

A Learning Trajectory in 6-Year-Olds' Thinking About Generalizing Functional Relationships

Maria Blanton
TERC

Bárbara M. Brizuela
Tufts University

Angela Murphy Gardiner
TERC

Katie Sawrey
Tufts University

Ashley Newman-Owens
Tufts University

The study of functions is a critical route into teaching and learning algebra in the elementary grades, yet important questions remain regarding the nature of young children's understanding of functions. This article reports an empirically developed learning trajectory in first-grade children's (6-year-olds') thinking about generalizing functional relationships. We employed design research and analyzed data qualitatively to characterize the levels of sophistication in children's thinking about functional relationships. Findings suggest that children can learn to think in quite sophisticated and generalized ways about relationships in function data, thus challenging the typical curricular approach in the lower elementary grades in which children consider only variation in a single sequence of values.

Key words: Algebraic reasoning; Early childhood; Teaching experiment

Current initiatives have identified algebra as a central concern in mathematics education in the United States and have recast it as a longitudinal strand of thinking for students in Kindergarten through Grade 12 (e.g., National Council of Teachers of Mathematics [NCTM], 2000, 2006; National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010; U.S. Department of Education, 2008). These and other efforts

The research reported here was supported in part by the National Science Foundation under Grant No. DRL-1154355. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

have not only elevated the focus on algebra in the elementary grades (hereafter, *early algebra*) as a critical path to students' mathematical success (Katz, 2007), but they have also pushed the study of functions in the elementary grades into the discourse on teaching and learning algebra.

Although functions have long been regarded as a powerful concept in mathematics, the study of functions has historically been confined to secondary grades because of the perception that it required a level of formal, abstract thinking that was only attainable for students of sufficient age. However, it has been framed in more recent years as an important unifying strand across the K–12 curriculum (Freudenthal, 1982; Hamley, 1934; Schwartz & Yerushalmy, 1992) and has been argued to be a critical route into the teaching and learning of early algebra (Carraher & Schliemann, 2007).

There is an important connection between functional thinking and the early algebraic thinking practices of generalizing, representing, justifying, and reasoning with mathematical relationships (Blanton, Levi, Crites, & Dougherty, 2011; Kaput, 2008). Specifically, functional thinking entails (a) generalizing relationships between covarying quantities; (b) representing and justifying these relationships in multiple ways using natural language, variable notation, tables, and graphs; and (c) reasoning fluently with these generalized representations in order to understand and predict functional behavior. Although this connection to early algebra ensures the significance of functions in the elementary grades, it also raises important questions regarding the nature of children's understanding of functional relationships and how this understanding emerges.

Research on Children's Functional Thinking

Early algebra research related to students' functional thinking has migrated from the secondary and middle grades into the elementary grades in order to make sense of students' thinking about historically complex ideas in new and younger aged settings. Much of the research in elementary grades, however, has focused on the upper elementary grades. Existing early algebra research provides important evidence about how children, primarily in Grades 3–5, notice, represent, justify, and reason with functional relationships (e.g., Blanton, Stephens, et al., 2015; Brizuela & Earnest, 2008; Carraher, Schliemann, Brizuela, & Earnest, 2006; Cooper & Warren, 2011; Moss, Beatty, Barkin, & Shillolo, 2008; Schliemann, Carraher, & Brizuela, 2007). However, younger children's understanding of concepts associated with functions is still not widely studied. In this article, we distinguish research on young children's¹ understanding of recursive patterns from research on their understanding of functional relationships. Although work on the former is abundant and has provided key insights into how children in the lower elementary grades make sense of numerical and geometric patterns in a single sequence of data, we are interested here in how they understand functional relationships between two quantities.

¹ We use the phrase *young children* to refer to children in Kindergarten through Grade 2.

Recent research suggests that a more systematic study of young children's functional thinking is warranted. Previous studies indicate that children in Kindergarten through Grade 2 can begin to account for relationships between quantities rather than only the simple recursive patterns originally thought to be within their reach (e.g., Blanton & Kaput, 2004; Cooper & Warren, 2011; Moss & Beatty, 2006; Moss & McNab, 2011; Warren & Cooper, 2005). Research findings also suggest that young children can begin to use variable notation to represent relationships between quantities (Brizuela, Blanton, Gardiner, Sawrey, & Newman-Owens, 2015).² Moreover, children's use of different representations to make sense of functional relationships can serve to mediate and support their thinking. For example, children in Kindergarten and Grade 1 can create function tables as a means to organize covarying data (Blanton & Kaput, 2004; Brizuela & Alvarado, 2010), and by Grade 2, they can begin to use this representation transparently as a tool for thinking about these data (Brizuela & Earnest, 2008; Brizuela & Lara-Roth, 2002).

We maintain that young children are not only capable of deeper functional thinking than previously thought but that the origins of these ideas appear at grades earlier than typically expected. The premise of our work is that children's capacity for noticing relationships between quantities, the ways they track and organize function data, and the types of representations they can use to make sense of these functional relationships can be developed from the very earliest grades at the start of formal schooling. What we did not know prior to this study was the extent to which young children might engage in these activities and how their learning might progress. Based on research in the upper elementary grades (e.g., Blanton, Stephens, et al., 2015; Cooper & Warren, 2011; Moss et al., 2008) and in spite of promising studies among younger children (e.g., Brizuela & Alvarado, 2010), our perspective prior to the study was that young children would be likely to recognize and articulate recursive patterns while possibly not generalizing other important relationships at all. Moreover, we presumed that recursive thinking would be a necessary bridge into functional thinking for young children. That is, understanding variation in a single sequence of values would be a prerequisite for children's understanding of relationships between quantities and could facilitate their thinking about these relationships.

From this perspective, our goal in the study reported here was to map conceptual terrains in young children's thinking as they engaged in the particular algebraic thinking practice (Kaput, 2008) of generalizing algebraic relationships with functions. Our research was framed around the following questions: *How do young children generalize functional relationships between two quantities? In particular, what kinds of relationships do children notice, and what are the levels of sophistication in their thinking about these relationships?*

² Studies also suggest that young children are capable of using variable notation to represent and reason with generalizations in domains of early algebra other than functional thinking, such as generalized arithmetic and quantitative reasoning (e.g., Carpenter, Franke, & Levi, 2003; Dougherty, 2008).

In exploring these questions, we did not adopt an a priori conceptual framework for analyzing data, although this is often a useful and reasonable approach. As mentioned earlier, very little is known about how young children generalize functional relationships. Although we know from a disciplinary perspective that relationships in covarying data might be thought of in terms of recursive patterns, covariational relationships, or correspondence rules (Smith, 2003), we did not know how young children would make sense of these mathematical structures or the levels of thinking they might exhibit relative to these. Our concern was that an existing framework developed for older populations might be problematic for the young population considered in this study.³ On the one hand, frameworks can provide a useful lens for data analysis; on the other hand, they can restrict what we notice about the data. As such, we wanted to avoid imposing an a priori point of view as we set out to describe children's generalizing about functional relationships. We wanted an exploratory approach that would allow us to identify nuances in children's thinking that we might not see if we were looking through a specific lens.

Research Design and Methods

Learning Trajectories as a Research Paradigm

In this study reported here, we use learning trajectories as the basis for systematically characterizing progressions in children's thinking about generalizing algebraic relationships in functions. Learning trajectories have become an increasingly important construct in the research on teaching and learning mathematics (Clements & Sarama, 2004, 2009; Maloney, Confrey, & Nguyen, 2014; Simon, 1995). Although there are differing views on what they entail, we take learning trajectories here to include three essential components, as characterized by Clements and Sarama (2004): (a) learning goals, (b) instructional activities or an instructional sequence, and (c) a developmental progression that specifies increasingly sophisticated levels of thinking in which students might engage.

In addition to these components, there are other important guiding principles for developing learning trajectories that informed our work. In particular, developing a trajectory in children's thinking involves using empirical findings to construct content-specific learning goals and an associated instructional sequence (e.g., Baroody, Cibulskis, Lai, & Li, 2004; Gravemeijer, 2004). In our study, we used findings from early algebra research regarding children's functional thinking (primarily from the upper elementary grades because the majority of this research has occurred in that context) to construct an initial instructional sequence. The successively more sophisticated ways of thinking that children exhibited as they advanced through this instructional sequence became the basis for a progression that we propose here regarding children's thinking about

³ Indeed, as we note in our discussion section, our informal assumptions going into the study about children's capacity for functional thinking—assumptions based on our own research in the upper elementary grades—did not play out as we expected with younger children.

generalizing relationships in functions (cf. Baroody et al., 2004; Clements & Sarama, 2014; Duschl, Schweingruber, & Shouse, 2007).

The design of an instructional sequence and the associated tasks is tightly coupled with hypotheses about students' learning within a task (Barrett & Battista, 2014; Simon & Tzur, 2004), and the progression that results represents only one possible path of learning (Baroody et al., 2004; Stevens, Shin, & Krajcik, 2009). Moreover, the levels of thinking constituting the developmental progression of a learning trajectory are not intended to be interpreted as stages through which one must progress in a linear sequence but as states of mind with fluid boundaries through which students move bidirectionally as their learning progresses (Clements & Sarama, 2014). Students may skip levels altogether or may revert to previous levels when faced with changes in tasks or even benign alterations in their instructional environment, such as relocating the child's physical placement in the room (Clements & Sarama, 2014).

Participants

The study reported here is part of a larger project in which we explored Kindergarten through Grade 2 children's thinking about functional relationships. To provide the full context, we discuss participants and data collection for the larger project. However, as described further below, we focused in this study on children in Grade 1 (approximately 6 years old).

Children in six classrooms from two elementary schools in the northeastern part of the United States participated in the study. In each of the two elementary schools, one classroom in each of Kindergarten through Grade 2 participated. School A consisted of an 8% nonwhite population, with about 20% of the district's school population categorized as having low socioeconomic status (SES) and 5% as English as a second language (ESL) students. School B consisted of a 98.6% nonwhite population, with 89.5% of the school's population categorized as low SES and 33.9% as ESL. A total of 115 children participated in the full study (21, 18, and 15 in Kindergarten, Grade 1, and Grade 2, respectively, at School A; 18, 22, and 21 in Kindergarten, Grade 1, and Grade 2, respectively, at School B).

After data collection, we narrowed our focus in the data corpus to first-grade children. There were pragmatic reasons for this choice. The large volume of data necessitated some form of data reduction (Chi, 1997). In limiting our data set, we wanted to focus on children who had had little formal schooling (thus, eliminating second grade) but who could also articulate their thinking in ways that we thought would be relatively clear (thus, eliminating Kindergarten). Our interest was in the boundary points of children's thinking as their notions about functions might emerge. From our classroom teaching experiment, we knew that first-grade children had solved tasks in ways that were similar to second-grade children, yet first-grade children's ideas were more emergent and thus, might potentially provide us with more detail about the genesis of children's thinking. At the same time, although we also observed that Kindergarten children solved tasks in ways that were similar to first-grade children, it was more difficult for them to clearly

articulate their thinking than it was for first-grade children. This sometimes constrained our ability to understand the nature of their thinking.

Also, because of the differences in school experiences between Kindergarten and second-grade children, we wanted a data set that reflected relatively homogeneous school mathematics experiences for children with regard to grade level. As a result, we did not include cross-grade data. We emphasize here that we are not using *homogeneity* to refer to the nature of children's mathematical thinking. Indeed, our research sites were chosen to reflect diversity in mathematical thinking. We are referring instead to factors such as the number of years of formal schooling participants had experienced and the similarity in the concepts they had been taught through formal instruction as defined by state frameworks. Focusing on one grade level helped ensure that children had had similar experiences in terms of both their time in formal schooling and in the curricular benchmarks that guided their regular classroom instruction.

Data Collection

Following a teaching experiment methodology (Steffe & Thompson, 2000), we engaged children in an 8-week set of classroom teaching sessions in conjunction with individual interviews (cf. Barrett & Battista, 2014). In particular, we conducted a classroom teaching experiment (CTE) in two classrooms at each grade level (i.e., Kindergarten, Grade 1, and Grade 2). CTEs are a common methodology in learning trajectory research because their incorporation of instructional design and planning and both informal analysis of ongoing classroom events and retrospective formal analysis of data sources generated through the experiment support the refinement of authentic (i.e., classroom-based) conjectures about progressions in children's thinking (Clements, Barrett, & Sarama, in press; Lesh & Lehrer, 2000). The instructional sequence that formed the basis of the CTEs was structured as a set of two 4-week, classroom-based instructional cycles (referred to here as Cycle 1 and Cycle 2). Each instructional cycle consisted of two 40-minute lessons per week, for a total of eight lessons (16 lessons for both cycles). The length and frequency of the cycles were intended to give children enough consistent interaction with the concepts addressed in order to gain some traction in their thinking.

At each school site, one member of our project team taught the lessons while other members of the project team observed and videotaped. We originally designed individual lessons to increase in complexity so that we could stop a lesson at a given point for a particular grade if needed. For example, prior to starting the CTEs, we assumed that children in Kindergarten might only go as far as constructing function tables and noticing recursive patterns. The task design allowed for a lesson to end at appropriate points—such as after recursive patterns were identified in a function table—prior to completion of the full lesson. However, we found that we were able to teach each lesson in its entirety at each of the grade levels. Local teams at each of the two school sites met weekly to discuss preliminary observations from classroom lessons, including concepts that

seemed particularly difficult or easy for children, as well as insights into how children were talking about or representing functional relationships. The full project team also met weekly to discuss similarities and differences in these observations across research sites and to revise subsequent lessons to account for this. This informal analysis of classroom data helped us to later focus our more fine-grained, formal analysis of interview data.

The CTEs included semiclinical, individual, pre-, mid-, and post-CTE interviews intended to capture details about shifts in individual children's functional thinking over the course of the CTEs that we might not have obtained from classroom settings. Pre-CTE interviews were conducted prior to the start of the two instructional cycles, mid-CTE interviews were conducted at the conclusion of Cycle 1, and post-CTE interviews were conducted at the conclusion of Cycle 2.

To select children to participate in the interviews, we used a combination of approaches. First, teachers identified children in approximately the top third of the class, as indicated by either their numeracy and verbal skills or other classroom factors such as participation. We caution that the notion of "top third" is relative. The schools were chosen because they were demographically and academically different from each other, and one of the schools was considered low performing based on standardized assessment scores. The research team also identified children who were observed to be vocal and articulate about their thinking. Using these two groups of children, we then selected interview participants to ensure that they would be able to productively engage with the interview tasks we had designed. That is, because the function concepts on which we focus in this study are generally not addressed at Kindergarten through Grade 2 and because there is little research on whether children—particularly in Kindergarten and Grade 1—might be able to engage with the concepts to be explored, we wanted to interview children who we hypothesized had the potential to complete some portion of the tasks and who could verbalize their understanding. In all, a total of 10 children from each grade level, Kindergarten through Grade 2 (30 children total), were selected to participate in each of the interviews.

Each interview lasted about 30 minutes and was videotaped. It was conducted by one member of the project team and, when possible, observed by one other member of the team in order to provide another real-time perspective on the child's thinking as she or he solved the task (cf. Steffe & Thompson, 2000). In some cases, the interviewer was also the instructor for the class. Interviews consisted of the child solving tasks similar to those used in the CTEs and describing his or her thinking aloud. A core set of questions, generated by the research team, was used in each of the interviews. The interview protocols (see Appendix) were intended to be a guide for questioning. As children solved interview tasks, the ideas that they generated did not necessarily follow the chronology in the line of questioning established in the protocols. As such, our priority was to follow children's thinking while ultimately addressing the questions in our protocols within the body of the interviews.

Instructional Sequence Used for the Classroom Teaching Experiment

Designing the sequence. We drew on existing empirical research regarding children's functional thinking to design the instructional sequence that formed the basis of our CTEs. Because much of this research has focused on the upper elementary grades, our hypotheses regarding learning goals and task design were informed by those research findings. In our view, this was a reasonable starting point. As Clements and Sarama (2014) noted, tasks associated with an instructional sequence can promote learning by being "just beyond the students' present level of operating, so they must actively engage in reformulating the problem or their solution strategies" (p. 14). Moreover, because our research goals entailed understanding how young children might think about concepts on tasks similar to those used in the upper elementary grades, we hoped our use of such tasks would help us to identify boundary points in younger children's thinking that could contribute to existing research by providing a more complete picture of the development of children's functional thinking across Kindergarten through Grade 5.

Further, because we had found anecdotally in our previous work that children in Grades 3–5 (particularly, Grade 3) seemed to have more difficulty with functions of the type $y = x + b$ than those of the type $y = mx$, we hypothesized that $y = mx$ type functions would be a more appropriate starting point. As a result, we designed the instructional sequence to focus initially on functions of the type $y = mx$, shift to functions of the type $y = x + b$, and end with a brief look at functions of the type $y = mx + b$. We note, however, that because participants had not formally studied multiplication, we expected that they would represent $y = mx$ functions additively (e.g., $y = 2x$ as $y = x + x$ or an equivalent verbal description), if at all.

Additionally, we incremented the complexity of the functions used in the tasks based on how numeracy is typically taught in the early grades. For example, children typically begin with counting by ones, and then they explore counting by twos and doubling. Also, teachers with whom we consulted regarding our task designs noted that arithmetic operations on odd numbers greater than one might be more complex than those on even numbers. We varied our choice of values for parameters m and b accordingly: m was assigned the values (in order across successive tasks with functions of the form $y = mx$) of 1, 2, and 4; b was assigned the values (in order across successive tasks with functions of the form $y = x + b$) of 1, 2, and 3. Regardless of the function type, each task was designed to include the following features:

- Generate a set of covarying data, given a problem scenario describing how two quantities covary;
- Organize the data in a table and explore the meaning of the values in the table;
- Explore any types of relationships noticed in the data depicted in the table;
- Describe what happens to the dependent quantity as the independent quantity increases by one unit;
- Predict near and far function values;

- Generalize a relationship between the two quantities and represent this relationship in words and variable notation; and
- Describe the meaning of the variable notation and the relationship represented with variable notation in terms of the problem context.

After we drafted our initial set of tasks, we piloted a subset of the tasks by teaching lessons in Kindergarten and Grade 1 classrooms and conducting individual interviews. Children participating in our pilot work were not participants in the study reported here. Our pilot work provided us with a sense of how children in these earlier grades made sense of the task components and helped us gauge the appropriateness of surface features of our tasks, such as our word choice and the types of numbers we used. It also helped us design instructional features in a way that we hypothesized might better sustain children's engagement with a task (e.g., how to intersperse group work with whole-class discussions, whether and how to use manipulatives or tactile objects as part of the lesson or interview design). As tasks were piloted through lessons and interviews, the project team met weekly to discuss initial observations and make final revisions for the instructional sequence.

Implementing the sequence. Figure 1 presents a summary of the sequence in which the function tasks used in the study were ordered as well as the particular relationship and type of function examined in each task. The first two lessons of Cycle 1 focused on the types of repeating and growing patterns that are typically addressed in Kindergarten through Grade 2. We hypothesized that starting with familiar content—variation in a single quantity—would help bridge children's thinking about relationships between quantities. Because the independent variable is typically implicit in such pattern tasks, we also used these lessons to introduce children to identifying and tracking the two related quantities in a task as a precursor to the function tasks that constituted the remainder of the instructional sequence.

Instruction used in the CTEs involved discussion and small-group work. At the start of each lesson, the teacher—a member of our research team—introduced a problem to children in a whole-group setting. Children then worked in small groups to discuss the problem, collect and organize their data, look for relationships, and represent the relationships through words or variable notation. As they worked in small groups, the teacher circulated among the groups to listen to discussion and to question children in ways that would lead to more productive talk (e.g., “How are you organizing your data? Do you see any relationships in your table? Tell me about the relationship between the number of desks and number of people who can be seated—what do you notice?”). After children had sufficient time to explore the problem in small groups, each lesson culminated in a whole-group discussion. Children often led the whole-group discussion with the teacher clarifying their ideas or asking questions to facilitate discussion.

One important goal of instruction was developing the use of function tables as a representation for organizing the function data. In the beginning of Cycle 1

Task Sequence and Relationship Explored	Function Type Addressed
Pre-CTE Interview	
Noses on Dogs: The relationship between the number of dogs and the number of noses on the dogs.	$y = x$
Cycle 1 (4 weeks; 8 lessons)	
Repeating and Growing Patterns	N/A
Pennies in a Jar: The relationship between the number of days and the number of pennies in a jar, if Sara gets one penny each day from her grandmother.	$y = x$
Brady's Birthday Party: The relationship between the number of square desks and the number of people that can be seated at the desks if the desks are joined end to end, no one sits on the ends, and one person can sit on each of two sides of a desk (Blanton, Stephens, et al., 2015).	$y = x + x$
People and Ears: The relationship between the number of people and the total number of ears on the people (assuming each person has two ears).	$y = x + x$
Dogs and Legs: The relationship between the number of dogs and the total number of legs on the dogs (assuming each dog has four legs).	$y = x + x + x + x$
Mid-CTE Interview	
Height With a Hat: The relationship between a person's height without wearing a hat and with wearing a 1-foot hat (Carraher et al., 2006).	$y = x + 1$
Cycle 2 (4 weeks; 8 lessons)	
Cutting String: The relationship between the number of cuts of a straight piece of string and the resulting number of pieces of string.	$y = x + 1$
Candy Boxes: The relationship between the number of candies John and Mary each have, if John and Mary each have a box of candies with the same number of candies and, in addition, Mary has one more candy on top of her box (Carraher, Schliemann, & Schwartz, 2008).	$y = x + 1$

Finding Pennies: If Sarah has a jar of pennies then finds 3 more pennies, the relationship between the number of pennies Sarah had before she found the pennies and after she found the pennies.	$y = x + 3$
Age Difference: If Janice is 2 years younger than Keisha, the relationship between Keisha’s age and Janice’s age (Carraher et al., 2006).	$y = x + 2$
Height With a Hat: The relationship between a person’s height without wearing a hat, wearing a 1-foot hat, and wearing a 2-foot hat (Carraher et al., 2006).	$y = x + 1$ $y = x + 2$
Growing Caterpillar: The relationship between the length of a caterpillar (number of body parts) and the number of days length is measured, if the caterpillar grows by adding two body parts each day and the head is not counted (Blanton, 2008).	$y = x + x$
Post-CTE Interview	
Growing Train: The relationship between the number of stops a train makes and the number of train cars it has if a train stops at stations along its route and picks up two train cars at each station. Assume the engine is not counted. (Additionally, consider how the relationship changes if the engine is counted.)	$y = x + x$ $y = x + x + 1$

Figure 1. Task sequence and the relationships and function types addressed in each lesson.

during whole-group discussion, the teacher often created a table on the board using data that children had generated from the problem. Children had many opportunities to develop and reason with tables during whole-group and small-group work. As a result, the teacher’s introduction of this new representation in Cycle 1 led to children’s spontaneous use of tables as their main organizational tool by the end of the CTE (Brizuela, Blanton, Gardiner, et al., 2015).

Another goal of instruction was to support children’s awareness of relationships between quantities. The teacher consistently stressed that many patterns and relationships could be found in a function table but that finding the relationship between the two quantities required children to look across the table instead of up and down a particular column. The teacher often highlighted a relationship by underlining a row in the table or color-coding data to help children visualize the two quantities they were comparing. One particularly useful strategy was writing an equation to the right of the table for each corresponding pair of values, illustrating a relationship between the values in the table. This seemed to help highlight the functional relationship underlying the values in the table for the children.

Data Analysis

As described earlier, the analysis reported here focused on identifying progressively more sophisticated levels of thinking in children's activity of generalizing algebraic relationships in functions, levels that would then constitute the developmental progression in a learning trajectory. Our methods involved a grounded theory approach (Strauss & Corbin, 1990) in which the sampling and analysis of data occurred in conjunction with the development of our progression, all in an iterative process that continued until new data did not change our emerging model of the levels of sophistication in children's thinking. The intent of our analysis was to look for qualitative profiles across our data that would represent levels of thinking characteristic of young children as they progressed through our specific instructional sequence.

We focused our formal analysis on first-grade children's interview data rather than videotapes of classroom instruction in the CTE because these data enabled us to more thoroughly examine a child's thinking throughout the entirety of a task. As noted above, 10 first-grade participants (four from School A and six from School B) were interviewed. Their interviews are the focus of the study reported here.

Formal analysis. The formal analysis began with an analysis of interview data for two first-grade participants. This allowed us to establish a "working" progression through close negotiation of the levels of sophistication in children's thinking comprising this emerging model. In this first phase of analysis, one member of the project team used open coding (Strauss & Corbin, 1990) to analyze video and transcript data from a first-grade child's pre-, mid-, and post-CTE interviews and to identify initial characteristics of this child's thinking related to the core dimension of generalizing. Data were examined to identify ways in which the child talked about relationships in function values—whether within a single sequence of values or between covarying sets of values. Transcripts were analyzed line by line, and theoretical memoing (Glaser, 1998) was used both to characterize incidents in which the child articulated any form of relationship and to refine characteristics of these incidents through a constant comparison of qualitatively different types of thinking and the relationships among them. The memos consisted of both detailed descriptions of the types of thinking that occurred with regard to a particular aspect of generalizing (e.g., the child's description of a recursive pattern) and excerpts of transcripts as supporting documentation of the descriptions.

Video and transcript interview data from a second first-grade child were then similarly analyzed to refine our characterizations of the types of thinking that occurred from the analysis of data for the initial first-grade child. Memos were revised to account for any discrepancies with or refinement of the original descriptions of incidents in thinking and were color-coded, based on whether they arose from the pre-, mid-, or post-CTE interviews, in order to visually capture any trends regarding where and when particular types of generalizing occurred.

The memos were then sorted (Glaser, 2002) according to qualitatively similar types of thinking, and each of the memo groups was given a preliminary

descriptive code that reflected the thinking within the memo group. For example, the code “emergent generalization—sees valid general relationship but cannot describe how one quantity relates mathematically to another” was a preliminary code reflecting children’s activity of generalizing in which they could describe a general relationship but not quantify the transformation in that relationship. The codes were then organized according to their approximate emergence⁴ in children’s thinking across the interviews as well as their level of mathematical sophistication (Battista, 2004). From this, a preliminary progression capturing the levels of sophistication in children’s generalizing was hypothesized.

Subsequently, the remainder of the project team independently and then collectively examined this initial progression. Through this collective analysis of the transcript data from two of the first-grade children, memos and descriptive codes capturing the levels of sophistication in children’s thinking were negotiated and further refined. A second member of the project team then independently reviewed the modified progression looking for confirming or disconfirming evidence of its levels in a broader subset of the interview data that included other participants. After this review was complete, the full project team again collectively reviewed the revised progression.

Our next step was to use the levels within this emerging progression to systematically code the full set of first-grade interview data. The levels-as-coding scheme was not treated as a static framework but as a flexible model that we expected to be revised (i.e., codes might be refined, collapsed, or discarded) as interviews with different children were analyzed. To facilitate this, the software Inqscribe© was used to transcribe interview data. Transcript data were parsed by using the speaker’s turn in conversation as the unit of analysis and imported to FileMakerPro v.11© for housekeeping purposes, which allowed searchability along profile data such as interview type and student.

Conversational episodes (i.e., a speaker’s turn or a sequence of speaker turns) that involved any activity of generalizing were flagged as such and were subsequently coded using the levels identified in the progression. Two members of the project team independently coded these transcribed episodes from the first-grade interviews. Interrater reliability between the two coders was determined by comparing coding decisions. In instances in which coding was not aligned, coding assignments were negotiated in order to resolve the discrepancy or to refine the coding scheme. Close monitoring of the video data and children’s work from the interviews was used to resolve the discrepancy. If a discrepancy in coding could not be resolved by the two coders, the full project team reviewed the analysis to reach a resolution and, if appropriate, refine the progression. The constant comparison and refinement of descriptive codes (i.e., levels of the progression) and memos characterizing incidents in students’ thinking continued until no new codes emerged in the data.

⁴ We organized codes based approximately on how they emerged across interviews, but this was not an exact emergence (for example, we have noted that children can move bidirectionally between levels).

Findings

Levels of Sophistication in Children's Thinking About Generalizing

In this section, we describe the key result of the study reported here—the levels of sophistication in 6-year-olds' thinking about generalizing algebraic relationships in function data. Our sequencing of the progression was based primarily on our analyses of children's thinking while keeping in mind the underlying logic of the mathematics to be learned. As Battista (2004) noted, levels of sophistication in a progression are based on empirical studies of children's thinking about a topic as well as a more canonical understanding of the mathematics being addressed. Indeed, the bidirectionality we observed in an individual's movement across levels made it problematic to use children's thinking alone as the means for ordering the levels of the progression. Thus, in our sequencing of the levels, we also considered what types of thinking might be viewed mathematically as more sophisticated as well as whether a type of thinking—such as noticing recursive patterns—appeared more organically, without strong instructional intent or design.

We reiterate two earlier points to bear in mind regarding learning trajectories. First, the levels of sophistication in children's thinking are tightly connected to the instructional sequence that gives rise to them and, as such, are not intended to represent the only possible progression in thinking about a particular concept (e.g., Barrett & Battista, 2014). Second, the levels in a progression are not intended to be viewed as “stages” through which children must progress in a linear sequence but as a characterization of the sophistication exhibited in their thinking (e.g., Clements & Sarama, 2014). A consequence of the latter point is that children might skip levels altogether. For example, we are not suggesting here that children needed to go through each of the levels relative to recursive thinking before they exhibited functional thinking. Moreover, in any one interview analyzed in this study, we sometimes found evidence of a child exhibiting multiple levels within the progression.

Although it is difficult to separate children's activity of generalizing from the representations (e.g., variable notation) that they used to characterize and reason with generalized relationships, we emphasize that our focus here is on the nature of the relationship children noticed and not the representation itself. Our analysis of the levels of sophistication in children's representations of functional relationships, particularly variable notation, is discussed elsewhere (e.g., Blanton, Brizuela, Sawrey, Gardiner, & Newman-Owens, 2015).

As noted earlier, within each interview task, children were asked to generate data for a given problem scenario and organize their information. If needed, they were prompted by the interviewer to construct a function table. (At the pre-CTE interview, the interviewer introduced function tables.) The function table then became a context for investigating children's thinking about relationships in the data, and from our analysis of these data, we hypothesized the levels in children's thinking. Here, we frame the levels in children's thinking around their observable actions—in this case, their inscriptions and words—as they solved function-based tasks as well as our subsequent inferences about their thinking regarding relationships in functions. Both of these are important, but we are mindful that our

inferences about children's thinking are necessarily constrained by what we observed in their actions. Finally, for clarity in illustrating the levels, we present data from children who more succinctly described their thinking. This should not be taken to mean they were necessarily more capable than other first-graders interviewed but instead that they could convey their thinking in ways that best help us illustrate the levels reported here. Moreover, although we use data from a variety of children to present the levels, we often use one student—Rebecca—as a way to illustrate a progression in one child's thinking.⁵

Level 1: Prestructural. Children who exhibited prestructural level thinking did not describe or even implicitly use any kind of mathematical relationship in talking about problem data. In particular, they did not describe any recursive pattern in a sequence of single data values or any relationship between two covarying quantities. Moreover, they did not implicitly evoke these relationships—even in the absence of explicitly characterizing them—to find successive data values or to construct corresponding pairs of values. We infer from this that children did not recognize that mathematical quantities could be related or notice that they were related by an underlying quantitative structure, nor did they understand how to articulate this structure. The following excerpt, which occurred during the mid-CTE interview Height With a Hat task, illustrates this type of thinking. In this excerpt, the interviewer had just asked Maggie to describe how much taller a person would be when wearing a hat 1 foot in height compared with that person's actual height.

- Maggie:* Mmmmm, um... I have no idea [shakes head side to side].
Interviewer: Do you remember how tall we said the hat was?
Maggie: One fff...oot?
Interviewer: One foot. Okay. So if we put the hat on somebody, how—
Maggie: It'll be 1 foot.
Interviewer: —much taller is it? Oh, okay. So, you would get 1 foot taller?
Maggie: Hmm [shrugs, smiles].
Interviewer: You would get [putting hat on Maggie's head] 1 foot taller?
Maggie: [Laughs.]
Interviewer: And how about me? How much taller would I get?
Maggie: ...[tilts head to side, hesitates] Six.
Interviewer: Six?
Maggie: [Nods.]

Although Maggie knew that the height of the hat was 1 foot, she did not describe any general relationship involving the two quantities—that is, the person's height with the hat or without the hat. Moreover, she did not implicitly evoke any relationship to reason about specific values. When asked by the interviewer how much

⁵ All student names are pseudonyms.

taller she (the interviewer) would be with the hat, Maggie seemed to randomly suggest “six.” There is no evidence that she used any observed structure or relationship—such as “the person’s height with a hat is 1 foot more than her height without the hat”—to produce this value. If she knew or even assumed that the interviewer’s height was 5 feet, we would expect evidence that she obtained the value of 6 by adding 5 and 1. However, not only did Maggie not describe how she obtained 6 feet, but she also did not convey the recognition that one would need to know the interviewer’s height without the hat in order to produce an actual value for her height with the hat. Even when the interviewer used Maggie’s language of “It’ll be 1 foot” to suggest that a person’s height with the hat would make them “1 foot taller,” Maggie shrugged and smiled but did not connect her statement “It’ll be 1 foot” to “1 foot *taller*” or use it to justify the interviewer’s height with the hat as 6 feet.

A secondary characteristic of this prestructural level included that children might describe an observed regularity in nonmathematical features of the objects compared but not a mathematical relationship. For example, Rebecca was asked if she saw any relationships in her function table for the Height With a Hat task at a point early in the interview when her table contained only numerical values (see Figure 2 for her complete written work from the interview). She noticed that the values along the diagonal in the table were the same: “Four-four, five-five, six-six.... And if we could do another one it would be seven-seven, then eight-eight [pointing at values in the table along the diagonal].” Rebecca seemed to be describing a regularity that would be true if the objects in the table had been any symbol set, not just the numerical values used here. That is, the regularity she noticed seemed to be based on the forms of the inscriptions themselves, not on quantities and their relationships. The symbols could have been nonnumerical, but the regularity she observed was in the physical sameness of the symbols.

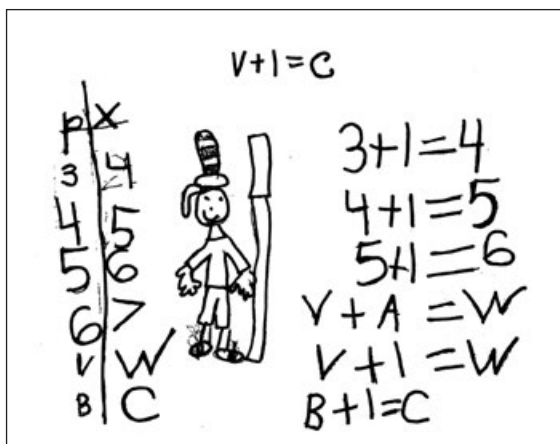


Figure 2. Rebecca's written work on the mid-CTE interview task, Height With a Hat.

Level 2: Recursive-Particular. Children who exhibited thinking at the *recursive-particular* level conceptualized a recursive pattern as a sequence of particular instances. In perhaps the same way a child might not yet view “5” as a composed object and instead see it as a partitioned set of 5 units (ones), we maintain that at this level, children had not yet composed the underlying recursive pattern as a generalization over a class of instances. In other words, children did not yet understand—or “see”—the *sameness* in each instance in a way that allowed them to articulate the pattern as a generality in a unitary form (Kaput, Blanton, & Moreno, 2008).

This was exhibited when a child noticed a recursive pattern in a sequence of values but described the pattern only in terms of a sequence of distinct, particular instances. For example, during the Noses on Dogs task in Rebecca’s pre-CTE interview, she was asked what value should go in the function table below the value “3” representing 3 dogs (see Figure 3 for a depiction of Rebecca’s written work).

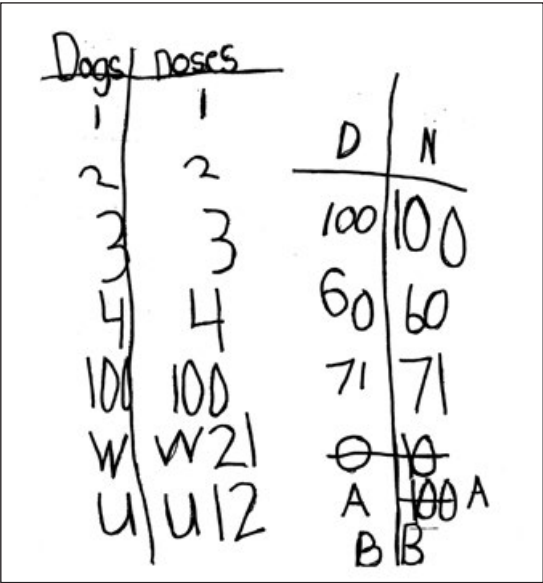


Figure 3. Rebecca’s written work on the pre-CTE interview task, Noses on Dogs.

Rebecca: Four.
Interviewer: How do you know?
Rebecca: Because it’s counting one-two-three-four [she moves her finger down the first column, pointing to the numbers 1, 2, 3, and 4 as she says them].

The usual nomenclature in the early elementary grades suggests that Rebecca's use of the term *counting* implied "counting by one." Moreover, her response of "four" as the number that would come after the value of 3 in the table suggests that she had a strategy for finding the next value in the sequence. Her words indicated the recognition of a recursive pattern that increases successive values by one. What is distinctive, however, is that she conceptualized this pattern as a counting process ("one-two-three-four") not as a general relationship (for example, "it's one more every time").

Liz also exhibited this type of thinking on the post-CTE interview Growing Train task (see Figure 4 for Liz's written work). After Liz had constructed a function table for this task, the interviewer asked Liz if she noticed any relationships.

S	C
1	2
2	4
3	6
4	8
B	C

$1+1=2$
 $2+2=4$
 $3+3=6$
 $4+4=8$
 $B+C=D$
 $B+B+1=0$

Figure 4. Liz's written work on the post-CTE interview task, Growing Train.

Liz: One, two, three, four [tapping the numbers going down the left column representing the number of train stops] and two, four, six, eight [tapping the numbers going down the right column representing the number of cars].

Interviewer: Good! How about a relationship between the number of stops and the number of train cars?

Liz: [Silence, then shakes her head to indicate no.]

Although Liz initially recognized a recursive pattern in each column,⁶ she framed

⁶ We infer from Liz's response to the question of whether she noticed any patterns that what she described was, to her, a pattern. That she framed the pattern she noticed in a rhythmic sequence (for example, "1-2-3-4," which would be consistent with counting by ones in first grade suggests a recursive pattern.

it as a particular sequence of values (e.g., “two, four, six, eight”) not as a generalized process (e.g., “add two every time”)—in effect, tapping the pattern out much like a child might represent “5” as a partitioned sequence of 5 taps rather than as a composed unit.

The context in which this level of thinking occurred varied for Rebecca and Liz: Rebecca was asked specifically to explain how she determined a value in the table, whereas Liz was asked to identify any relationships she noticed. However, their characterization of structure in the data exhibited what we describe as recursive-particular thinking. For us, this is reminiscent of Sfard’s (1991) account of how a child with an *operational conception* of number would respond to the question “How many objects are there?” by repeating a procedure of counting rather than stating the total number of objects (see also Piaget, 1952). In a similar way, we might frame recursive-particular thinking cognitively as an operational conception in that children’s understanding of the relationship was still represented as a counting process (e.g., Liz’s “two-four-six-eight”). In this, children seemed engaged in a mental activity of interiorization (Meel, 2003; Piaget, 1970) in which they were operating on objects but had not yet “condensed” these actions as an integrated whole. Using Simon, Tzur, Heinz, and Kinzel’s (2004) elaboration of Piaget’s reflective abstraction, we might characterize children whose understanding was at the recursive-particular level as being at the first stage of projection in which each attempt to reach a goal was “preserved as a mental record of experience (von Glasersfeld, 1995)” (p. 319) and the learner simply sorted the records based on the results but did not yet identify patterns based on the comparison of similar experiences.

Level 3: Recursive-General. Children who exhibited thinking at the *recursive-general* level conceptualized a recursive pattern as a generalized rule between arbitrary successive values without reference to particular instances. Although children might at some point have used numerical cases to explain their thinking, their primary description of the pattern reflected a general rule.

During the post-CTE interview Growing Train task, the interviewer asked Raymond what value he would record in his function table if the train made four stops.

Raymond: Well, here’s what I would do. I would put 4 stops, 8 cars [writes 4 and 8 in his function table].

Interviewer: Okay, great, and how’d you know to do that?

Raymond: Because—because you count by twos, ’cause you add two every time.

In order to get the total number of cars for the given number of stops, Raymond went on to say: “At every stop, add two cars to the number of cars you have now.” We note that, although Raymond’s covariational description (Confrey & Smith, 1991) of the two quantities he was comparing is significant, he was not attending

to their correspondence relationship. Instead, he used a recursive pattern to determine the total number of stops (“you add two every time”; “add two cars to the number of cars you have now”). A significant difference between the thinking that Raymond exhibited and the thinking characterized at the recursive-particular level is that he was able to express the recursive pattern in a generalized, unitary form independent of particular values and use this generalized rule to find specific function values. That is, he conceptualized the relationship as “you add two every time” not as, for example, “two, four, six, eight.” Using a process-to-object lens (Sfard, 1991), the distinction is that the relationship had been condensed although not yet reified in Raymond’s thinking; that is, the child could now “view that process as a whole” (Zandieh, 2000, p. 109).

Level 4: Functional-Particular. Children who exhibited thinking at the *functional-particular* level conceptualized a functional relationship as a set of particular relationships between specific corresponding values. A critical marker of this level was that—similar to the recursive-particular level—children could describe a relationship within specific cases but not as a generalized functional relationship over a class of instances.

During the mid-CTE interview Height With a Hat task, Zara was given the specific heights of different people and asked about their heights when wearing the hat. After Zara provided the correct height with a hat in each of these specific cases, the following exchange occurred.

Interviewer: Okay. . . . Could you tell me...what you did every time? When I gave you the height, what did you do every time?

[After a pause, the interviewer continued.]

Interviewer: So how’d you get from three to four, and five to six, and six to seven, and eight to nine [pointing to respective pairs of corresponding values in the table (see Figure 5)]?

Zara: Eight plus one will equal nine.

Interviewer: Okay [writing down “ $8 + 1 = 9$ ” to the right of right table, next to the (8, 9) pair of values].

Zara: And six plus one will equal to seven. And three plus four will equal four.

Interviewer: Oops. Three plus four equals four?

Zara: Three plus one more.

Zara was clearly able to calculate the height of a person wearing the hat when the height of the person was known. That is, he was able to determine one’s height wearing a hat within a specific case for a set of cases. However, he could not articulate how two generalized quantities—an arbitrary person’s height without wearing a hat and height wearing a hat—were related. Instead, he represented the relationship as a set of particular instances (e.g., “Eight plus one will equal nine. . . . And six plus one will equal to seven. And three plus four [*sic*] will equal four.”), or what might

Height of person		Number	
of fact			
3	14	4	$3+1=4$
5	12	6	$5+1=6$
6	19	7	$6+1=7$
8	26	9	$8+1=9$
y		x	
4	5		$4+1=5$

Figure 5. Zara’s written work on the mid-CTE interview task, Height With a Hat. The middle column of numbers (14, 12, 19, and 26) are transcriptions from an earlier portion of the interview in which Zara used drawings to determine a person’s height with the hat. When he and the interviewer started working through the functional method for calculating height with the hat, Zara did not want to erase the middle column, and those numbers were left in the table.

be described as a “quasi-generalization” (Cooper & Warren, 2011). In this sense, we view Zara’s thinking about a functional relationship as particular in nature because he did not generalize this relationship over a class of instances in a unitary form.

In a manner similar to recursive-particular thinking (Level 2), children’s understanding at this level might be framed cognitively as a mental action of interiorization because children operated on familiar objects in an initial step towards condensation, and ultimately reification, of the relationship (Sfard, 1991). For example, Zara operated on specific covarying quantities, adding the height of the hat (1 foot) to specific heights of particular people and constructing a set of equations to model these specific actions.

This operational phase, in which children’s conceptualization of a relationship was constrained by examining specific values and indicating operations on these values, might be viewed as a preconceptual stage (Sfard, 1991). Alternatively, we might frame it as an initial step toward reflective abstraction but with the child still at the projection stage (Simon, Tzur, Heinz, & Kinzel, 2004). Monk (1992) described this point-wise view of a functional relationship as one’s ability to produce a function’s output value when given its input value. It is, however, a

less-than-full process conception because the child could not circumvent the process of acting on input values and thus had not condensed that action as one that could simply be imagined (Zandieh, 2000).

Level 5: Primitive Functional-General. At the *primitive functional-general* level, children could conceptualize a general relationship between two quantities across a set of cases (as opposed to only within a set of particular cases) although their representations had primitive characteristics. A critical distinction of this level is that although children could describe a general relationship, they could not articulate a relationship that identified the specific quantities being compared in their generalized form or the specific mathematical transformation between them. Yet, when given such a rule, they could evaluate its correctness.

During the pre-CTE interview Noses on Dogs task, the interviewer asked Rebecca, "So what are we noticing here with the relationship [pointing to the values in the function table]?" (see Figure 3). Rebecca responded, "The same thing because they're all the same numbers." Pointing to corresponding values in the rows of the function table, she continued, "One and one, two and two, three and three, four and four, one hundred and one hundred." Her gesturing across each of the pairs of related values in the table underscored that the relationship she noticed was functional in nature. We also note that although she referenced specific cases in describing the relationship, she did so to justify the general claim she had already provided ("they're all the same numbers").

In our view, this reflected a transition beyond the functional-particular level in which children could only characterize a functional relationship using case-based language. That is, what distinguished her thinking here from the functional-particular level was that her description of the relationship ("they're all the same numbers") referred to an observation across the cases. However, we characterized her thinking as primitive because she did not articulate—in words or symbolic notation—a mathematical transformation between the generalized quantities "number of dogs" and "number of noses." For example, she did not express the relationship as "the number of dogs is the same as the number of noses" or as, for example, $D = N$, in which D represents the number of dogs and N represents the number of noses. However, even though Rebecca did not produce such a relationship when asked, she did recognize an accurate verbal representation of the relationship.

Interviewer: OK, would you tell [your teacher] that the number of dogs is less than the number of noses?

Rebecca: No.

Interviewer: Would you say the number of dogs is greater than the number of noses?

Rebecca: No.

Interviewer: Would you say the number of dogs is the same as or equal to...?

Rebecca: Yes.

- Interviewer:* The number of dogs is the same as or equal to the number of noses?
- Rebecca:* Yes.

This type of thinking also occurred in tasks in which the relationship was more complex than the identity relationship in the Noses on Dogs task. For example, during the Height With a Hat task, when Raymond was asked to describe a person's height with a hat in comparison to that person's height without a hat, he said, "It has to be taller." He characterized the difference in height qualitatively ("taller") but did not quantify that difference (e.g., "1 foot taller") or specify the quantities being compared. Thus, although he saw a generalized relationship between the quantities, it was primitive in its characterization.

More generally, we might frame this level of thinking as children advancing in the process of condensation. Although they seemed to be beginning to condense particular instances into an integrated unit (here, a generalized relationship), all of the distinct parts—the quantities and operations on them—had not yet crystallized in their thinking. This is perhaps not surprising. Kieran (1993) noted, "Interiorization and condensation are lengthy sequences of gradual, quantitative... changes" (p. 195). As such, we expect that levels subsequent to this level (primitive functional-general) would reflect a shift toward a more abstract or condensed understanding of the functional relationship. However, children who exhibited thinking at the level previous to this (functional-particular) seemed to be merely "getting used to certain operations" (Sfard, 1991, p. 13), such as calculating and representing relationships between specific values, through relatively routine processes that did not yet reflect the integration of these processes into a generalized, unitary form.

Level 6: Emergent Functional-General. Children's thinking at the *emergent functional-general* level reflected the emergence of key attributes of a generalized functional relationship, although their representation of the relationship was incomplete. Their representations at times conveyed either the generalized quantities but not the mathematical relationship between them or the mathematical relationship but not the generalized quantities. In some cases, their representations were in the form of an expression that referenced only one quantity.

For example, during the post-CTE interview Growing Train task, children who exhibited thinking at this level identified the quantities being compared as they described their functional relationship but did not specify the transformation between these quantities. When Neil was asked to describe a relationship he noticed in his own words, he responded, "I add the number of stops to get the number of cars that he had." Although he specified the two quantities ("number of stops" and "number of cars") and had a general understanding that he would operate ("add") on an arbitrary quantity to produce an arbitrary quantity, he did not describe the particular mathematical transformation of one that would produce

the second (e.g., “I *double* the number of stops to get the number of cars”).

Additionally, in representing the relationship using variable notation, children sometimes conflated the functional relationship with a recursive pattern. During the Growing Train task, Neil described the relationship symbolically as $A + 2 = C$, in which A represented the number of stops and C represented the total number of cars, instead of $A + A = C$. We infer from Neil’s attempt to characterize the mathematical transformation that he inadvertently mixed a representation for a recursive pattern in the number of cars (“add 2”) with the concept of doubling. Children in first grade had not studied multiplication, so additive notation was used in depicting relationships (e.g., $A + A$ rather than $2 \times A$). Thus, although it might be argued that Neil’s rule was simply a representational error in which he used the more familiar symbol $+$ but instead meant $\times$, his explanation of why he represented the rule as $A + 2 = C$ (“so [A] might be like any number, so add 2 more to A , then that would go up at C ”) suggests that, in that moment, he saw the relationship as “add 2 more to the number of stops” rather than “double the number of stops.”

In contrast, in the mid-CTE interview Height With a Hat task, Rebecca described an arbitrary person’s height with the 1-foot hat in relation to that person’s height without the hat as “It’s always increasing by one.” In this, she was able to characterize the mathematical transformation between two general quantities (“always increasing by one”) but did not specify the quantities (the height of the person without wearing the hat and the height of the person wearing the hat) being compared in the relationship.

It might be suggested that Rebecca was unaware of what quantities were being compared and, thus, could not name them. Alternatively, she might have clearly understood the quantities being compared but thought it redundant to name them (for example, if she or the interviewer had already named them in the exchange closely surrounding any discussion of relationships in the data). However, the interview data do not support this interpretation. First, earlier in the interview, Rebecca had constructed a function table for Height With a Hat data in which she used p to represent a person’s height without wearing a hat and x to represent the person’s height when wearing a hat. So, she had an awareness of the quantities and what her variable notation represented. Moreover, the interviewer did not name the quantities when she asked Rebecca what relationships she saw in the data in the table (“Do you see any types of relationships in the t-chart?”⁷) or in the conversation that followed that led to Rebecca’s characterization of the relationship as “It’s always increasing by one.” Thus, it seems unlikely that Rebecca thought that naming the quantities was redundant. We think, instead, that this reflects a progression in Rebecca’s thinking in that she had “pieces” of knowledge (Wagner, 2006) regarding the relationship at hand, but her understanding was not yet fully condensed in a way in which she would articulate the quantities being compared, or the pieces, into a coherent whole.

⁷ The term *t-chart* is sometimes used in the elementary grades to refer to a function table.

Finally, at this level, children sometimes represented a relationship only as an expression, leaving the dependent variable unspecified. For example, during the post-CTE interview Growing Train task, children sometimes described the function rule as simply the expression “ $A + A$ ” in which A represented the number of stops. Or they sometimes used language such as “add the number of cars to itself” when describing the relationship in their own words. Although either of these representations is an important step toward a full understanding (and would understandably be viewed as acceptable in most elementary grades settings), they do not clearly represent both of the two generalized quantities and the relationship between them.

As this situation suggests, another distinction of this level was that children described an emergent functional relationship using either words or variable notation. We are not saying that children needed to first be able to describe a functional relationship in words before they could use variable notation. On the contrary, as reported elsewhere, we observed that children sometimes took up variable notation as a useful way to describe relationships that were difficult to articulate in natural language (Brizuela, Blanton, Sawrey, et al., 2015). Representing the mathematical transformation in a functional relationship requires one to first see the underlying transformation between quantities before it can be represented, regardless of the form of representation used. In contrast to the previous level, in which children could articulate general relationships but not specify a mathematical transformation on quantities, children who exhibited emergent functional-general thinking and could characterize a mathematical transformation chose natural language or variable notation to represent it.

The significance of this level is that it seems to illustrate the cognitive space in which children’s thinking about functional relationships was beginning to condense and that their conceptualization of the attributes of a functional relationship—and the precision with which they could describe these relationships—was beginning to crystallize. Although children exhibited an emergent awareness of significant components of functional relationships at this level, they did not yet clearly articulate these relationships in their own language or in symbolic language. We viewed these “missteps” not as an endpoint in children’s thinking—children often progressed to subsequent levels within a single interview—but as a benchmark in their progression of learning.

Level 7: Condensed Functional-General. Children who exhibited thinking at the *condensed functional-general* level conceptualized functions as a generalized relationship between two arbitrary and explicitly noted quantities. That is, they could explicitly describe generalized quantities and the mathematical transformation between the quantities in functional relationships represented through both words and variable notation. A distinction of children’s thinking at this level centered on their awareness of what constituted a functional relationship. No longer using only expressions to represent functional rules, they described relationships that explicitly identified the two quantities in the problem scenario and the transformation between them, accurately capturing mathematical operations on quantities. When using

variable notation, they characterized the rule as an equation. When representing the rule in words, they used complete (although sometimes awkwardly worded) sentences that identified both the quantities and their relationship.

In the post-CTE interview Growing Train task, after Raymond had constructed a function table to organize his data, he was asked to find a relationship between the number of stops and the number of cars. He wrote the equation $S + S = C$, then had the following exchange with the interviewer.

- Raymond:* Stop is one. Plus another stop is one more. Then I would get C , because that's the number I would get if you added the stop plus stop.
- Interviewer:* And S represents the number of stops, you just said, and what does C represent?
- Raymond:* It represents the number of cars.

During her interview, Rebecca represented the relationship as $R + R = V$, in which R represented the “number of stops” and V represented the “number of cars.” When asked to describe this rule in her own words, she responded, “Whatever number, how many stops it made, if you doubled it, that's how many cars it would have.” Our point is not that precision is the goal, although mathematical precision is important (NGA & CCSSO, 2010). Instead, our point is that children's representations of their thinking at this level reflected an understanding that two quantities were related and an understanding of how to represent the transformation between these quantities, both of which pointed to a more sophisticated level of thinking than that exhibited in the previous level.

Moreover, children whose thinking was characteristic of this level exhibited flexibility in their understanding of the relationship that was not characteristic of the previous level. That is, once they had constructed a rule, they could unpack its constituent parts—for example, a generalized quantity or a mathematical operation—as it related to the problem context. This flexibility in their thinking about the functional relationship reflects important practices of *decontextualizing* and *contextualizing* (NGA & CCSSO, 2010) in that children could abstract a situation and represent it symbolically while also probing the referents for the symbols used to represent the situation. For example, during the post-CTE interview Growing Train task, Rebecca decontextualized by abstracting a general relationship ($R + R = V$) from her collection of equations representing relationships between two specific, corresponding function values (for example, “ $2 + 2 = 4$ ”; see Figure 6). When asked why she described her rule as “ $R + R = V$,” Rebecca pointed to the equations she had written in her function table and said, “So I want it to match and be the same.” We see this as similar to Radford's (2006) characterization of “true” generalizing as noticing a local commonality and generalizing across all instances (see also Cooper & Warren, 2011).

Raymond and Rebecca were both able to contextualize the variable notation symbolizing the relationship by describing what the constituent parts of their

$R+R=V$

S	h
1	2
2	4
3	6
4	8
R	V

Equations between columns: $1+1=2+1$, $2+2=4$, $3+3=6$, $4+4=8$

Bottom equation: $V+V R+R=V$

Figure 6. Rebecca's written work on the post-CTE interview task, Growing Train.

equations represented in terms of the problem scenario. For example, when asked by the interviewer "What does R represent here in your equation [referring to $R + R = V$]?" Rebecca responded, " R represents how many stops the cars—the train—made."

Although Level 4 (functional-particular) might be seen as a mental action of interiorization as children operated on more familiar objects, Levels 5–7 seemed to reflect the process of condensation, in much the same way that a child might gradually compose an understanding of "5." That is, as children progressed through these levels—with an expected amount of cognitive messiness—they began to encapsulate a process of generalizing functional relationships and to think of a relationship as a whole unit (Cottrill et al., 1996; Sfard, 1991).

Level 8: Function as Object. There are at least two characteristics of children's thinking at the *function-as-object* level that distinguished their thinking from that of previous levels. First, children perceived boundaries concerning the generality of the relationship. That is, although at previous levels children might have described a generalized relationship, their understanding of the generality of their claim—including both when the generalization was true and when it failed to be true—was not characteristic of descriptions of their thinking. At the function-as-object level, however, they not only seemed to understand that the relationship held across some set of values but that it would no longer be valid if there was a

perturbation in the problem situation. In our view, this type of thinking is consistent with the argument that generalizing involves understanding “what is preserved and what is lost between the specific structures which have some isomorphism (Getner & Markam, 1994; Halford, 1993)” (Cooper & Warren, 2011, p. 191).

During the post-CTE interview Growing Train task, Rebecca exhibited this kind of thinking when she was asked when her rule $R + R = V$ would hold.

- Interviewer:* Do you think that rule is going to work all the time in this problem?
- Rebecca:* Yes.
- Interviewer:* OK, why?
- Rebecca:* Only if you count this as a part [she points to the engine].
- Interviewer:* OK, if we count it or don't count it?
- Rebecca:* If we count it, it changes the t-chart.

Rebecca's point was that if we now counted the engine as part of the total number of cars the train would have after each stop, then the rule describing the relationship between the number of stops and number of cars would change (the train engine was not counted in the original statement of the problem).

This thinking—the understanding of the boundary points of a generalization—reflects a more significant characteristic of this level. That is, children conceptualized the relationship structurally, as an object in its own right on which new processes could be performed. We maintain that, although the gradually increasing sophistication in children's thinking exhibited in Levels 4–7 reflected the initiation of a mental action of interiorization that transitioned as the child's thinking about the relationship condensed, the subsequent function-as-object level represented an ontological shift in perspective—a reification of a process (Sfard, 1991)—that we suggest is qualitatively different than the previous four levels. This was exhibited in Neil's thinking about the Growing Train task. During the portion of this postinterview task for which the engine was not counted, Neil represented the functional relationship as $A + A = C$. He explained that A represented “the number of stops that it was at” and that C represented “the number of cars that he has with him.” When asked how his rule would change if he now counted the engine, the following conversation occurred.

- Neil:* I just add one, plus one.

[Neil wrote “1 +” in front of all of the equations he had constructed during his work on the problem (see Figure 7), transforming the rule “ $A + A = C$ ” to “ $1 + A + A = C$.”]

- Interviewer:* And why did you...
- Neil:* 'Cause you plus the car, so it's this, 'cause if this wasn't, they'd have this [removes the engine from the toy train on the table in front of him]. It'll only count these and if it had it [he reattaches the engine to the cars], it would count it.

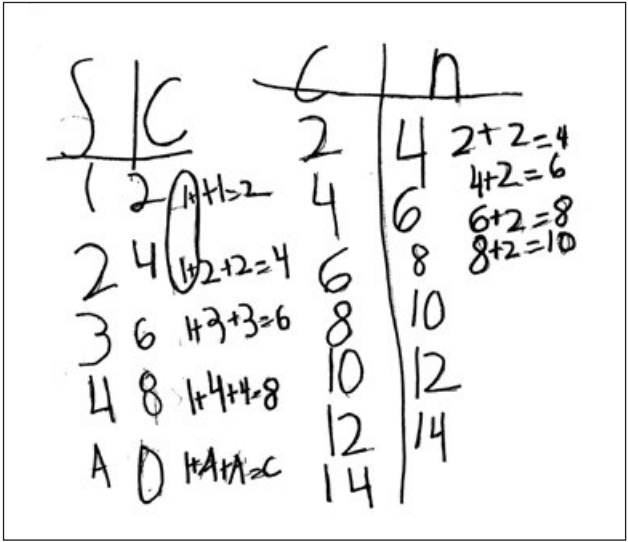


Figure 7. Neil’s written work on the post-CTE interview task, Growing Train.

Rebecca also exhibited this type of thinking. Near the end of her post-CTE interview, she was asked how her function rule ($R + R = V$) would change if she now counted the train engine. Like Neil, her response that she would just “add one” to the function rule she had produced in the original problem—an action that she represented as “+1 $R + R = V$ ” and then as “+1 $+ R + R = V$ ”—suggests that she saw her original rule ($R + R = V$) as an object that could be operated on to produce a new rule. In other words, Rebecca and Neil no longer seemed to view the original rule as a *process* of doubling the number of stops to get the number of cars (Sfard & Linchevski, 1994). If they had, we would expect that in order to answer the question of what would happen to the rule if the engine were counted, they would go back into the meanings of the referents, reconstruct a new table that accounted for adding the engine, and construct a new function rule from that. Instead, they not only understood that they could transform the original rule, but they were also able to construct that transformation (Cottrill et al., 1996; Gilmore & Inglis, 2008).

Observations About the Chronology and Frequency of Levels

Although our focus here is on identifying and describing the levels of thinking that children exhibited as they progressed through our instructional sequence, there are other secondary issues that are worth noting. Figure 8 provides a chronology of the levels of thinking exhibited by one child, Rebecca, across her

⁸ Figure 8 addresses only the occurrence of levels, not their duration. Moreover, there were segments of transcripts that did not address generalizing at all. These are not analyzed in this study.

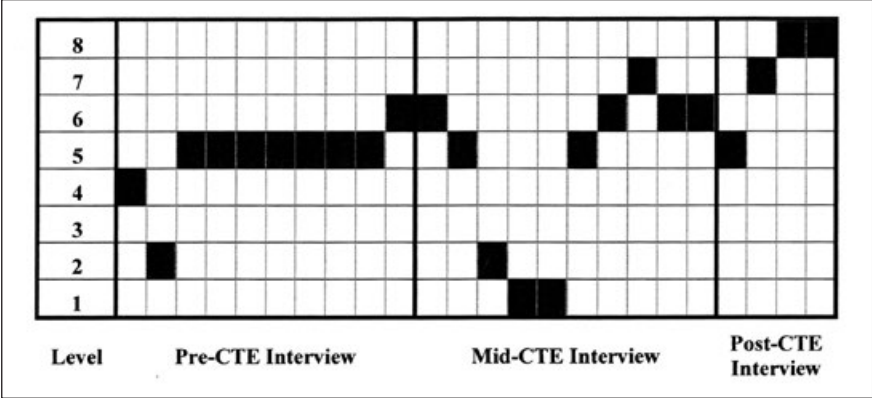


Figure 8. Chronology of the occurrence of levels of thinking exhibited in transcripts from Rebecca’s pre-, mid-, and post-CTE interviews.

pre-, mid-, and post-CTE interviews. As noted earlier, interview transcripts were analyzed, using the speaker’s turn as the unit of analysis, according to the level of thinking that occurred. In the natural flow of conversation, a chunk of that conversation (i.e., a segment constituting a sequence of speaker turns) might speak to issues relative to a particular level. Figure 8 illustrates how those levels occurred across the three interviews.⁸

Rebecca’s thinking is representative of other participants in that her thinking moved bidirectionally between levels, sometimes reverting back to lower levels either within an interview or as she encountered new tasks in subsequent interviews. For example, in the pre-CTE interview Noses on Dogs task, we observed that her understanding of a general relationship at this early point was fragile. That is, even though she had described a primitive form of a general relationship (e.g., “they’re all the same numbers”), she later abandoned this in the face of information that suggested otherwise. For instance, near the end of her pre-CTE interview, she noted the place in her function table (see Figure 3) where she had erroneously written corresponding values of 0 and 10 for the number of dogs and number of noses, respectively, and observed, “Some can be even different like, zero and ten [pointing to the row with 0 and 10].” When the interviewer suggested that this might be an error in the table, Rebecca again revised her thinking: “They’re like, the same.” It is also interesting that Rebecca exhibited lower levels of thinking on her mid-CTE interview than on her pre-CTE interview (although she exhibited higher levels of thinking as well). It is possible that this is a reflection of the more difficult type of task used in the mid-CTE interview (i.e., one of the form $y = x + b$ as opposed to the identity function used in the pre-CTE interview).

This underscores to us that levels in a progression are not stages through which one progresses; children may skip levels entirely or revert to previous ones when encountering new tasks (including the use of different function types or different types of questions about the tasks) or even superficial changes in their environment

(Clements & Sarama, 2014). At the same time, Figure 8 does indicate an upward trend that suggests the progression in Rebecca’s thinking over time is consistent with a mathematical perspective. For example, in the pre-CTE interview, she did not advance beyond Level 6, but in the mid-CTE interview, her thinking advanced to Level 7. By the post-CTE interview, which involved the most complex task, she operated most frequently at Level 8, the most advanced level. Moreover, unlike during the first two interviews, she exhibited no thinking lower than Level 5 on the post-CTE interview.

A final point we want to underscore is the extent to which first graders (6-year-olds) were able to generalize and reason with functional relationships. Our perspective prior to the study was that at most, first-graders might be able to recognize and articulate recursive patterns in their own words (essentially, exhibit Level 3 thinking). We did not anticipate the extent to which they would be able to generalize, represent symbolically, and reason with functional relationships. Figure 9 indicates that 32% of the conversational episodes for the 10 first-grade interview participants across the pre-, mid-, and post-CTE interviews were coded between Levels 1 and 3. That is, 32% of the time, children exhibited either no recognition of quantitative relationships or, at most, recognition of recursive patterns expressed in natural language. In stark contrast, 68% of the occurrences were between Levels 4 and 8, all of which were related to children’s recognition of functional relationships. In other words, children exhibited substantially more functional thinking than recursive thinking.

Certainly, we do not claim that all children exhibited the most sophisticated level of thinking in the progression we propose here. Indeed, they did not, as indicated by the variation in the data shown in Figure 9. However, in heterogeneous groupings of children, we would expect to find variation in classroom academic ability for any new concept in instruction. We argue that, in spite of this, an

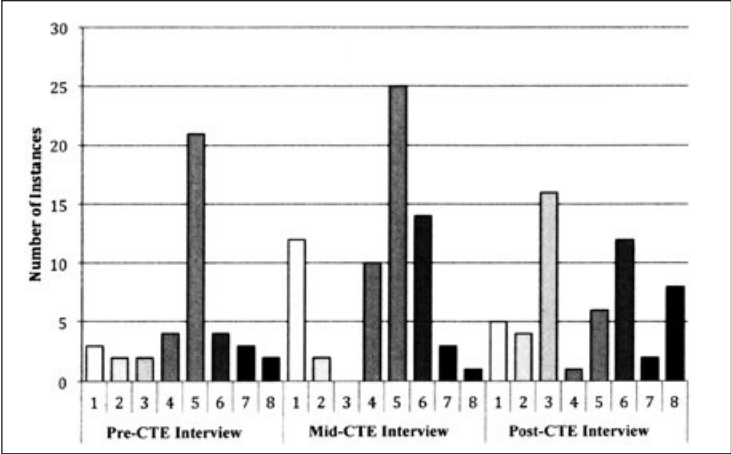


Figure 9. Frequency of the occurrences of Levels 1–8 of the progression for each CTE interview with all 10 first grade interview participants.

introduction to functional thinking in such early grades can initiate an arc of learning that could positively change children's opportunities for success as they encounter formal algebra studies in the middle grades.

Discussion

Our analysis of children's activity of generalizing shows that it was framed around two core types of relationships: recursive and functional. What struck us initially about our findings was that there were similar cognitive characteristics in children's thinking between these two core types. In both recursive and functional thinking, children seemed to initially understand the relationships from a perspective of the particular as described in Levels 2 and 4, conceptualizing a recursive pattern as a counting sequence (e.g., "two-four-six-eight") and a functional relationship as a sequence of numerical relationships between two specific quantities (e.g., "Eight plus one will equal nine—six plus one will equal to seven—three plus [one] will equal four"). This perspective, rooted in a conceptualization of a relationship as a sequence of specific cases, transitioned into a generalized understanding of these relationships that exhibited increasingly sophisticated forms and for which—in the case of functional relationships—the relationship itself became an object for transformation.

A qualitative distinction between these two ways of conceptualizing the relationship—that is, having either a particular or a general view of the relationship—might be understood in terms of one's capacity for symbolization. Kaput, Blanton, and Moreno (2008) argued that generalizing and symbolizing are inherently linked in that generalizing requires one to use an expression—either a conventional or unconventional representation—to symbolize multiple instances in a unitary way. We suggest that although children who conceptualized a relationship as a particular sequence of cases (i.e., Levels 2 or 4) perceived an underlying relationship, they did not yet have a representational means to compress multiple instances into a unitary form that could symbolize these instances.

From a process-to-object lens, that children transitioned through similar levels of thinking for recursive patterns and functional relationships seems plausible due to the repetitive nature of the process of interiorization, condensation, and reification. In Sfard's (1991) account of the historical development of number, she wrote that "approximately the same sequence of events could be observed time and again, whenever a new kind of number was being born" (p. 13). Presuming that the formation of any concept will go through this or some other such cycle (e.g., Radford, 2003) regardless of the grain size or time span in which it occurs, then as children shift between a recursive lens and a functional lens—in our view, qualitatively different forms of thinking for first graders—we might expect that they would start over in the development of their thinking about a specific type of relationship.

There are a few points that we would like to make about this. First, a reasonable question might be why children's thinking about recursive patterns did not become as sophisticated as their thinking about (the arguably harder) functional

relationships. We see this, at least in part, as a reflection of the instructional sequence that we developed as part of our research goals. We were primarily interested in how children understood functional relationships, so our instructional sequence focused on these and not on recursive thinking. Although we were not trying to eliminate recursive thinking, we did not actively pursue instruction that would push children's thinking further along these lines. Given that a progression in children's thinking is tightly coupled with the instructional sequence that gives rise to it (Simon & Tzur, 2004), we think it is reasonable that children did not advance in their understanding of recursive patterns in the way that they did with functional relationships. If we had focused on recursive patterns in our instructional sequence, we might expect a similar path towards reification with recursive patterns as that which occurred with functional relationships.

In addition, the data suggest that sometimes within a specific interview, children observed some form of a generalized functional relationship yet spoke about a recursive pattern as a sequence of particular values (e.g., as in Rebecca's pre-CTE interview with the Noses on Dogs task). In other words, we found that children might exhibit less sophisticated thinking when working with recursive patterns than with functional relationships. From a process-to-object perspective, we would argue that children never reified their understanding of a recursive pattern, so it could not theoretically represent a foundational object for reifying a functional relationship. This suggests to us that an understanding of functional relationships can coemerge with that of recursive patterns.

Finally, although we used primarily a process-to-object framework to interpret the nature of children's generalizing observed in our study, we recognize that there are other equally suitable frameworks that could serve as a lens for the data (e.g., Radford's [2003] characterization of generalizations as either factual, contextual, or symbolic). The important issue for us is that we did not use a cognitive lens *a priori*. That is, we took a grounded approach in our analysis from which we identified the levels in children's thinking, looking first to see what children were saying about relationships in the data and what this reflected regarding their capacity for generalizing. Then, we used Sfard's (1991) process-to-object framework for concept formation as an additional lens for interpreting our findings about the progression in children's thinking. In this, we were not attempting to fit data to a theory but rather to see how theory could enhance our understanding of the data.

In our view, this study has important implications for mathematics in the elementary grades. Based on our own history of research on children's understanding of functions in the upper elementary grades, we brought to this study deeply held views that the study of recursive patterns was somehow a necessary bridge in the development of children's functional thinking. It seemed reasonable that children could not be expected to understand a relationship between two quantities without first understanding variation in a single sequence of values. This view is certainly consistent with the abundance of pattern tasks traditionally found in the lower elementary grades—in which children explore recursive patterns exclusively—coupled with the lack of attention to relationships between

covarying quantities at these grades. It is also consistent with other research on children's functional thinking in the upper elementary grades that suggests that children's understanding of generalizing develops first through their understanding of recursive patterns (e.g., Cooper & Warren, 2011).

Although we do not question that there are connections between these types of thinking, our findings suggest that there is not a linear dependence between the two. In contrast, data from our study support that young children can begin to think in sophisticated ways about functional relationships in spite of a limited understanding of recursive patterns. Indeed, as we engaged in the CTEs, it became quickly evident in children's thinking about our tasks that we did not need to pursue recursive patterns first. This seems consistent with practices in mathematically high-achieving countries in which recursive patterns are not emphasized in elementary grades curricula in the way they have been in the United States (Daro, Stancavage, Ortega, DeStefano, & Linn, 2007; U.S. Department of Education, 2008). Thus, although recursive thinking sometimes arose naturally in conversations with children, we did not pursue its development as an intentional focus of instruction.

We did find it noteworthy that children in our study, at the start of their formal schooling, exhibited at times both sophisticated thinking about functional relationships and more primitive thinking about recursive patterns. We wonder if one factor might be that at the time our study occurred, participants had limited experience with typical recursive patterning tasks that are traditionally emphasized in the lower elementary grades. We also wonder whether the age of our participants and their still emerging arithmetic notions allowed for the coemergence of arithmetic and algebraic ideas in the early grades in a way that was mutually beneficial in their mathematical development. Is it the case that by the upper elementary grades, this window of opportunity has lessened, and children have been funneled towards a recursive mode of thinking that must be unpacked in order to attend to relationships between quantities? Our intention is not to minimize the importance of recursive thinking. Recursive functions play a critical role in mathematics and are advocated in the *Common Core State Standards for Mathematics* (NGA & CCSSO, 2010) for secondary grades in the United States. Our point instead is that the results of our study call into question a long-held assumption in elementary school mathematics that children cannot begin thinking of functional relationships from the earliest grades and must instead have lengthy experiences with recursive patterns before they can consider relationships between covarying quantities.

Conclusion

Our goal in the study reported here was to identify the levels of sophistication in children's thinking about generalizing functional relationships. At the same time, we do not want to lose sight of the extent to which children participating in this study were able to generalize functional relationships in quite sophisticated ways. There are several points to make concerning this. First, this study provides an existence proof of young children's capacity for much deeper mathematical

thinking than we had previously thought possible. As we noted earlier, some of the presumptions we held as we began this work have been challenged. We started with typical patterning tasks that are often used in the lower elementary grades as a bridge to functional relationships. In retrospect, we saw no compelling mathematical need to do this (although there might be valid pedagogical reasons for doing so). We assumed that recursive patterns would likely be an important starting point—and ending point—for many children in our study. Instead, we found that many children could circumvent recursive thinking and reason about functional relationships. That is, they did not seem to need a deep understanding of variation in a sequence before they could explore a relationship between two quantities.

Because of their age, we assumed that children would struggle more with the challenges and misconceptions that are prevalent in the literature regarding older students—for example, the object–quantity confusion associated with variables (Carpenter et al., 2003; McNeil et al., 2010) or the difficulty in shifting students' perspective away from recursive thinking to functional thinking (Cooper & Warren, 2011). We found instead that many children did not seem to become entangled with these challenges in ways that we anticipated.

Our participants came from diverse academic and demographic settings; however, we caution that our findings here are not intended to be generalized across all populations of students. We can only make claims here for the children participating in our study. Although our results are certainly promising, more research is needed to understand how comprehensive this type of thinking can be with differing populations of students. Moreover, we intentionally focused on how first-grade children made sense of specific types of functions that were selected with the intent to leverage children's arithmetic work (e.g., function types that would elicit children's work with doubling in arithmetic). As such, we caution that the types of generalizations children made are within this context.

Although we view this study as evidence that young children are capable of making sense of functional relationships, of representing these relationships in sophisticated ways, and of reasoning with symbolic relationships in novel situations, we acknowledge that these findings may not resolve the question of whether we should teach these concepts in such early grades. We recognize that there are different answers to this question and that regardless of what approach one might advocate, we expect that it stems from a thoughtful look at research and a clear intent of increasing children's mathematical success. Along with others (e.g., Dougherty, 2008), our perspective is that although children in the lower elementary grades might not be expected to master algebraic concepts, it is important that they be given sustained, multiyear opportunities to systematically interact with these ideas and that they have access to a variety of representations that allow them to mathematize unknown quantities and their relationships. In our view, it is through sustained interaction that algebraic habits of mind can be forged that can serve students well in a formal study of functions in the middle grades.

Moreover, because of algebra's gatekeeper status (Schoenfeld, 1995) in school

mathematics, we do not see decisions about whether one should teach content that does not have such status (e.g., statistics) to young children as equivalent to decisions about teaching early algebraic concepts. As reiterated most recently in the *Common Core State Standards for Mathematics* (NGA & CCSSO, 2010), algebra is foundational to K–12 mathematics, and this elevates its role and importance across school mathematics in ways that other areas of mathematical content might not share (U.S. Department of Education, 2008). Its seminal role in school mathematics, in conjunction with the well-documented difficulties and deep marginalization that adolescents have had with formal algebra (e.g., Hiebert et al., 2005; U.S. Department of Education, Office of Educational Research and Improvement, National Center for Education Statistics, 1996, 1997, 1998), requires us to think about new and novel ways to alleviate its gatekeeper status.

This study points to further questions for research. First, our focus here has been on identifying increasingly sophisticated ways that children generalize relationships in functions over time and the types of reasoning in which they engage within these levels. More analysis is needed, however, to understand the mechanisms that promote shifts in children's thinking as they interact with function-based relationships.

Second, we intentionally focused on first-grade children in this study; in our view, the lack of research in the lower elementary grades on children's understanding of functional relationships warranted an initial focus on one grade span. However, a complete story of young children's functional thinking at the start of formal schooling should arguably span multiple grades (Barrett & Battista, 2014). We see research studies that focus on how stable the progression provided here is across multiple grades (e.g., Kindergarten through Grade 2) and for different beginning learners of the concepts addressed here as an avenue of future work.

Additionally, although we focused on children's functional thinking in the context of their analysis of function tables, there is also important research regarding how students reason about functions figurally and geometrically (e.g., Moss & McNab, 2011; Rivera & Becker, 2008). This raises the question of how these other contexts might affect the progression proposed here.

Finally, in light of the question raised earlier regarding the feasibility of developing functional thinking with such young learners, research is needed to understand the affordances of developing children's algebraic thinking practices through the study of functions in early grades vis-à-vis other curricular demands. This might include understanding how the development of children's algebraic thinking serves their arithmetic thinking or how a trajectory of learning initiated in the lower elementary grades, at the start of formal schooling, deepens as children advance into the upper elementary grades and into a more formal study of algebra in the middle grades.

This study provides further evidence that the practices of early algebraic thinking—generalizing, representing, justifying, and reasoning with mathematical relationships and structure—can be successfully integrated into the lower elementary grades. Young children have the capacity to engage in these ways of

thinking—even in mathematical contexts that are rarely, if ever, introduced in the lower elementary grades, such as functional relationships between two quantities. From its inception, early algebra has been framed as a way of thinking that should start in Kindergarten (and before), yet we have been hesitant to fully embrace early algebraic ideas in such early grades based on the presumption that children are not ready. With the high standards advocated in the *Common Core State Standards for Mathematics* (NGA & CCSSO, 2010), we cannot afford to miss this opportunity.

References

- Baroody, A. J., Cibulskis, M., Lai, M.-L., & Li, X. (2004). Comments on the use of learning trajectories in curriculum development and research. *Mathematical Thinking and Learning*, 6(2), 227–260. doi:10.1207/s15327833mtl0602_8
- Barrett, J. E., & Battista, M. T. (2014). Two approaches to describing the development of students' reasoning about length: A case study for coordinating related trajectories. In A. P. Maloney, J. Confrey, & K. H. Nguyen (Eds.), *Learning over time: Learning trajectories in mathematics education* (pp. 97–124). Charlotte, NC: Information Age Publishing.
- Battista, M. (2004). Applying cognition-based assessment to elementary school students' development of understanding of area and volume measurement. *Mathematical Thinking and Learning*, 6(2), 185–204. doi:10.1207/s15327833mtl0602_6
- Blanton, M. L. (2008). *Algebra and the elementary classroom: Transforming thinking, transforming practice*. Portsmouth, NH: Heinemann.
- Blanton, M., Brizuela, B. M., Sawrey, K., Gardiner, A. M., & Newman-Owens, A. (2015). *A learning trajectory in young children's understanding of variable*. Manuscript submitted for publication.
- Blanton, M. L., & Kaput, J. J. (2004). Elementary grades students' capacity for functional thinking. In M. J. Hoines & A. B. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 135–142). Bergen, Norway.
- Blanton, M., Levi, L., Crites, T., & Dougherty, B. (2011). *Developing essential understanding of algebraic thinking for teaching mathematics in grades 3–5*. In R. M. Zbiek (Series Ed.), *Essential Understanding Series*. Reston, VA: National Council of Teachers of Mathematics.
- Blanton, M., Stephens, A., Knuth, E., Gardiner, A. M., Isler, I., & Kim, J.-S. (2015). The development of children's algebraic thinking: The impact of a comprehensive early algebra intervention in third grade. *Journal for Research in Mathematics Education*, 46(1), 39–87. doi:10.5951/jresmetheduc.46.1.0039
- Brizuela, B. M., & Alvarado, M. (2010). El trabajo de alumnos de primer grado en problemas aditivos con el uso de diferentes herramientas notacionales [First graders' work on additive problems with the use of different notational tools]. *Revista IRICE*, 21, 37–43.
- Brizuela, B. M., Blanton, M., Gardiner, A., Sawrey, K., & Newman-Owens, A. (2015, April). *First grade students' uses of tables as they explore functional relations*. Paper presented at the Research Conference of the Annual Meeting of the National Council of Teachers of Mathematics, Boston, MA.
- Brizuela, B. M., Blanton, M., Sawrey, K., Newman-Owens, A., & Gardiner, A. M. (2015). Children's use of variables and variable notation to represent their algebraic ideas. *Mathematical Thinking and Learning*, 17(1), 34–63. doi:10.1080/10986065.2015.981939
- Brizuela, B. M., & Earnest, D. (2008). Multiple notational systems and algebraic understandings: The case of the “best deal” problem. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 273–301). New York, NY: Lawrence Erlbaum.
- Brizuela, B. M., & Lara-Roth, S. (2002). Additive relations and function tables. *Journal of Mathematical Behavior*, 20(3), 309–319. doi:10.1016/S0732-3123(02)00076-7
- Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Portsmouth, NH: Heinemann.

- Carraher, D. W., & Schliemann, A. D. (2007). Early algebra and algebraic reasoning. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 2, pp. 669–705). Charlotte, NC: Information Age Publishing.
- Carraher, D. W., Schliemann, A. D., & Schwartz, J. L. (2008). Early algebra is not the same as algebra early. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 235–272). Mahwah, NJ: Lawrence Erlbaum.
- Carraher, D. W., Schliemann, A. D., Brizuela, B. M., & Earnest, D. (2006). Arithmetic and algebra in early mathematics education. *Journal for Research in Mathematics Education*, 37(2), 87–115.
- Chi, M. T. H. (1997). Quantifying qualitative analyses of verbal data: A practical guide. *Journal of the Learning Sciences*, 6(3), 271–315. doi:10.1207/s15327809jls0603_1
- Clements, D. H. & Sarama, J. (2004). Learning trajectories in mathematics education. *Mathematical Thinking and Learning*, 6(2), 81–89. doi:10.1207/s15327833mtl0602_1
- Clements, D. H., & Sarama, J. (2009). *Learning and teaching early math: The learning trajectories approach*. New York, NY: Routledge.
- Clements, D. H., & Sarama, J. (2014). Learning trajectories: Foundations for effective, research-based education. In A. P. Maloney, J. Confrey, & K. H. Nguyen (Eds.), *Learning over time: Learning trajectories in mathematics education* (pp. 1–30). Charlotte, NC: Information Age Publishing.
- Clements, D. H., Barrett, J. E., & Sarama, J. (in press). Measurement in early and elementary education. In J. E. Barrett, D. H. Clements, & J. Sarama (Eds.), *Children's measurement: A longitudinal study of children's knowledge and learning of length, area, and volume*. *Journal for Research in Mathematics Education* monograph series. Reston, VA: National Council of Teachers of Mathematics.
- Confrey, J., & Smith, E. (1991). A framework for functions: Prototypes, multiple representations, and transformations. In R. G. Underhill (Ed.), *Proceedings of the 13th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, p. 57–63).
- Cooper, T. J., & Warren, E. (2011). Years 2 to 6 students' ability to generalise: Models, representations and theory for teaching and learning. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 187–214). Heidelberg, Germany: Springer. doi:10.1007/978-3-642-17735-4_12
- Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K., & Vidakovic, D. (1996). Understanding the limit concept: Beginning with a coordinated process scheme. *Journal of Mathematical Behavior*, 15(2), 167–192. doi:10.1016/S0732-3123(96)90015-2
- Daro, P., Stancavage, F., Ortega, M., DeStefano, L., & Linn, R. (2007). *Validity study of the NAEP mathematics assessment: Grades 4 and 8*. Washington, DC: American Institutes for Research. Retrieved from http://www.air.org/sites/default/files/downloads/report/Daro_NAEP_Math_Validity_Study_0.pdf
- Dougherty, B. (2008). Measure up: A quantitative view of early algebra. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 389–412). New York, NY: Lawrence Erlbaum.
- Duschl, R. A., Schweingruber, H. A., & Shouse, A. W. (Eds.). (2007). *Taking science to school: Learning and teaching science in grades K–8*. Washington, DC: National Academies Press.
- FileMaker Pro Advanced (Version 11.0) [Computer software]. Santa Clara, CA: FileMaker. Retrieved from www.filemaker.com
- Freudenthal, H. (1982). Variables and functions. In G. von Barneveld & H. Krabbendam (Eds.), *Proceedings of the Conference on Functions* (pp. 7–20). Enschede, the Netherlands: National Institute for Curriculum Development.
- Gilmore, C., & Inglis, M. (2008). Process- and object-based thinking in arithmetic. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sepúlveda (Eds.), *Proceedings of the Joint Meeting of the 32nd Conference of the International Group for the Psychology of Mathematics Education (PME) and PME-NA XXX* (Vol. 3, pp. 73–80). Morélia, Mexico.
- Glaser, B. G. (1998). *Doing grounded theory: Issues and discussions*. Mill Valley, CA: Sociology Press.

- Glaser, B. G. (2002). Conceptualization: On theory and theorizing using grounded theory. *International Journal of Qualitative Methods*, 1(2), 23–38. Retrieved from <http://ejournals.library.ualberta.ca/index.php/IJQM/article/view/4605/3757>
- Gravemeijer, K. (2004). Local instruction theories as means of support for teachers in reform mathematics education. *Mathematical Thinking and Learning*, 6(2), 105–128. doi:10.1207/s15327833mtl0602_3
- Hamley, H. R. (1934). *Relational and functional thinking in mathematics: The 9th yearbook*. New York, NY: Bureau of Publications, Teachers College, Columbia University.
- Hiebert, J., Stigler, J. W., Jacobs, J. K., Givvin, K. B., Garnier, H., Smith, M., . . . Gallimore, R. (2005). Mathematics teaching in the United States today (and tomorrow): Results from the TIMSS 1999 video study. *Educational Evaluation and Policy Analysis*, 27(2), 111–132. doi:10.3102/01623737027002111
- Inqscribe (Version 2.1.1) [Computer software]. Chicago, IL: Inquirium. Retrieved from www.inqscribe.com
- Kaput, J. J. (2008). What is algebra? What is algebraic reasoning? In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 5–17). New York, NY: Lawrence Erlbaum.
- Kaput, J. J., Blanton, M. L., & Moreno, L. (2008). Algebra from a symbolization point of view. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 19–55). New York, NY: Lawrence Erlbaum.
- Katz, V. (Ed.). (2007). *Algebra: Gateway to a technological future*. Washington, DC: Mathematical Association of America.
- Kieran, C. (1993). Functions, graphing, and technology: Integrating research on learning and instruction. In T. A. Romberg, E. Fennema, & T. P. Carpenter (Eds.), *Integrating research on the graphical representation of functions* (pp. 189–237). Mahwah, NJ: Lawrence Erlbaum.
- Lesh, R., & Lehrer, R. (2000). Iterative refinement cycles for videotape analyses of conceptual change. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 665–708). Mahwah, NJ: Lawrence Erlbaum.
- Maloney, A. P., Confrey, J., & Nguyen, K. H. (Eds.). (2014). *Learning over time: Learning trajectories in mathematics education*. Charlotte, NC: Information Age Publishing.
- McNeil, N. M., Weinberg, A., Hattikudur, S., Stephens, A. C., Asquith, P., Knuth, E. J., & Alibali, M. W. (2010). A is for apple: Mnemonic symbols hinder the interpretation of algebraic expressions. *Journal of Educational Psychology*, 102(3), 625–634. doi:10.1037/a0019105
- Meel, D. E. (2003). Models and theories of mathematical understanding: Comparing Pirie and Kieren's model of the growth of mathematical understanding and APOS theory. In A. Selden, E. Dubinsky, G. Harel, & F. Hitt (Eds.), *Issues in mathematics education: Research in collegiate mathematics education V* (Vol. 12, pp. 132–181). Providence, RI: American Mathematical Society.
- Monk, S. (1992). Students' understanding of a function given by a physical model. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 175–194). Washington, DC: Mathematics Association of America.
- Moss, J., & Beatty, R. (2006). Knowledge building in mathematics: Supporting collaborative learning in pattern problems. *International Journal of Computer-Supported Collaborative Learning*, 1(4), 441–465. doi:10.1007/s11412-006-9003-z
- Moss, J., Beatty, R., Barkin, S., & Shillolo, G. (2008). "What is your theory? What is your rule?" Fourth graders build an understanding of functions through patterns and generalizing problems. In C. Greenes & R. Rubenstein (Eds.), *Algebra and algebraic thinking in school mathematics: 70th yearbook of the National Council of Teachers of Mathematics (NCTM)* (pp. 155–168). Reston, VA: NCTM.
- Moss, J., & McNab, S. L. (2011). An approach to geometric and numeric patterning that fosters second grade students' reasoning and generalizing about functions and co-variation. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 277–301). Heidelberg, Germany: Springer. doi:10.1007/978-3-642-17735-4_16
- National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

- National Council of Teachers of Mathematics. (2006). *Curriculum focal points for prekindergarten through grade 8 mathematics: A quest for coherence*. Reston, VA: Author.
- National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common core state standards for mathematics*. Washington, DC: Author. Retrieved from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf
- Piaget, J. (1952). *The child's conception of number*. London, Great Britain: Routledge & Kegan Paul.
- Piaget, J. (1970). *Genetic epistemology* (E. Duckworth, Trans.). New York, NY: W. W. Norton.
- Radford, L. (2003). Gestures, speech, and the sprouting of signs: A semiotic-cultural approach to students' types of generalization. *Mathematical Thinking and Learning*, 5(1), 37–70. doi:10.1207/S15327833MTL0501_02
- Radford, L. (2006). Algebraic thinking and the generalization of patterns: A semiotic perspective. In S. Alatorre, J. L. Cortina, M. Sáiz, & A. Méndez (Eds.), *Proceedings of the 28th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 2–21). Mérida, México.
- Rivera, F. D., & Becker, J. R. (2008). Middle school children's cognitive perceptions of constructive and deconstructive generalizations involving linear figural patterns. *ZDM: The International Journal on Mathematics Education*, 40(1), 65–82. doi:10.1007/s11858-007-0062-z
- Schliemann, A. D., Carraher, D. W., & Brizuela, B. M. (2007). *Bringing out the algebraic character of arithmetic: From children's ideas to classroom practice*. Mahwah, NJ: Lawrence Erlbaum.
- Schoenfeld, A. H. (1995, May). Is thinking about “algebra” a misdirection? In C. B. Lacampagne, W. Blair, & J. Kaput (Eds.), *The algebra initiative colloquium. Volume 2: Working group papers* (pp. 83–86). Washington, DC: U.S. Department of Education, Office of Educational Research and Improvement, National Institute on Student Achievement, Curriculum, and Assessment. Retrieved from <http://files.eric.ed.gov/fulltext/ED385437.pdf>
- Schwartz, J., & Yerushalmy, M. (1992). Getting students to function in and with algebra. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 261–289). Washington, DC: Mathematics Association of America.
- Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22(1), 1–36. doi:10.1007/BF00302715
- Sfard, A., & Linchevski, L. (1994). The gains and the pitfalls of reification—The case of algebra. *Educational Studies in Mathematics*, 26(2–3), 191–228. doi:10.1007/BF01273663
- Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, 26(2), 114–145. doi:10.2307/749205
- Simon, M. A., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning*, 6(2), 91–104. doi:10.1207/s15327833mtl0602_2
- Simon, M. A., Tzur, R., Heinz, K., & Kinzel, M. (2004). Explicating a mechanism for conceptual learning: Elaborating the construct of reflective abstraction. *Journal for Research in Mathematics Education*, 35(5), 305–329. doi:10.2307/30034818
- Smith, E. (2003). Stasis and change: Integrating patterns, functions, and algebra throughout the K–12 curriculum. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to Principles and Standards for School Mathematics* (pp. 136–150). Reston, VA: National Council of Teachers of Mathematics.
- Steffe, L. P., & Thompson, P. W. (with von Glasersfeld, E.). (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 267–306). Mahwah, NJ: Lawrence Erlbaum.
- Stevens, S. Y., Shin, N., & Krajcik, J. S. (2009). *Towards a model for the development of an empirically tested learning progression*. Paper presented at the Learning Progressions in Science (LeaPS) Conference, Iowa City, IA. Retrieved from <http://www.education.msu.edu/projects/leaps/proceedings/Stevens.pdf>
- Strauss, A., & Corbin, J. (1990). *Basics of qualitative research: Grounded theory procedures and techniques*. Newbury Park, CA: SAGE.

- U.S. Department of Education. (2008). *Foundations for success: The final report of the National Mathematics Advisory Panel*. Retrieved from <http://www2.ed.gov/about/bdscomm/list/math-panel/report/final-report.pdf>
- U.S. Department of Education, Office of Educational Research and Improvement, National Center for Education Statistics. (1996). *Pursuing excellence: A study of U.S. eighth-grade mathematics and science teaching, learning, curriculum, and achievement in international context: Initial findings from the Third International Mathematics and Science Study* (NCES 97-198). Washington, DC: Author. Retrieved from <http://nces.ed.gov/pubs97/97198.pdf>
- U.S. Department of Education, Office of Educational Research and Improvement, National Center for Education Statistics. (1997). *Pursuing excellence: A study of U.S. fourth-grade mathematics and science achievement in international context* (NCES 97-255). Washington, DC: Author. Retrieved from <http://nces.ed.gov/pubs97/97255.pdf>
- U.S. Department of Education, Office of Educational Research and Improvement, National Center for Education Statistics. (1998). *Pursuing excellence: A study of U.S. twelfth-grade mathematics and science achievement in international context: Initial findings from the Third International Mathematics and Science Study* (Revised, NCES 98-049). Washington, DC: Author. Retrieved from <http://nces.ed.gov/pubs98/98049.pdf>
- Wagner, J. F. (2006). Transfer in pieces. *Cognition and Instruction*, 24(1), 1–71. doi:10.1207/s1532690xci2401_1
- Warren, E., & Cooper, T. (2005). Introducing functional thinking in year 2: A case study of early algebra teaching. *Contemporary Issues in Early Childhood*, 6(2), 150–162. doi:10.2304/ciec.2005.6.2.5
- Zandieh, M. J. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), *Issues in mathematics education: Research in collegiate mathematics education IV* (Vol. 8, pp. 103–127). Providence, RI: American Mathematical Society.

Authors

Maria Blanton, TERC, 2067 Massachusetts Avenue, Cambridge, MA 02140; Maria_Blanton@terc.edu

Barbara M. Brizuela, Department of Education, Paige Hall, Tufts University, Medford, MA 02155; barbara.brizuela@tufts.edu

Angela Murphy Gardiner, TERC, 2067 Massachusetts Avenue, Cambridge, MA 02140; Angela_Gardiner@terc.edu

Katie Sawrey, Department of Education, Paige Hall, Tufts University, Medford, MA 02155; Katharine.Sawrey@tufts.edu

Ashley Newman-Owens, Department of Education, Paige Hall, Tufts University, Medford, MA 02155; Ashley.Newman_Owens@tufts.edu

Submitted February 18, 2014

Accepted June 30, 2014

APPENDIX

Interview Protocol for Pre-CTE Interview Task, Noses on Dogs

Task: Noses on Dogs
<i>Start with a specific example</i>
Sample questions: - Let's say we have a group of 3 dogs. - How could we figure out how many noses that group of dogs has? - What do you need to know? - Is there anything that will help you figure it out? - Can you show me what you are thinking?
<i>Explore to what extent the student can think about a generalized case</i>
Sample questions: What if there were a different number of dogs? How could you show a friend how to figure out how many noses there are?
<i>Construct and explore meaning of a function table</i>
Sample questions: - Let's suppose there is 1 dog. How can we find how many noses there are? What if there are 2 dogs? 3 dogs? 4 dogs? - Can you organize your information in a function table (t-chart)? Using the students' t-chart, ask the following: - What do the parts of your table (t-chart) represent? - Why did you use these headings in your table? (If they use variables to represent the quantities, start with 7; if not, start with 10) - What do your variables represent? - What can the value of your variables be? - Why did you choose different variables (assuming they did)? - Can you tell me why you put the numbers in as you did? What do they mean? - How many dog noses would there be for 3 dogs? How do you know? How can you use your table to figure this out? - How many dogs do I have if I know there are 2 noses? How do you know? How can you use your table to figure this out? - How many dog noses would there be on 20 (or 50, 100) dogs? How do you know? How can you use your table to figure this out? - How many dogs do I have if I've counted 24 (or 48 or something larger) noses? How do you know? How can you use your table to figure this out?

<i>Explore relationships in data</i>
Sample questions: 15. Do you see any relationships in the numbers in your table? Can you describe them? 16. Will these relationships always work? How do you know? 17. How would you describe the change in the number of dog noses each time we add one more dog? 18. Do you think this will always happen? Why? 19. Can you find a relationship between the number of dogs and the number of dog noses? 20. How can you use your table (t-chart) to do this? 21. How would you show your friend how to figure out how many noses there are for some number of dogs? 22. How does your table (t-chart) help you do this?
<i>Explore the nature of the functional relationship the student has shared</i>
Sample questions: 23. Is this relationship always going to be true? How do you know?
<i>Explore use of variable notation to represent the relationship</i>
Sample questions: 24. Is there a rule that shows the same relationship? 25. How can you represent any number of dogs? (Or, how did you represent any number of dogs in the function table?) 26. Can you use a variable to represent this? 27. What could the values of your variable be? 28. How can you represent the number of noses for any number of dogs? (Or, how did you represent the number of dog noses in your function table?) 29. Did you need to use different letters? Why? 30. Can you use this to write a rule that shows the relationship between the number of dogs and number of dog noses?
<i>Explore how the student uses the generalization he or she developed</i>
Sample questions: 31. Can you tell me again what the variables in your rule represent? 32. How would you use your rule to find the number of dog noses for 10 dogs? 33. If your friend counted 16 dog noses, how would you tell your friend how to figure out the number of dogs there must be? 34. If your friend said he had 5 dogs and counted 6 dog noses, do you think he counted correctly? How do you know?

<i>Close out the interview</i>
Sample questions: 35. Is there anything else you notice? 36. Do you have any questions for me?

Interview Protocol for Mid-CTE Interview Task, Height With a Hat

Task: Height With a Hat
<i>Introduce the hat</i>
Sample questions: 1. What is this in front of us? 2. This hat is labeled with its height. What height is on the label?
<i>Explore to what extent the student can think about a generalized case</i>
Sample questions: 3. What happens if I put this hat on my head? How much taller do I get? 4. What happens if I put this hat on your head? How much taller do you get? 5. What happens to anyone who puts this hat on his/her head? How much taller does that person get?
<i>Construct and explore meaning of a function table</i>
Sample questions: This morning, I measured the heights of 4 people in the cafeteria. Help me figure out the following: 6. If Charlotte is 3 feet, how tall will she be with the hat? How do you know? Can you write an equation to show me how you got your answer? 7. If Shandra is 4 feet, how tall will she be with the hat? How do you know? Can you write an equation to show me how you got your answer? 8. If Nate is 5 feet, how tall will he be with the hat? How do you know? Can you write an equation to show me how you got your answer? 9. If Mr. Marcus is 6 feet, how tall will he be with the hat? How do you know? Can you write an equation to show me how you got your answer? 10. Can you use a table to organize each person's height and height with the hat? 11. What do the parts of your table (t-chart) represent? 12. Why did you use these headings in your table? (If they use variables to represent the quantities, start with 13; if not, start with 16)

13. What do your variables represent?
14. What can the value of your variables be?
15. Why did you choose different variables?
16. Can you tell me why you put the numbers in as you did? What do they mean?
17. If someone is 3 feet tall, what is his height when he is wearing the hat? How do you know? How can you use your table to figure this out?
18. If you know someone's height with the hat is 4 feet tall, how tall is he without the hat? How do you know? How can you use your table to figure this out?
19. If someone is 5 feet tall, what is her height when she is wearing the hat? How do you know? How can you use your table to figure this out?
20. If you know someone's height with the hat is 6 feet tall, how tall is she without the hat? How do you know? How can you use your table to figure this out?

Explore relationships in data

Sample questions:

21. Do you see any relationships in the numbers in your table? Can you describe them?
22. Will these relationships always work? How do you know?
23. How would you describe the change in the total height (person's height with hat) each time the person's height increases by 1 foot?
24. Do you think this will always happen? Why?
25. Can you find a relationship between a person's height and that person's total height with the hat?
26. How can you use your table (t-chart) to do this?
27. How would you show your friend how to figure out a person's total height with the hat?
28. How does your table (t-chart) help you do this?

Explore the nature of the functional relationship the student has shared

Sample questions:

29. Is this relationship always going to be true? How do you know?

Explore use of variable notation to represent the relationship

Sample questions:

30. Is there a rule that shows the same relationship?
31. What if you didn't know someone's height? How could you represent this?
32. Could you use a variable to represent his height?
33. What could the values of your variable be?

- | |
|--|
| <p>34. How could you represent that person's total height with the hat? (Or, how did you represent the total height with the hat in your function table?)</p> <p>35. Did you need to use different letters? Why?</p> <p>36. Can you use this to write a rule (equation) that shows the relationship between a person's height and their total height with the hat?</p> |
|--|

<i>Explore how the student uses the generalization he or she developed</i>

Sample questions:

- | |
|---|
| <p>37. Can you tell me again what the variables in your rule represent?</p> <p>38. How would you use your rule to find a person's total height with the hat if you knew she was 6 feet tall?</p> <p>39. If your friend told you she was 5 feet tall with the hat on, how would you tell your friend how to figure out her height?</p> |
|---|

<i>Close out the interview</i>

Sample questions:

- | |
|--|
| <p>40. Is there anything else you notice?</p> <p>41. Do you have any questions for me?</p> |
|--|

*Interview Protocol for Post-CTE Interview Task, Growing Train***Task: Growing Train***Construct and explore meaning of a function table*

Sample questions:

There was a train that ran the same route every day. As it went along, it picked up two train cars at each stop. (Assume we do not count the engine.)

1. How many train cars did it have after stop 1?
2. How many train cars did it have after stop 2?
3. How many train cars did it have after stop 3?
4. How do you know?
5. Can you use a table to organize the information about the number of train cars the train has after each stop?

Using the student's table, ask the following:

6. What do the parts of your table (t-chart) represent?
7. Why did you use these headings in your table?

(If they use variables to represent the quantities, start with 8; if not, start with 11)

8. What do your variables represent?
9. What can the value of your variables be?
10. Why did you choose different variables?
11. Can you tell me why you put the numbers in your table as you did?
12. If the train makes 4 stops, how can you use your table to figure out the number of train cars it would have after stop 4?
13. If you know there are 6 train cars, how can you use your table to figure out how many stops the train made?
14. If the train makes 10 stops, how can you use your table to figure out how many train cars it will have all together?

Explore relationships in data

Sample questions:

15. Do you see any relationships in the numbers in your table? Can you describe them?
16. Will these relationships always work? How do you know?
17. How would you describe the change in the number of cars each time the train makes a stop?
18. Do you think this will always happen? Why?
19. Can you find a relationship between the number of stops the train makes and the total number of cars on the train?
20. How can you use your table (t-chart) to do this?
21. How would you show your friend how to figure out the number of train cars for any number of stops?
22. How does your table (t-chart) help you do this?

<i>Explore use of variable notation to represent the relationship</i>
Sample questions: 23. Is there a rule that shows the same relationship? 24. What if you didn't know the number of stops the train made? How could you represent this? 25. Could you use a variable to represent the number of stops? 26. What could the values of your variable be? 27. How could you represent the total number of cars on the train for any number of stops? (Or, how did you represent the total number of cars?) 28. Did you need to use different letters? Why? 29. Can you use this to write a rule (equation) that shows the relationship between the number of stops and the numbers of cars on the train?
<i>Explore how the student uses the generalization he or she developed</i>
Sample questions: 30. How would you use your rule or table to find out the total number of train cars if your train makes 20 stops? What if your train makes 100 stops? 31. How would you use your rule or table to figure out the number of stops the train made if we knew there were 100 cars all together?
<i>Extension:</i> Repeat the above task by that we count the engine in our count. How would this affect our table? Our equations? Our rule?
<i>Close out the interview</i>
Sample questions: 32. Is there anything else you notice? 33. Do you have any questions for me?