors do caution that the children found the use of the mouse challenging—again, this is a hardware limitation that touchscreen technology can mitigate—and the additional affordances of *Kidpix* sometimes distracting. Sarama and Clements (2002) describe a geometric *manipulable* and *constructive* digital technology called “Piece Puzzler”. It was intentionally designed to contain screen versions of pattern blocks and tangram shapes, which children can manipulate to create or duplicate larger composite shapes. The authors report that the “use of the tools encourages children to become explicitly aware of the actions they perform on the shapes” (p. 123) since, unlike physical pattern blocks and tangram shapes, children cannot just tacitly pick up and move the pieces with their hands.

Another type of digital technology that has gained wide appeal at the primary school level is the dynamic geometry environment (DGE) such as *Cabri-géomètre* and *The Geometer’s Sketchpad*. Because of the nature of this kind of digital technology, which enables continuous transformation through dragging in which only non-critical attributes of a shape can change, but critical ones are preserved, an a priori analysis of DGE affordances suggests that they could both (1) help learners see and make a large example space of geometric shapes such as long, skinny triangles, and (2) help learners appreciate aspects of the inclusive relations in the sense that it is possible to transform a constructed parallelogram, for example, into a rectangle. Within the small but growing area of research on the use of DGEs with young learners, the typical activity structure involves teachers and students interacting with pre-made sketches rather than constructing shapes on their own.

Battista’s research on his *Shapemakers* unit, which was designed for grade 5 students, shows how the use of *Sketchpad* was central in helping the students identify, classify and even define quadrilaterals. Battista (2007) theorizes the effectiveness of dragging in terms of a two-folding assumption: first, “unconscious visual transformations are one of the mental mechanisms by which we spatially structure shapes” (p. 150). For example, the opposite sides of a parallelogram are unconsciously seen as parallel through a translation of one side onto another. Battista’s second assumption, which he called the *transformational-saliency hypothesis*, related more centrally to dragging. This hypothesis essentially stated that people notice invariance. As students drag the rhombus maker, they notice what stays invariant, namely the fact that all four sides are equal. For Battista, it was not just that one *might see* invariance in dragging but that one cannot help but notice it. He thus conjectured that “investigating shapes through *Shape Maker* transformations make the essence of the properties more psychologically salient to students than simple comparing examples of shapes as in traditional instruction” (p. 152). Dragging thus changes the way shapes are perceived, moving from a static visual apprehension to that of a temporal attention on what remains invariant. This hypothesis might explain the tendency students have to compare shapes using transformations, as reported both in DGE environments (Jones 2000; Sinclair & Moss 2012) and in non-DGE ones (Lehrer et al. 1998). However, it is important to underline the way in which the *Shapemakers* tasks and the teacher’s guidance promoted this attention to invariance.

In the context of working with much younger children (aged 4 and 5), Sinclair and Moss (2012) found that the use of *Sketchpad* enabled these young children to move rather quickly beyond the prototypical images of triangles, which prior research has shown to be challenging for many students (Clements and Battista 1992; Hershkowitz 1989). Before describing the results of the study, it is important to signal their choice of theoretical framework, which differs from the prior research described so far. They adopt the Sfardian’s approach, in which thinking becomes a form of communication, so that instead of speaking about van Hiele levels of thinking, which are based on the Piagetian idea of thinking as manipulating mental structures, they speak of types of discourse. As they explain, “the view of geometric thinking as a form of communication entails that this thinking arises as a result of interactions with expert participants of the activity. This position is incommensurable with the Piagetian view of transitions from one level to another as a matter of the child’s “natural” development (and this assumption seems to underlie van Hiele’s work in spite of his insistence on the principal role of the teacher in the pacing of the level-to-level transitions)” (p. 30). They propose the three types of discourse, adapted to the particular

case of triangles, which have obvious connections to van Hiele levels: the discourse of elementary discursive objects involving physical reality; the discourse of concrete discursive objects; and, the discourse of abstract objects.

Using this analytical framework, the authors show how the classroom discourse transitions quite quickly (over a 30 min period) from 1st type to 2nd type discourse, prompted initially by the teacher's introduction of a (non-intuitive) example of a triangle (appearing 'upside-down'). In trying to decide whether or not such a shape is a triangle, the children begin to appeal to its attributes, including the three sides and vertices. One child, after a few minutes of discussion, states that "Every triangle could be, um, a different shape but it just has three corners", a statement that does not just show her willingness to call the variety of shapes appearing on the screen into the class of "triangle" but also a sense that there are a multitude of such shapes ("every triangle") and that there are some sufficient conditions for describing them (just has three corners"). The authors are careful to point out how the teachers' guidance—particularly her insistence on using attributes to describe the shapes on the screen and her use of counter-examples—is crucial in these children's work with the dynamic triangles.

The study reported in Kaur (this issue) extends this research to the context of triangle comparison and inclusive relations, by adding to the Sfardian theoretical framework, the role of gestures and diagrams in the children's thinking. This paper has important connections with research on classifying and defining that has been undertaken at the secondary level, both with and without dynamic geometry technologies—and suggests possibilities for better continuity between the usual focus on naming regular shapes in early primary school to more complex work with definitions in secondary school.

Also focusing on digital technology, Sourcy-Lavergne and Maschietti (this issue) report on teaching experiments using *Cabri-Elem*, which is a dynamic geometry software in which teachers and researchers can create microworlds focused on particular concepts. Their approach involves a dual interaction with both physical and digital artifacts that is very important in the context of the primary school classroom where physical manipulatives are common. These authors focus on children's orientation and movement on a grid.

Programming is an important topic that some countries, such as Italy and the UK, include explicitly in the curriculum indications for preschool or primary school.¹ However, after the research on Logo at the primary school level (such

as Clements and Battista 1989), little research has been published on the teaching and learning of mathematics through computer programming. Currently, a variety of programmable toys (such as Bee-bot, Probot and Lego NXT) exist and are being used in classrooms around the world. Studies have examined the use of robotic toys to foster problem solving, mapping, and measurement activities (Highfield 2009, 2010; Highfield and Mulligan 2009). In their study of young children (aged 6–7 years old) programming of robots, Bartolini-Bussi and Baccaglioni-Frank (this issue) focus on the emergence of definitions of square and rectangle in a classroom experiment that was intended to foster students' transition from a dynamic perception of paths to seeing paths as static wholes, boundaries of figures with sets of geometric characteristics. As with Sourcy-Lavergne and Maschietti (this issue), Bartolini-Bussi and Baccaglioni-Frank (this issue) draw on the theory of semi-otic mediation (Bartolini Bussi and Mariotti 2008) and also focus explicitly on the articulation between physical and digital environments, as well as on the interplay between static and dynamic reasoning in the teaching and learning of geometry.

3 Future directions

With the growing potential for primary school geometry opened up as a result of the trends described above, the possibilities for increasing the range and depth of geometry education have and will continue to grow. It seems that one of the challenges that mathematics education researchers are facing is to articulate the important educational question of *what* should be taught and *why*. Drawing on Dewey, Schwartz (1999) outlines three different purposes of schooling: preparing people for the world of work, aiding the personal growth and development of citizens and transmitting the culture and values of the society. Many researchers working within the domain of spatial reasoning seem to focus on the first purpose as they point to the importance it plays in many STEAM fields. However, developing spatial reasoning can also be seen as aligned with the second purpose of personal growth and development, given recent work that shows that young children have strong informal understandings of shape and space that could be developed in schooling. Those who seek to expand the teaching and learning of geometry to focus on mathematical ideas such as symmetry and invariance seem more aligned with the third purpose. But there are tensions between transmitting the culture and values embedded in historical traditions of geometry (as in Euclidean geometry) in contrast with contemporary ideas, topics and methods. Interestingly, digital technologies seem to be able to support each of these different

¹ Several new programming languages have been specifically developed for young children, such as *Squeak*, *ToonTalk* and Giorgi & Baccaglioni-Frank's (2011) *Mak-Trace* app for the iPad.

purposes, as can be seen in the papers in this special issue, where they are used in the context of traditional planar geometry concepts as well as for more recent topics and applications.

Of course, decisions about what should be taught and why might also take into account Bruner's (1969) definition of what is worth knowing: "whether the knowledge gives a sense of delight and whether it bestows the gift of intellectual travel beyond the information given" (p. 39). The focus on key ideas certainly aims to fulfill the second part of Bruner's definition. In almost all of the papers in this special issue, the children's "sense of delight" is quite palpable. However, current theories, which focus on cognition, development, semiotics and discourse, make it difficult for researchers to inform the decision of what is worth knowing and why in relation to delight—and although this special issue attempts to begin the discussion, this would seem to be an area of fertile future work.

Another area of fertile, future work is certainly that of teacher preparation and professional learning. Only one paper in this special issue focuses explicitly on the teachers who teach geometry (Tsamir, et al. 2015), although several other papers point to the specific roles that teachers can play, as is the case, for example, in the paper by Bartolini-Bussi and Baccaglini-Frank (this issue), who underline the way that the teacher must support the transition from artifact signs to mathematical signs. As part of their theory of semiotic mediation, these authors also provide some insights into the way that a teacher might productively design a lesson, elements of which also appear in the papers by Kaur (this issue) and Ng & Sinclair (this issue). The lesson study professional learning model reported by Moss, Hawes, Naqvi and Caswell (this issue) offers promise as a method for supporting teachers in developing deeper content knowledge and related lessons that are both playful *and* mathematically powerful. Much work needs to be done to understand how teachers might be better prepared to play the kinds of roles that have been highlighted in these studies.

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