

Two aspects of young children's thinking about different types of dynamic triangles: prototypicality and inclusion

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Abstract This paper illustrates how young children (ages 7–8, grade 2/3) can use the potential of dynamic geometry environments to develop an understanding of, and reasoning about, the properties and behaviours of different triangles (scalene, isosceles, equilateral). It provides a detailed description of a geometry unit, during which children worked both in a whole classroom setting in which they could interact directly with Sketchpad (through an interactive whiteboard) and individually at tables with paper-and-pencil. Using Sfard's communicational approach, an attempt is made to extend the work of Battista to show how children developed a reified discourse on triangles, which involves inclusive descriptions of classes of triangles (in other words, they thought about equilateral triangles as special types of isosceles triangles). This discourse focused on the behaviour of pre-constructed triangles as the children dragged their sides and vertices, thus generating many non-prototypical examples. During the teacher-led explorations and discussions, the children's routines moved from informal dynamic descriptions to formal geometric properties as well as from particular to more general discourse about different types of triangles.

Keywords Dynamic geometry environment (DGE) · Triangles · Inclusion · Prototypicality · Primary school

1 Introduction

This paper reports on four lessons focused on working with different types of triangles in a split class of grade 2/3

children (ages 7–8) using sketches pre-constructed with *The Geometer's Sketchpad* (Jackiw 1991, 2009). The focus of this research is to study how the use of this Dynamic geometry environment (DGE) affects the children's thinking about triangles, including how they attend to various aspects of the dynamic sketches, how they talk and gesture about the moving objects on the screen, how they reason about the behaviour of different classes of triangles (i.e. scalene, isosceles and equilateral triangles) and how the mathematical explanations they offer evolve as they become more experienced both with geometry and with the software.

Previous studies in ZDM have recognised the role of DGEs in fostering an inquiry-based approach in the teaching of mathematics (Gueudet & Trouche 2011; Abdelfatah 2011; Mariotti 2013) and the potential of drag-mode nature in DGE to open an explorative space for learners to acquire mathematical knowledge (Leung 2011). The present study is also an attempt to explore the use of DGE for improving geometry instruction, especially at primary level. The study presented in this paper is part of a larger project aimed at designing and implementing geometric activities for young children that take advantage of the affordances of DGEs to engage children in describing, comparing, reasoning and arguing about geometric shapes. The project builds on the work of Battista (2008), which shows that the use of the *Sketchpad*-based Shape Makers microworld was effective in helping grade 5 students reason about the relationship between quadrilaterals, and of Sinclair and Moss (2012), which shows that kindergarten children were able to identify triangles based on their properties also (and not only their 'prototypical' visual appearance) with the help of teacher-led Sketchpad-based explorations and discussions. The goal was thus to begin the work of classifying shapes by starting with different

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types of triangles (isosceles, equilateral and scalene) based on their sides.

2 Children's understanding of shapes and their classification

The learning of geometrical concepts is a complex process that includes both visual and property-based reasoning, i.e. identifying different properties of a shape. Many researchers have shown that students are not very successful at identifying non-prototypical triangles: For example, Hannibal and Clements (2000) examined the processes used by young children (3–6 years of age) to decide on whether given shapes are triangles and found a high rate of rejection of non-prototypical triangles. Hershkowitz (1989) showed that young children use prototypes and imposed properties (such as orientation of the shape on the page) either to accept or reject the categorisation of a given geometric figure into a named class of shapes. This difficulty of correctly identifying non-prototypical shapes persists even for middle school students (Clements & Battista 1992). As Duval (2005) has argued, reasoning about shapes involves a dimensional deconstruction in which, for example, a 2D shape is described in terms of its 1D constituents (lines, angles, vertices). But young children often have very little formal exposure to these 1D objects.

Although much research focuses on shape identification in the early grades, children are eventually expected to engage in classifying and defining activities. Here too, research has shown that students have difficulty in moving from identifying to classifying and defining (Gal & Linchevski 2010). De Villiers (1994) suggests that classifying is closely related to defining (and vice versa) and can be seen either as hierarchical (by using inclusive definitions, such as a trapezoid is a quadrilateral with at least one pair of sides parallel—which means that a parallelogram is a special form of trapezoid) or as partitionial (by using exclusive definitions, such as a trapezoid is a quadrilateral with only one pair of sides parallel, which excludes parallelograms from being classified as a special form of trapezoid). In general, in mathematics, inclusive definitions are preferred as they are more useful. A number of studies have reported on students' difficulties with the hierarchical classification of quadrilaterals (Fuys, Geddes & Tischer 1988; Jones 2000; Fujita & Jones 2007). Fujita and Jones (2007), in their study with elementary trainee teachers (18-year-old or more), found that excessive reliance on prototypes cause difficulty in understanding inclusion relations of shapes. They reported that many learners refused to accept inclusive relations of quadrilaterals due to the influence of prototype images of quadrilaterals. For example, a prototype

rectangle with 'two longer sides and two shorter sides' hinders them to accept a square as a rectangle.

While the act of naming serves as a catalyst to form different categories (Waxman 1999), studies have also shown that children assume a given object will have one and only one name (e.g. Markman and Watchtel 1988). This assumption may cause difficulties in accepting the hierarchical inclusion of geometric figures where a square is also a rectangle and a quadrilateral. Although no research has been conducted on students working with hierarchical definitions of triangles, one would expect that young children would find it difficult to say that an equilateral triangle is also isosceles.

Sinclair and Moss (2012) describe the aspects of DGE that are relevant to the possibility of developing a discourse around the identification of triangles for young children. For example, in a DGE, diverse triangles on computer screen may be perceived as one triangle changing shape over time under dragging. Since the triangle preserves its identity throughout change, the experience with continuous transformations widens the range of shapes children are ready to call the same name. In the context of work with older children, de Villiers (1994) observes the great potential of dynamic geometry software for conceptually enabling many children to see and accept the possibility of hierarchical inclusions. For example, a dynamic parallelogram can be transformed into a rectangle, rhombus or square by dragging its vertices. Jones (2000) examined students' (12-year-old) interpretations of quadrilaterals and relationships when using dynamic geometry software and suggests that carefully designed tasks, sensitive teacher input and a classroom environment that encourages conjecturing and a focus on mathematical explanation can be helpful in students' progress with formulating mathematical explanations and coming up with inclusive definitions. Battista's (2008) *Sketchpad*-based Shape Makers microworld provides grade 5 (11- to 12-year-old) students with screen manipulable shape-making objects. For instance, the Parallelogram Maker can be used to make any desired parallelogram that fits on the screen, no matter what its shape, size or orientation—but only parallelograms. Battista theorises the effectiveness of Shape Makers in allowing "flexible imagery" (p. 140) constrained by increasingly sophisticated knowledge of geometric properties of shapes. The representational aspect of Shape Makers microworld provides a powerful representation system for thinking about shape categories and their defining properties. For example, as students drag the rhombus maker, they notice what stays invariant, i.e. in case of rhombus the fact that all four sides are equal. Dragging thus changes the way shapes are perceived, moving from a static visual apprehension to a temporal attention to what remains invariant.

Thus far, most of the research on shape classification has been related to quadrilaterals (e.g. Fuys et al. 1988; Jones 2000; Battista 2008) with a focus on students from upper primary and high school. There has been limited research on the classification of shapes with young children (lower primary K-3), in particular on classification of triangles. One goal of this research was to see whether children could develop flexible prototypes of different classes of triangles (isosceles and equilateral) and use these to compare classes of triangles in terms of inclusive relations.

3 Theoretical perspective

The van Hiele (1986) model provides a distinction of five levels for students' learning about shape. The use of the word "levels" suggests a hierarchical and essentialist view of thinking and learning, which has been criticised by many researchers (see Battista 2007). Further, the conflicting results in the research suggest that the geometric ways of thinking identified by van Hiele are not sufficiently operationalised. Sfard (2008) offers a more fluid communicational approach in her discursive framework, which is well suited to this study and has been shown to be effective by other researchers (Sinclair & Moss 2012; Wang & Kinzel 2014) because it enables researchers (and teachers) to make claims about students' thinking in terms of how students communicate.

Sfard's approach is based on a participationist vision of learning, in which learning mathematics involves initiation into the well-defined discourse of the mathematical community. The mathematical discourse has four characteristic features: word use (vocabulary), visual mediators (the visual means with which the communication is mediated), routines (the meta-discursive rules that navigate the flow of communication) and narratives (any text that can be accepted as true such as axioms, definitions and theorems in mathematics). Learning geometry can thus be defined as the process through which a learner changes her ways of communicating through these four characteristic features.

In the context of identifying shapes, Sfard has proposed the following three orders of discourse characterised by different types of routines and word uses, which Sinclair and Moss (2012) use in their study of children's interactions with DG triangles: (a) The discourse of visual object recognition: the word 'triangle' is used as a proper noun. Shapes will be called triangles if they are similar to previously seen (such as prototypical) triangles. The routine of identification involves visual object recognition. (b) The discourse of informal properties: the word 'triangle' is used as a family name, that is, the name of a category of elementary objects; identification is made according to visual family recognition as well as through an informal

properties check. Children can justify their identification (because it has three sides) but their justifications may also include extraneous properties. (c) The discourse of definitions: the word 'triangle' is used as the name of a category of objects, and identification is made through visual family resemblance first, and then, verification/refinement of properties. Students at this level use definitions as necessary and sufficient conditions.

The identification of scalene, isosceles and equilateral triangles could proceed in a similar fashion through the three orders of discourse. Since this study also focuses on finding how children compare these different classes of triangles, the three orders used by Sinclair and Moss above can be extended by paying attention both to identification and comparison. Thus for comparison, in the first order, children can compare shapes using visual identification of both. They might pay attention to comparing the spatial/graphical properties of colour, size (skinny, long, small, big) and orientation (up, down). At the second order, children can justify their comparison through visual recognition and informal properties check of both shapes. These informal properties may include the tool-based dynamic properties such as movement of a point (characteristic of its trajectory, speed), movement dependencies between points (points that can be dragged directly and moving freely, point that cannot be grabbed but may move by the mean of another point) as well as the spatial descriptions about the size (small, big, thin) or orientation (up, down), etc. of the triangles. For example, while comparing scalene and isosceles triangles in DGE, children may notice the different movement behaviour of triangles under dragging or they can recognise some geometric properties like symmetry in isosceles and non-symmetry in scalene triangles. This initial informal properties check can be further helpful in recognising formal geometric properties. The geometrical properties (symmetrical movement, equal angles, equal sides) and geometrical vocabulary are the signs of discourse of definitions. At the third order, since the condition specified by one definition (i.e. of equilateral) may be an extension of the condition in another (i.e. of isosceles triangle), children can make use of inclusive definitions (term suggested by De Villiers 1994). For example, defining an isosceles triangle as a triangle with at least two equal sides includes equilateral triangle, which can be thought of as a special kind of isosceles triangle. This research is particularly interested in investigating how the children move between different word uses, routines and narratives during the lessons.

Since spoken discourse is multimodal, Sfard's four features of mathematical discourse sometimes fail to account for the full set of resources used by young children to communicate. Given the importance of gestures in communication of abstract ideas (Cook & Goldin-Meadow 2006),

and their potential to communicate temporal conceptions of mathematics (Núñez 2003; Sinclair & Gol Tabaghi 2010), it is necessary to focus not just on the words or the visual mediators that the children use, but also on their gestures. This is not just true for young children, of course, as Kita (2000) has pointed out that the production of a gesture helps speakers organise rich spatio-motoric information, where spatio-motoric thinking organises information differently than analytic thinking (which is used for speech). Given the visual and dynamic nature of the learning environment of this study, gestures are likely to be an important component of the classroom communication about triangles.

4 Method of research

The research was undertaken in the context of a classroom-based intervention to achieve the aim of developing an empirically tested and theory-based solution to address problems in the teaching and learning of triangles. Stylianides and Stylianides (2013) argue that such interventions increase the likelihood of the applicability of the results of the research by shedding light on how and why certain situations work because “the research is conducted in the ‘world’ of practitioners (i.e. in actual classrooms)” (p. 334). In the sections below, the description of the participants involved in the study is followed by the design of the sketches that were used and, finally, the teaching sequence that was undertaken.

4.1 Participants and data collection

This teaching experiment is part of a larger project that involves the study of children’s geometric thinking in the primary grades. We (the research team) worked in a grade 2/3 (ages 7–8 years) split classroom with children and two teachers from a pre-K-6 school in an urban middle SES district in Western Canada. There were 24 children in the class from diverse ethnic backgrounds and with a wide range of academic abilities. We worked with the children and teachers on a bi-weekly basis on a variety of geometric concepts (in order: symmetry, angles, triangles, parallel lines, tessellations) over the course of 7 months. The geometry unit on the topic of triangles included the following *Sketchpad*-based lessons:

Lesson 1: concept of triangle and exploring the triangle Shape Makers.

Lesson 2: exploring relationships between different triangle Shape Makers.

Lesson 3: triangle art in *Sketchpad* (filling a rectangular area using only triangles).

Lesson 4: construction of isosceles and equilateral triangles.

Each lesson lasted between 60 and 90 min and was conducted with the children seated on a carpet in front of an interactive whiteboard. The first and second lessons were followed by paper-and-pencil activities. All lessons were videotaped and transcribed. Our research team consisted of three people: a teacher and two research assistants whose responsibilities were to video record the lessons. The leader of the research team took the role of the teacher for these interventions. The classroom teacher, as well as a student teacher, was also present in each lesson and helped to manage various aspects of the lesson. From now on, the word ‘teacher’ will only be used for the researcher who acted as the teacher during the intervention. Previous lessons with *Sketchpad* involved the concepts of symmetry and angles, but the children had never received formal instruction about classification of triangles before.

This paper presents the results from lessons 1 and 2 in detail, as the focus of this paper is to explore the children’s thinking of various types of triangles in DGE. These lessons lasted together ~2.5 h. From the videotaped data, two excerpts from the first lesson and two excerpts from the second lesson will be presented in detail, as well as some examples from the paper-and-pencil drawings made by the children. No excerpt from the lessons 3 and 4 are presented in this paper, as those activities extend beyond the purview of this paper. The excerpts from the first lesson show the children’s evolving discourse about the behaviour of the dynamic triangles upon dragging, where the focus of the lesson was primarily on recognising the properties of three different triangles. The two excerpts from the second lesson illustrate how the children learned to communicate about properties of different triangles and eventually moved to more generalised statements in the discussion led by the teacher. The examples of the children’s drawings help reveal the contrast between the children’s triangle conceptions in the static and dynamic environments.

4.2 Design of sketches

In consonance with the work of Battista (2008), we (the research team) developed the triangle Shape Makers sketches (see Figs. 1, 2, 3) for different types of triangles (scalene, isosceles, equilateral triangles, right triangle). Each triangle type had a different colour (pink for scalene, red for equilateral, blue for isosceles and green for right), which enabled the teacher and the children to refer to the triangles without having to introduce their formal geometric names. The sketch in Fig. 1 was intended to help the children to attend to the similarities and differences

between the three triangles with the help of dragging facility. While initially very similar-looking, these triangles behave very differently under dragging.

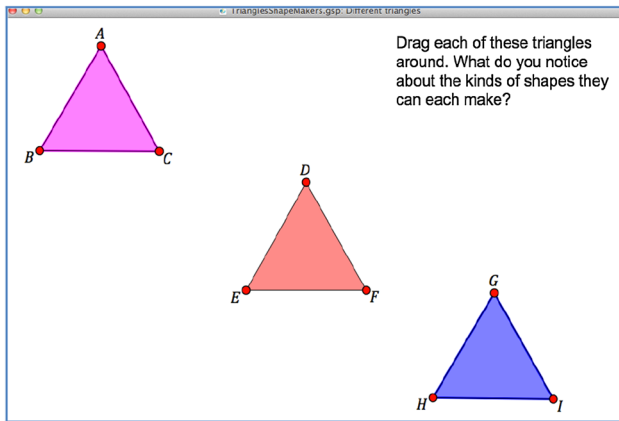


Fig. 1 Different triangles Shape Maker sketch (colour figure online)

4.2.1 Behaviour of dynamic scalene, isosceles and equilateral triangles

In the sketch shown in Fig. 1 only the middle (red) triangle is constructed to be equilateral; the bottom right (blue) one is constructed to be isosceles while the top left (pink) one is scalene. Although no vertex was labelled in the sketches, this is done so in Fig. 1 to explain the dragging behaviour of different triangles. In the scalene (pink) triangle, dragging any one vertex (A, B or C) does not move the other two vertices, whereas in equilateral (red) triangle, dragging vertex E or F (which determine the size of the triangle) rotates and enlarges the entire triangle around vertex F or E, respectively; dragging vertex D simply translates the triangle from one place to another. In the isosceles (blue) triangle, dragging vertex I does not move the other two vertices, dragging vertex H or G moves the entire triangle except the vertex G or H, respectively. Vertex I can only be dragged along a circular trajectory with GH as the radius of the circle.

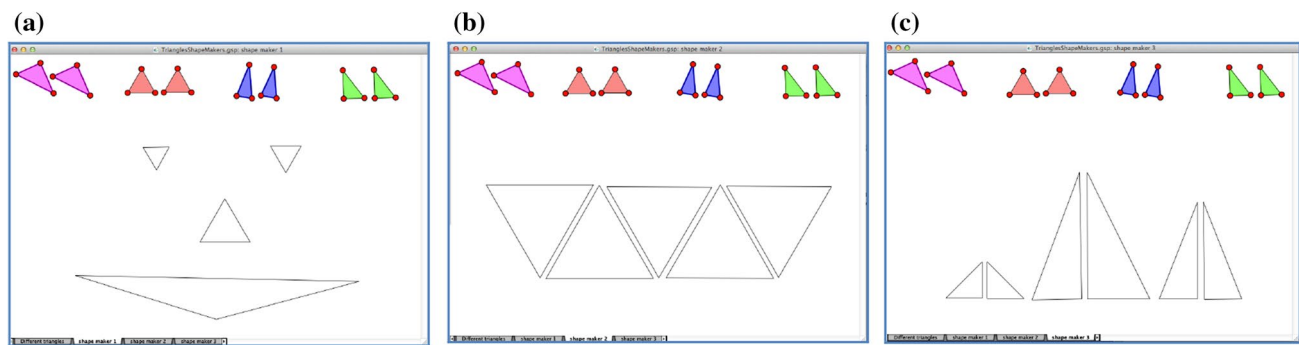
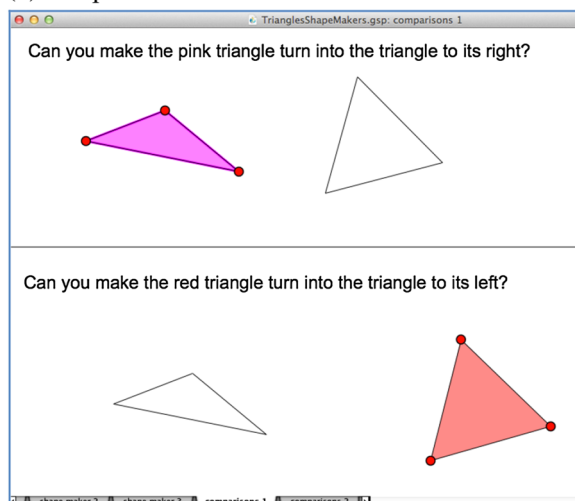


Fig. 2 a–c Shape Makers with outlines (colour figure online)

(a) Comparison 1



(b) Comparison 2

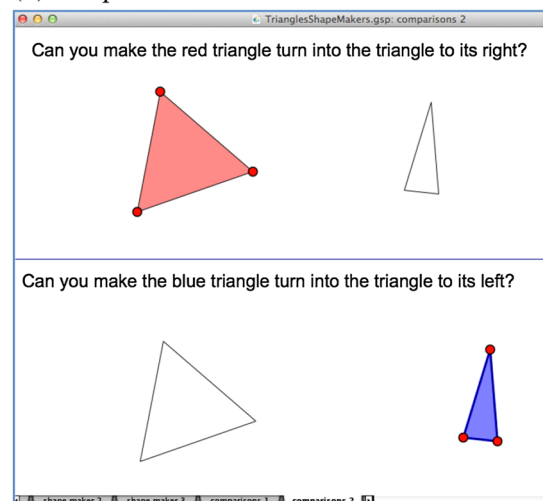


Fig. 3 a, b Comparison sketches

4.2.2 Shape Makers with outlines

For the sketches shown in Fig. 2a–c, the children were asked to explore which coloured triangles could fit into the given triangle outlines. In each of the sketches, the children can only use eight given coloured triangles. The sketch shown in Fig. 2a is adapted from Battista’s work. In this sketch, for the happy face, the eyes and nose outlines are made of equilateral triangles and the mouth is made of isosceles triangle. The intent of this sketch was to help children see that any triangle except the green (right) triangle could fit in the eyes and nose interchangeably, whereas in the mouth only pink (scalene) and blue (isosceles) triangles could fit. This would help in drawing children’s attention to the differences/similarities in the eyes, nose and the mouth outlines.

For the sketch shown in Fig. 2b, all the outlines look like equilateral triangles, so all the triangles can be used for filling the outlines except the right triangles. This design was intended to draw children’s attention to the fact that any kind of triangle can be used interchangeably in these particular (equilateral) outlines. This would implicitly initiate an understanding of inclusion of equilateral triangles in isosceles and scalene triangles. It is worth noting that there was no usage of green (triangles) during children’s working with the sketches shown in Fig. 2a, b.

For the sketch shown in Fig. 2c, all the outlines are made of right triangles of different sizes. Note that the equilateral triangles cannot be used in sketch 2(c), and isosceles triangles will fit only into the two left-most outlines. It was anticipated that children would first use the green (right) and pink (scalene) triangles based on visual similarity of green triangle with the outlines and their experience of easy morphing of scalene triangle during previous sketches, respectively. This would lead to a ‘stuck’ moment, which would help in prompting children to reconsider their choices and would implicitly help them to see the inclusion relationships. This is, in fact, exactly what happened in the classroom, where the children fitted the green (right) triangles into the two right-most outlines and the pink (scalene) triangles in the two left-most outlines. This prompted the need to reconsider the fitting of pink (scalene) triangle from left-most outlines to two middle outlines.

4.2.3 Shape Makers for comparison

The sketch in Fig. 3a was used as a way of focusing attention on the inclusive relations, after the children had worked with the challenges shown in Fig. 2. It focused on exploring whether a scalene triangle can fit into the given equilateral triangle (top) and whether an equilateral triangle can fit into a given scalene triangle (bottom).

The sketch in Fig. 3b is focused on exploring whether an equilateral triangle can fit into the given isosceles triangle

(top) and whether an isosceles triangle can fit into a given equilateral triangle (bottom). These two sketches differ from the previous sketches, as their intent is to make the explicit comparison between the pairs of different types of triangles.

4.3 Teaching sequence within the lessons

In first lesson, the teacher started with the static sketch in Fig. 1 and then asked the children to drag and explore the dragging behaviour of the three triangles. The teacher prompted children for noticing similarities and differences, but did not yet introduce the terms equilateral, isosceles and scalene for the red, blue and pink triangles, respectively. The children were then asked to fill the outline shown in Fig. 2a, and then the one shown in Fig. 2b—this was done by inviting children to come up one at a time to chose a triangle and drag it into place. After successful completion of these two tasks, the teacher asked the children to create paper-and-pencil drawings that were made up of only triangles.

In the second lesson, the outline shown in Fig. 2c was introduced, which the children completed, but this time with more difficulty. Then, the teacher introduced the sketches shown in Fig. 3a, b. After working with the sketches, she invited the children to draw families of the different triangles in their mathematics journals. The instructions were to draw a “mom”, “dad” and two other “family members” for each type of triangle. In the third lesson, the teacher asked the students to fill a rectangular area using only triangles. This was done in pairs with the children working at the computer (not sitting on the carpet), learning how to use Sketchpad themselves). In the fourth lesson, the teacher led a whole classroom exploration of how to construct both isosceles and equilateral triangle in Sketchpad. Then the children were asked to construct these triangles themselves using Sketchpad in pairs.

Throughout the lessons, the teacher expected the children to provide explanations for actions and claims and to try to convince their peers with their arguments. At times, suggestions were offered when children did not know how to proceed. In keeping with the “constructor” way of seeing that Duval (2005) identifies as the best entrée into geometric reasoning, a conscious decision was made to focus the interventions on the geometrical properties of the triangles and not to encourage the use of the measuring tools (such as length and angle).

5 Findings

In this section, a detailed analysis is provided of the excerpts that show the changing discourse over the two lessons, including the drawings produced at the end of each lesson.

5.1 Comparing the dragging behaviour of different triangles

The teacher presented the sketch (Fig. 1) to the children, in which all the three triangles appear to have the same size and shape. The teacher asked about the ways in which the triangles appear different and the same. Then, the teacher appointed three children to drag the vertices of each of the three coloured triangles. The children seated on the floor were asked to be “detectives” and to “describe what kinds of triangles can be made”, “what can change and what stays the same” in each of the triangles.

The first child Neva dragged the pink (scalene) triangle into various sizes and orientations, i.e. skinny and long, upside down, small and big triangles. Then Andy dragged the red (equilateral) triangle and the teacher asked if he could make it long and skinny. Several children said “no”. The teacher asked the children why it was not possible to make the red triangle long and skinny.

Excerpt 1 Children's discourse about the dynamic pink (scalene) and red (equilateral) triangles

- T (Pointing to one student) How come he can't make it long and skinny?
- Egan Because the red one, it's different than that (pointing to the pink triangle) and I think it can only go by a perfect triangle.
- T It can only... what do you mean by a perfect triangle? What's a perfect triangle? What's special about this triangle here (placing the marker at the red triangle)? Why is he calling it perfect? Rabia?
- Rabia Because the other triangle (pointing to the pink triangle) can move at a point but this one (pointing at red triangle) can move bigger or smaller differently.
- Jace Everything moves with it except one point.
- T Yes, everything moves with it (dragging one vertex of red triangle) whereas when I move the pink one, only one of the vertices move (dragging vertex of pink triangle). Even when it is getting bigger and smaller (dragging the vertex of red triangle and changing its size slowly), is there anything that stays the same as I make it bigger and smaller? (Many children put their hands up) Neva?
- Neva The angles.

The children started to notice the changes in the red triangle as Andy dragged one of the vertices. Egan's statement “it's different than that” shows that he started to notice the

differences between the red and the pink triangles, even though they initially looked the same in their static configurations. Egan's response ‘only go by a perfect triangle’ suggests that he notices something special about the triangle that makes it look, perhaps, like the prototypical triangle. Further, in Rabia's description “the other triangle can move at a point” and in Jace's statement “everything moves with it except one point”, the action words ‘move at a point’, ‘everything moves’ shows that the children are paying attention to the particular kinds of movements effected by each triangle. The dragging tool seems to have initiated this kind of reasoning so that the transforming triangles functioned as new visual mediators. Rabia's statement “move bigger or smaller differently” shows that she was noticing the different kind of size-changing behaviour of the equilateral triangle as compared to the scalene triangle. This shows her attention to the spatial and dynamic properties of the triangles. The use of words ‘red one’, ‘it’, ‘the other triangle’ and ‘this one’ to address the red and pink triangles shows that the children are thinking about one particular triangle as opposed to the family of those triangles. In addition, Neva noticed that the angles are staying the same under dragging. Thus, she is starting to notice the informal dynamic properties (based on dragging behaviour) as well as formal geometric properties (invariance of angles in the red triangle) of the different triangles.

This classroom discourse thus combines the first and second order of discourses, with a back and forth between attention to the visual appearance of the triangle(s) as a whole and to at least one property of those triangle(s).

5.2 Exploring the overlap of one triangle over the other fixed triangle

After the children explored dragging the three different triangles, the teacher asked them if they could fit the blue (isosceles) triangle over the given orientation of the pink (scalene) triangle without dragging the pink triangle. During the exploration, Rabia first matched the two vertices of the blue triangle to one side of the pink triangle (Fig. 4a) and then tried to drag the third vertex (the green dot in Fig. 4b) in an upward direction and concluded that she could not fit the blue triangle onto the pink one.

Excerpt 2 Children's discourse about dynamic blue (isosceles) triangle

- T You think you can't? How come you can't?
- Rabia Because I think if I move that one (placing marker at <black dot> vertex in Fig. 4b), that one also moves (placing marker at green dot in Fig. 4b).

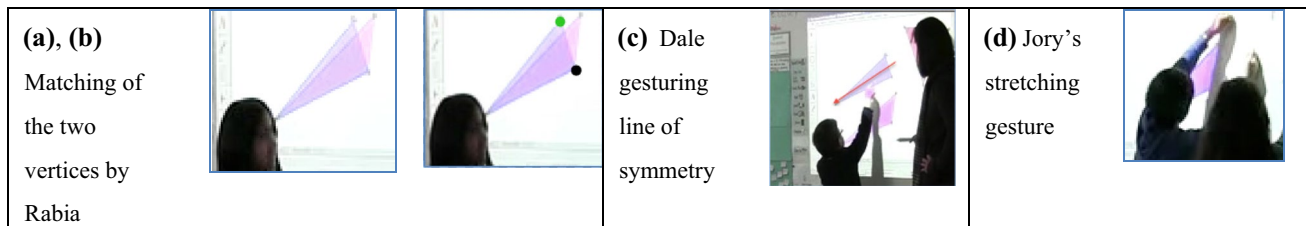


Fig. 4 a–c Snapshots of Rabia's overlapping attempt and Jory's gesture (colour figure online)

- T If you move this one here (*placing marker at <black dot> vertex*), this blue one, the other blue vertex moves too, right? Actually if you move this vertex (*dragging green dot vertex*), it (*black dot vertex*) moves too. How come you can't do it? How come you can't put the blue one on the pink one? *<Some students raise their hands> Yeah (pointing at Dale)?*
- Dale Because the blue one can only move symmetrical.
- T Because it can only move symmetrical, the blue one. That's an interesting thing. Where do you see the line of symmetry in this blue one? (*Dragging the blue triangle away from pink on the screen*).
- Dale There is (*comes near the IWB*) a line of symmetry right there (*gesturing to show the line of symmetry in middle of blue triangle [in the direction as shown by the arrow in Fig. 4c]*).

Two children were still unconvinced, so the teacher invited them to try on the screen. Both were unsuccessful. So the teacher asked whether anyone had a different argument that could be more convincing. Jory offered the following argument:

- Jory (*Coming to the IWB*) So, this one [*placing right index finger at green dot (Fig. 4b)*] wherever you move it, then this one [*placing right index finger at black dot (Fig. 4b)*] moves with this [*placing left index finger at green dot (Fig. 4b)*], so when you move, it will go that way (*stretching his arms upwards along the two longer sides of blue triangle (Fig. 4d)*).

After some more discussion about the overlap of the blue (isosceles) over the pink (scalene) triangle, the teacher started dragging the blue triangle separately on the screen and asked

- Teacher Okay, so I really like both of your arguments. Both the symmetry argument and the stretching argument (*dragging blue triangle away from*

pink triangle). I am going to call it the stretching argument. And what I want you to focus on is as I drag this point out (*dragging the vertex of blue triangle back and forth*), exactly like both Jory and Dale were saying it stays symmetric and this is growing at the same time as this (*pointing at the longer sides of blue triangle*). So these, if you look at these two sides right here (*pointing at the longer sides of blue triangle*), the two sides. What can we say about the two sides as I drag this out (*dragging blue triangle to different sizes*)?

Jace They are staying symmetrical.

Jory The lines...umm like they stay, they stay the same length (*pointing at the two sides of the blue triangle*). But in the pink one, the top line is long, they all are different, the side line... (Inaudible).

Later, the teacher marked the two equal sides in blue (isosceles) triangle with the same coloured lines and she led the same kind of discussion for the red (equilateral) triangle to help children notice the invariances.

Rabia's statement "if I move that one, that one also moves" shows that she is paying attention to the causal type of movement relationship between the different parts of the triangle. Dale's explanation "it can only move symmetrical", suggests that he has noticed invariance in the isosceles triangle, either holistically, or as a function of the movement of the congruent sides. The systematic dragging of the vertex of one of the longer sides of the isosceles (blue) triangle by Rabia acted as a visual mediator and seemed to help Dale see the property of symmetry. Jory's statements "wherever you move it, then this one moves with this" and "so when you move, it will go that way" along with the stretching arms gesture shows that he is also thinking about the simultaneous change in length of two arms of the isosceles triangle. Jory's arm stretching gesture provides evidence that using speech only to talk about the behaviour of dynamic triangles was not enough for him.

Most of the children agreed with the Dale's "symmetry argument" and Jory's "stretching argument", which they indicated by raising their hands or nodding their heads.

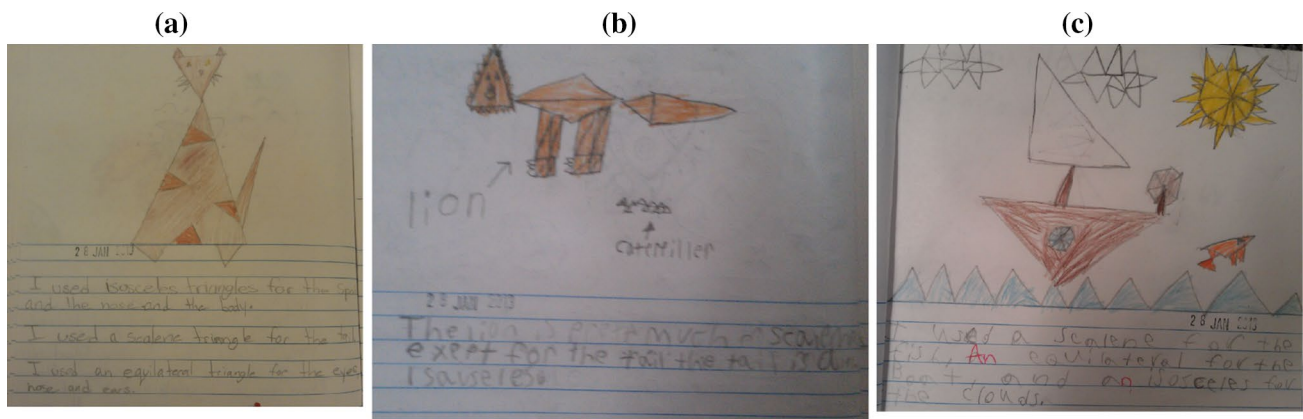


Fig. 5 a–c Examples of designs made by children (by Lida, Dale and Jory, respectively)

Note that Jace also talks about the lines “staying symmetrical” later on. Overall, the children came up with three arguments: (1) the dragging vertex argument (dragging one point moves the other point), (2) the symmetry argument (blue one can only move symmetrical) and (3) the stretching side argument (with arm stretching gesture where two arms act as two sides of triangle). These arguments involve a second order of discourse because they refer to the informal/formal properties of the isosceles triangle.

In the above excerpts, both the children and the teacher used “if...then” statements extensively to talk about the behaviour of the different triangles in the dynamic environment. The frequent use of “if...then” statements suggests that the children were making attempts to grasp the underlying rule governing the movement of each triangle and as a result were successful in identifying some geometric properties in each triangle.

Through teacher-led discussion, the children identified the properties of different sides (and angles) that stayed the same or changed under dragging. After these invariances were stated explicitly, the teacher introduced the “special names” equilateral, isosceles and scalene for the red, blue and pink triangles, respectively, by saying them aloud and also writing them on the IWB.

5.3 Triangle designs made by children

After the first lesson, the teacher asked the children to make drawings that only made use of triangles. Most of the designs involved animals such as cats, a lion, a dragon and a butterfly. Figure 5a–c shows the typical examples, made by Lida, Dale and Jory, respectively. These three drawings illustrate the examples of the aspects of inclusion and prototypically, respectively, in children's drawings. Lida (Fig. 5a) wrote: “I used the isosceles triangles for the spots and the nose and the body. I used a scalene triangle for the tail. I used an equilateral triangle for the eyes, nose

and ears”. It is interesting that Lida assigned the equilateral and isosceles category interchangeably for the nose, which looks very similar to the nose of the happy face in Fig. 2b, for which the children used an isosceles triangle. Lida may not be aware of the inclusive relation but she is willing to give the same shape two different names, which will prepare her for a third order of discourse.

Dale (Fig. 5b) wrote: “The lion is created much of scalene except for the tail. The tail is a isauseles”. Jory (Fig. 5c) wrote: “I used a scalene for the fish, an equilateral for the boat, and an isosceles for the clouds”. Like Lida's design, there is an extensive use of non-prototypical triangles in children's drawings.

5.4 Fitting triangles into the given outlines

In the second lesson, the children worked on the sketch in Fig. 2c, where they used the green (right) triangle for the first time. The teacher gave the “special name” right triangle to the green triangle.

Excerpt 3 The emergence of ‘paralysed triangles’

Thom tried to fit the blue (isosceles) triangle into one of the middle-right triangle outline (see Fig. 2c), but failed to do so. The teacher asked

- T How come you can't do it, Thom?
- Thom Because this one is (*pointing at bottom right vertex (see Fig. 6a of blue triangle and trying to drag it in upward direction)*) just paralysed to go up.
- T And is there any other reason why we can't do it with the blue one?
- Thom Because this is only allowed to spread out [*by spreading blue triangle by dragging bottom right vertex (see Fig. 6b)*]. This one would stay in spot

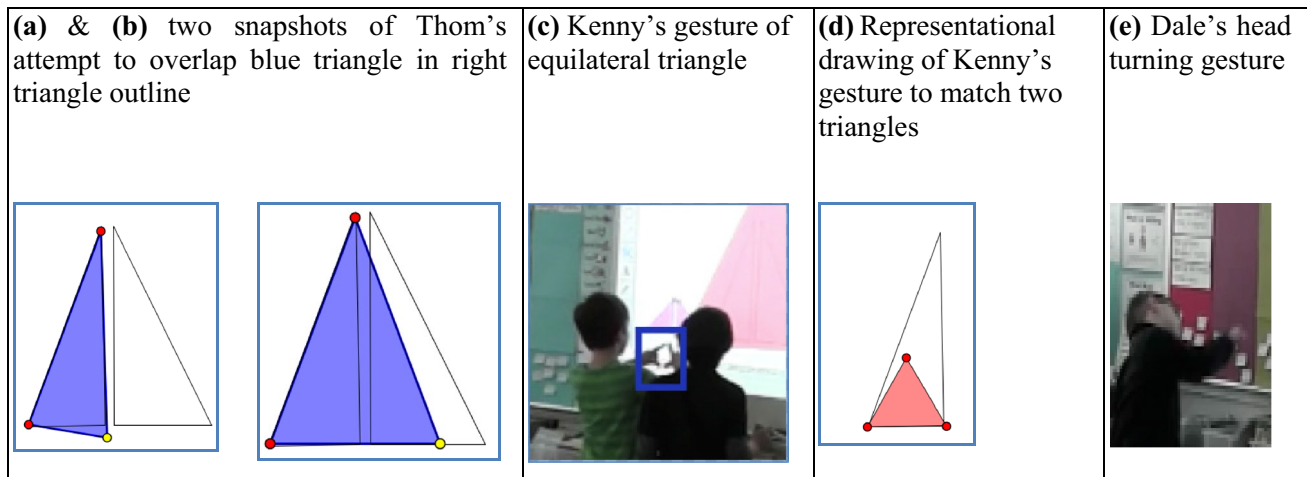


Fig. 6 a–e Snapshots of various instances and gestures referred in excerpt 3

(moving marker at the side made by two red vertices).

Fig. 6e) your head, two sides are the same (pointing at two sides with right hand index finger).

Later, Thom tried to overlap the equilateral (red) triangle onto the same right triangle, but was unsuccessful.

- T Can somebody help Thom out of his misery and explain why he can't use the red one?
- Kenny Because the red one, only this one (*pointing at left bottom vertex*) and this one (*pointing at right bottom vertex*) can move and not the top one. And if you started (*gesturing equilateral triangle with fingers [see blue box in Fig. 6c]*), it's like this down (*placing his gestured equilateral triangle on screen at right triangle by matching horizontal base [see drawing in Fig. 6d]*) like a triangle and then it's not exactly up (*pointing at top vertex*) there. So it won't work.

Troy then tried to overlap the equilateral triangle onto the left-most outline (isosceles right triangle) in Fig. 2c. When asked why the equilateral triangle could not fit, Thom said:

- Thom It's because mostly that won't work because that one of them is seemed to be paralysed or something and doesn't want to move from its seat. But the pink one... the scalene ...can move anywhere it wants and the only one. I think it's the only one that can get inside the shape. I think it's only the green and pink that can make that shape, but the others are just paralysed.
- T Okay, so Dale what do you think?
- Dale I think that the isosceles thing can work, because if you turn (*turning his head sideways as in*

Thom used the phrases 'paralysed to go up', 'only allowed to spread out', 'stay in spot' 'doesn't want to move' for the equilateral and isosceles triangles, whereas for the scalene triangle he used the words 'can move anywhere', 'can get inside the shape'. Clearly, this vocabulary emerged after observing the restricted and/or free movements of the dynamic triangles and prompted him to make connections with real-life experiences of the restrictive mobility of humans. Kenny's statement "only this one and this one can move and not the top one" shows a tool-based dynamic description of the equilateral triangle. Then his attempt to place a gestured equilateral triangle (Fig. 6c) over the right triangle (Fig. 6d), along with his statement "it's not exactly up there" while pointing at the top vertex suggests that he is implicitly talking about the symmetry of the equilateral triangle (and the non-symmetry of the right triangle). Additionally, Thom's 'the pink one', 'the only one', 'only the green and pink' and Kenny's utterance 'the red one' provide evidence for their thinking about particular coloured triangle as one specific thing as they both use the singular form and the definite article 'the' in their utterances.

Thom and Kenny's recognition of the informal, behavioural property of the restrictive movement of the dynamic triangles shows a second order of discourse, whereas the usage of the definite article 'the' seems to indicate they are thinking of one particular triangle, rather than a family of triangles. This shows that the process of discourse change does not occur in a smooth transition; rather, it oscillates between orders of discourse, depending on the task, and resulting in a hybrid form of geometric communication (as shown also in Sinclair & Moss 2012).

Dale's use of the words "isosceles thing can work", "if you turn your head, two sides are the same" along with head turning gesture (Fig. 6e) is also suggestive of use of morphing by him and indicates understanding of non-prototypical isosceles triangles. This kind of dynamic gesture might be evoked in part by the dynamic environment, where he sees the different orientations of the triangles under dragging. Again, in the above excerpt Thom and Kenny seems to use a different discourse than that of Dale, focusing more on properties.

Jory's arm stretching gesture (Fig. 4d), Kenny's equilateral triangle gesture with fingers (Fig. 6c) and Dale's head turning gesture (Fig. 6e) provides evidence that speech only was not sufficient for the children and that gestures played an important role in their communication. Often these gestures seemed to mimic the actions the children had made on the IWB so that the dynamic images were giving rise to new ways for the children to communicate.

5.5 Working with Shape Makers for comparison

Excerpt 4 occurred when the children worked with the comparison sketches (Fig. 3a, b). During the overlapping task of scalene and equilateral triangles, the children promptly recognised that the scalene can fit into an equilateral outline, whereas the equilateral triangle could not fit into the scalene outline.

Excerpt 4 Children's shift towards generalised statements

T How come scalene can make equilateral triangle?
Jory Because scalene...um...they can create any shape of triangles.

Dale tried to fit the equilateral triangle over the scalene outline of sketch (Fig. 3a).

T Jory, what are you thinking?
Jory You can't do it, because if you move one of the sides, then the other side moves with it. And they have to... And all of the sides have to be equal, so you can't do it.

During the exploration of the sketch (Fig. 3b), after the students successfully placed an isosceles triangle into an equilateral outline, but failed to fit an equilateral triangle into an isosceles outline, the teacher asked

T Why can we turn isosceles into equilateral, but we can't turn equilateral into isosceles, Lida?
Lida Isosceles can be turned into equilateral because two sides have to be the same, but that doesn't

mean that all three sides can't be the same. At least two sides should be the same.

Jory's utterances "scalene...um...they can create", "they have to" and Lida's "isosceles can be turned" give evidence that they were referring to the family of scalene, equilateral and isosceles triangles, respectively. This generalised discourse might have been initiated in part due to the use of the terms scalene, isosceles and equilateral. Furthermore, the words 'create' and 'turned' by Jory and Lida are suggestive of the use of morphing by them, which is explicitly shown by the dynamic behaviour of the triangles. Lida's statements "At least two sides should be the same" and "that doesn't mean that all three sides can't be the same" give evidence of her thinking in terms of inclusive relations, which reaffirms the initial inclusivity noted in her drawing (Fig. 5a). Jory's statement "they can create any shape of triangles" for scalene triangles may be suggestive of his description of the behaviour of all scalene triangles as opposed to one particular scalene triangle. Also this description of scalene being able to create any shape of triangles makes inclusion of isosceles and equilateral triangles evident. These statements emerged as a result of the children's attempts to overlap or fit different triangles on each other in the sketch. Thus, Lida and Jory's arguments are showing their shift towards a third-order discourse.

5.6 Drawings of different types of triangle families by children

After the second lesson, the teacher asked the children to draw families of different triangles. Figure 7 shows the examples of drawings of different triangles by three children.

Analysis of the children's drawings reveals that twelve children drew prototypical (with horizontal base) triangles in the case of the equilateral family, four children drew prototypical isosceles family and two children drew prototypical right triangles. Cory's drawing in Fig. 7a presents an example of the non-prototypical drawings. It is interesting to note that most of the children showed their understanding of different non-prototypical triangles in their explanations while working with the dynamic sketches, whereas they used the prototypical triangles in their drawings. This is evident in Dale's isosceles and equilateral triangle drawings (Fig. 7b), even though he recognised the fact that a right triangle could also be isosceles by turning his head sideways (Fig. 6e) in excerpt 3. Similar discrepancy can be seen in Jory's drawings (Fig. 7c).

Also, an interesting contrast is found in children's drawings after lesson 1 and lesson 2. For example, Dale used non-prototypical isosceles triangle in his previous drawing (Fig. 5b), whereas he drew a prototypical isosceles triangle

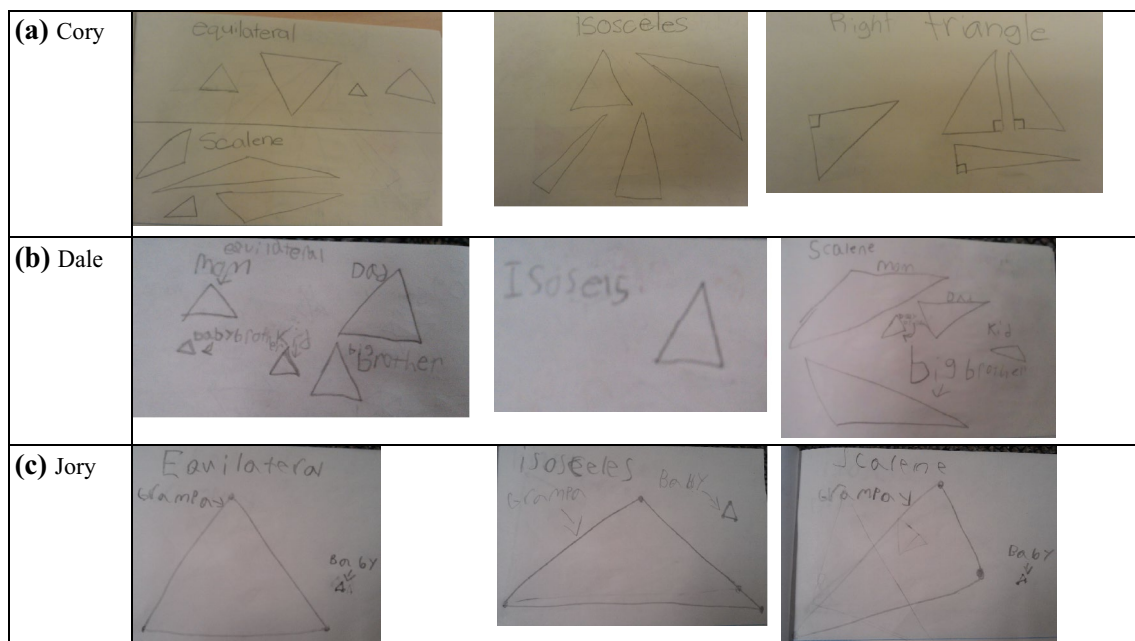


Fig. 7 a–c Drawings of family of *different triangles* by Cory, Dale, and Jory, respectively

in his second drawing (Fig. 7b). Such differences are also found in other children's drawings, where they used prototypical triangles after the second lesson even though they had used non-prototypical ones after the first lesson (for example: Jory's drawings in Figs. 5c, 7c). In fact, the prototypical drawing of an isosceles triangle is prototypical in part because it is a good representative of isosceles triangles. So while we may want children to be able to identify and produce non-prototypical drawings as they are learning about triangles, there may come a point where drawing prototypical ones can be seen as evidence of them using a third order of discourse. In other words, it may be that their drawings at the end of a learning process can actually be similar to their initial ones, which means that drawings alone cannot determine the kind of discourse children have developed.

6 Discussion and conclusions

The above analysis shows that, during the teacher-led explorations and discussions with dynamic sketches, the children's routines moved from tool-based, informal descriptions to more formal properties as well as from particular to more general discourse about types of triangles. Children's reasoning started with describing the informal dynamic aspects of the triangles based on movement patterns like "Everything moves with it except one point" and "If I move that one, that one also moves" and then

eventually shifted to formal geometric properties such as "the angles are staying same", "it moves symmetrical" and "the lines are the same". They used a mixture of first- and second-order discourse to begin with. Dragging the vertices acted as a visual mediator and helped the children to develop the routine of looking at movement behaviour and eventually shifting towards formal geometrical properties. Jory used his arms as sides of isosceles triangle for justifying why isosceles triangle cannot fit into the outline of a triangle whose sides are different, whereas Dale used the head-turning gesture to recognise the equal sides of in right isosceles triangle. Both of these cases illustrate the use of body-based visual mediators by children in their geometric thinking. Indeed, the children used their gestures again and again (see Figs. 4d, 6c, e) during the intervention, which helped them to communicate visual and dynamic information about the objects on the screen.

The use of action verbs "go", "moves", "staying", "paralysed", "getting bigger or smaller" and 'if-then' statements shows the children's propensity to reason in terms of motion when comparing different types of triangles, which was clearly initiated by the dynamic and temporal elements of DGE. Thom's description of the isosceles and equilateral triangles in terms of being 'paralysed' was quite novel and clearly emerged after looking at the restrictive type of movement shown by these triangles, which reaffirms Healy and Sinclair's (2007) claim that the temporality of dynamic mathematical presentations offers striking opportunities for narrative thinking.

The children produced new words (e.g. paralysed triangles), gestures (e.g. Figs. 4d, 6c, e) and diagrams (i.e. triangle designs) in communicating about different types of triangles in the dynamic geometry environment. The role of prototypical triangles in this classroom intervention also suggests that the nature of static media may prompt a particular kind of drawing, which may affect the kind and/or range of triangles that children produce. Of course, while it is not surprising that the task and the medium should affect children's thinking, it is worth investigating in more detail the transitions between static and dynamic media to better understand what each has to offer both in terms of exploration and of assessment. Along the same lines, the initial colour-coding of the triangles may have worked against the idea of inclusive relations since, for example, the blue isosceles triangle did not turn red once it was dragged into an equilateral shape. While the colour was useful as an initial means of communication, it may also have prevented some children from recognising the inclusive relations between the types of triangles.

Finally, the children's communication depicts a shift from specific to more generalised statements as intervention progressed. Jory and Lida's use of generalised statements in excerpt 4 may be due to the repeated use of the terms scalene, equilateral and isosceles by the teacher during the second lesson. This is also in concert with recent work in early childhood psychology research (see, e.g. Fulkerson & Haaf 2006), which has reported about the powerful link between object naming and object categorisation. Naming distinct objects with the same word highlights commonalities among the objects and promotes object categorisation. In our case, the use of words scalene, equilateral and isosceles seems to have promoted the triangle categorisation among the children and helped them to shift to 3rd order discourse during the second lesson. Also, Lida and Jory's use of inclusive descriptions emerged as a result of dynamic actions of dragging during an attempt to superimpose one triangle over another. This reaffirms the results of Sinclair and Moss (2012) by providing evidence of how a dynamic environment can help students to move to different orders of discourse. The children productively used the ideas of symmetry and angles to understand the behaviour of different types of triangles during the intervention. Thus, this study also provides initial evidence that the teaching of concepts like symmetry and angles in early years can offer children new means of identifying and describing shapes (and triangles in particular).

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