

THE EMERGENT PERSPECTIVE IN RICH LEARNING
ENVIRONMENTS: SOME ROLES OF TOOLS AND ACTIVITIES IN
THE CONSTRUCTION OF SOCIOMATHEMATICAL NORMS

ABSTRACT. The emergent perspective (Yackel and Cobb, 1996) is a powerful theory for describing cognitive development within classrooms. Yackel and Cobb have shown that the formation of social and sociomathematical norms, and opportunities for learning are intertwined. The present study is an attempt to extend the range of application of the emergent perspective to middle high school classrooms. The learning environments we consider are rich in the sense that (i) the tasks in which students are engaged are open-ended problem-situations (ii) the activities around the tasks are multiphased, consisting of small group collaboration on problem solving, reporting and reflection in a classroom forum with the teacher (iii) the tools used are multi-representational software. We identify here some practices rooted in such rich environments from which several sociomathematical norms stemmed. The present study shows that socio-mathematical norms do not rise from verbal interactions only, but also from computer manipulations as communicative non-verbal actions.

1. INTRODUCTION: THE EMERGENT THEORY AND CLASSROOM
RESEARCH

Cobb, Yackel and Bauersfeld (Cobb and Bauersfeld, 1995; Yackel and Cobb, 1996) have challenged a radical Vygotskian perspective according to which, mostly, social interactions facilitate knowledge acquisition and learning (Hatano, 1993). At the same time, they challenge a radical version of constructivism: they maintain that doing mathematics is a social activity as well as an individual construction activity. They subscribe to an emergent perspective that involves the explicit coordination of interactionism and constructivism. This perspective treats the development of norms and beliefs as complementary. The teacher has the central role of initiating and guiding elaboration in the formation of these norms. However, the individual student has an active role in this formation as well, and students' beliefs (as well as teacher's beliefs) are key factors in negotiations on classroom norms. Cobb, Yackel and Wood (1992) have identified general social norms, social constructs that involve a taken-as-shared sense of *when* it is appropriate to contribute to a discussion. These norms sustain classroom cultures characterized by explanations, justifications and argumentation. Examples of such norms in classrooms (where the 'spirit of the reform' is present) are that students cooperate to solve problems, that



Educational Studies in Mathematics **39**: 149–166, 1999.
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meaningful activity is valued more than correct answers, and that partners should reach consensus as they work on activities. Yackel and Cobb (1996), who focused on the study of conditions that create opportunities for learning mathematics, identified another class of norms that are about the actual *process* by which students and teacher contribute. They called such norms *sociomathematical*, to designate the classrooms social constructs specific to mathematics that individuals negotiate in discussions to develop their personal understandings. The emergent perspective places face-to-face classroom social norms against general beliefs, sociomathematical beliefs against mathematical beliefs and values, and classroom mathematical practices against mathematical conceptions.

According to Yackel and Cobb, sociomathematical norms form when explanations and justifications are made acceptable. The acceptability itself is made possible when explanations and justifications can be interpreted by students in terms of actions on mathematical objects that were practiced: 'the currently taken-as-shared basis for mathematical communication served as the backdrop against which students explained and justified their thinking' (Yackel and Cobb, 1996, p. 461).

In the present study we try to extend the range of application of the emergent theory to middle high school classrooms in rich environments. The following is a short description of the environment in which our study was conducted, and of what we mean by 'rich'.

2. THE COMPUMATH PROJECT

The CompuMath Project is a large scale educational program for junior high-school students. It reflects our beliefs about, and experience with the meaningful learning of mathematical constructs. It is based on an environment characterized by rich social interactions supporting high-level discursive activity, and by the extensive use of multirepresentational software. Students work in small groups on open-ended tasks, the majority of which are problem-situations. These tasks are integrated into activities organized into multiple phases. The teacher invites the groups of students to report on the tasks, and to discuss their solution processes and those of others, with the whole class. Also, she facilitates the consolidation of concepts, processes and procedures produced by the groups. The environment is described as rich, because of the following characteristics: (i) the tasks are open-ended problem-situations; (ii) the design of the activities around the tasks consists of multiple phases (small group problem solving, reporting and reflection); (iii) the use of computerized tools.

The topics covered in the CompuMath Project are those traditionally included in the junior high-school syllabus: early algebra, algebra, statist-

ics, geometry and functions. The project teachers in its experimental stage were highly motivated and less concerned with performance than with understanding. In all mathematical topics, students have computerized tools at their disposal such as graphic calculators or graphers, spreadsheet and dynamic geometry software. All computerized tools are really tools, in the sense that they enable the user to construct objects. These tools enable dynamic linkage among representations (algebraic, graphical, tabular, pie chart, or figural), and actions within each of them (e.g., drawing, plotting, computing, scaling, or dragging). Representations and actions depend on the topic covered, although a certain unity of functioning is common to all the tools. For example, dragging is central when using the geometry software or the spreadsheet (although the context is quite different in geometry and in (pre-)algebra). Drawing and scaling are operations common to all tools.

3. CLASSROOM RESEARCH IN RICH ENVIRONMENTS

The adoption of the emergent perspective in rich environments requires us to reconsider the role of tools in classroom activity. Whatever the tools, they allow the establishment of practices and the taken-as-shared basis for mathematical communication in explanations and justifications. But what is the direct role of multirepresentational software in communication; i.e., the role of the non-verbal interactions between students and tools in classroom activities? Another concern arising not only from the use of this tool, but also from other aspects of rich environments, is the impact of the environment on the process of mathematization. This issue is particularly important because computerized software has been intentionally designed to represent mathematical objects and to afford actions on these objects. Also what happens when the teacher orchestrates a process of mathematization from the experience of the manipulation of objects with computerized software, to mental actions on previous material actions? The direct role of tools in communication and the role of rich environments in mathematization have been mostly approached in case study setting. In these studies, some of which are reviewed in the following, mathematization and communication relate to 'the construction of meaning'.

For example, in a recent book, *Windows on Mathematical Meanings* (Noss and Hoyles, 1996), the authors describe the development of proportional reasoning for one student acting in a LOGO environment. The successive tasks the student undergo with the same tool enable her to connect the task encountered to previous tasks (what they call to 'web'). In another study, Noss, Healy and Hoyles (1997) showed how a microworld

(‘Mathsticks’) helped a couple of students construct mathematical meaning by providing new means of expression (visual and symbolic) and new connections.

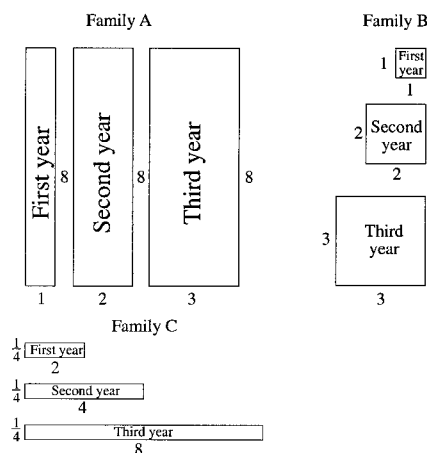
The idea of making meaning for students with computerized tools whose design afford certain actions, has been a central theme in a series of studies. Roschelle (1992) analyzed interactions between students working in a collaborative mode on a system for acquiring concepts in physics. He reached the conclusion that multirepresentational software may trigger conceptual change because they create favorable ground for developing metaphors, and because students collaborating together can gradually construct shared meaning. Similarly, Schwarz (1997) analyzed how pairs of students coordinated actions to use the affordances of a microworld (an intermediate abstraction) to construct meaning for algebraic operator while solving algebra word problems. Nemirovski and Noble (1997) studied how one student used a computer-based tool to create graphs of certain dimensions of a ‘real object’ as a function of distance and orientation. Their study deepened the relation between subject and tool and pointed at the development in the learner’s experiencing of transitional objects and live-in spaces.

All cited studies indicate that computerized tools (as well as conversation) may serve as communicative means for constructing social and cognitive entities. The studies themselves focused on case studies of pairs of students or of individuals using computerized tools. Our purpose here is to study some social constructs developing in classrooms with rich environments. This study broadens then the scope of two lines of research: we investigate socio-mathematical norms at junior high-school level, and we observe activities of students using computerized tools within a classroom. As we adopted the emergent perspective according to which cognitive constructs may develop in parallel with social norms, we paid attention to the use of tools as means of verbal/non-verbal communication and to the transformation of actions/objects during activities organized in multiple phases. The next section presents some typical examples of the emergence of sociomathematical norms in the CompuMath project.

4. SOCIOMATHEMATICAL NORMS IN THE COMPUMATH PROJECT

The sociomathematical norms identified by Yackel and Cobb (1996) are always presented as a result of a classroom forum verbal discussion between teacher and students. The basis for intersubjectivity is the taken-as-shared experience of manipulation of mathematical objects. The examples we present constitute some departure from this situation: Manipulating mathematical objects with computerized tools while investigating problem situ-

Each of the following three families of rectangles has its growth pattern:



In family **A** the width grows each year by one unit; the length remains constant at 8 units. In **B** the width and the length of the rectangle grow each year by one unit. In **C** the length doubles each year, and the width remains equal to $\frac{1}{4}$.

Try to find various hypotheses comparing change in area, perimeter etc. For example, answer the following questions: In which family will the area be largest and when? When will the area of each family exceed 1000?

Check your hypotheses with the tools available and explain!

Figure 1. The 'Growing Rectangles'.

ations is to interact with a tool and activities specially designed to convey mathematical ideas. The constitution of certain norms in this rich environment is often not the result of interactions between students and teacher, as shown in the following examples.

First example: What constitutes evidence when solving problem-situations (or the status of computer displays)

This example is taken from an introductory one-year-long course on functions in Grade 9. It happened at the sixth week of the course. At that stage, students had practiced the passage between representations, actions within representations such as walking on a graph with the pointer and reading the coordinates of the points on the graph. They had practiced the substitution of the x-value of points on a graph in the algebraic representation to obtain the corresponding y-value. Thus, students had intensively practiced actions and displays within specific representations and passing to other representations to view corresponding actions and displays. This practice was not obtained in activities focused on the systematic acquisition of skills, but rather, students were engaged in activities such as the Growing Rectangles,

in which operations and actions were given context for meaning while investigating problem situations. This activity is given in Figure 1.

The students are first invited to suggest hypotheses without using the computerized tool, then to use it to check them. Students tried to figure out which family will eventually take the lead by using intuition and/or by computing ‘by hand’ the areas of the three families of rectangles for a few values. After allowing a little time for investigation, the teacher initiated a whole class discussion on possible hypotheses concerning the changes in areas. Most students guessed that family **B** will take the lead, and family **C** will always remain last. One student, Zivit, argued that **C** will take over sooner or later:

The rectangle in family **C** doubles each year and in family **A** it grows by 8, so we will reach a situation where **C** will overtake **A**.

The teacher confronted students on this issue and asked all students to vote on Zivit’s **C** hypothesis against the formerly held **B** hypothesis. Many students voted for **B**, two voted in favor of Zivit’s hypothesis and about ten students chose to abstain. Then the class turned to work in small groups using graphical calculators. In many groups, students translated the situation into algebraic representations (with some difficulties in translating the function-area in **C**, $y = 1/4 * 2^x$), and then obtained the graphs with their graphical calculators (see Figure 2).

After the groups had written their reports, the teacher discussed with the class the mathematical findings and strategies. Students reported that on the eighth year ($x = 8$), the three rectangles have the same area, and that from that day on, family **C** takes the lead from family **A**. Family **B** remains in between. The evidence provided by the different representations was accepted and no student declared the computer wrong. On the contrary, they tried to reinterpret the situation. Most of the students seemed to be satisfied by the conclusion that ‘it grows faster than the other families and it catches them up’. The following excerpt is of a student (Miri) who had difficulty accepting this conclusion. At the end of the lesson, she turned to the research observer (Obs), who was not part of the class community, to initiate the following discussion:

Miri: I was the one who guessed that family **C** will always be behind the other two. I found from the graphs in the computer that it’s just the opposite, but I don’t understand why.

Obs: What can you do in order to convince yourself?

[Miri talked for a while with her pal Hanna, then substituted $x = 9$ to obtain 81, 72 and 128].

Miri: In spite of what we got, I don’t understand, because it is something multiplied by $1/4$, so it has to be smaller.

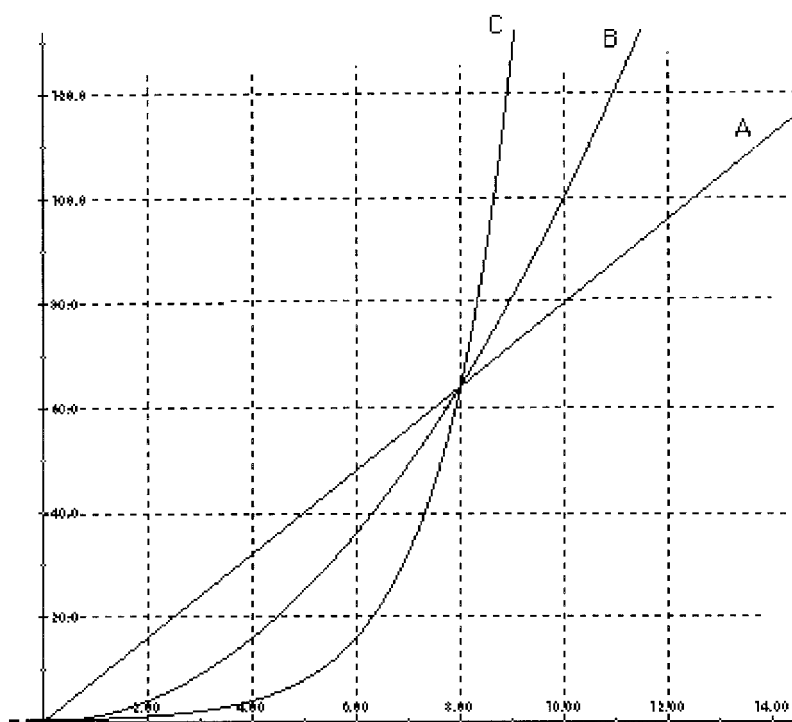


Figure 2. The Graphs of 'Growing Rectangles'.

Miri: [after a while, with some light in her eyes]. ... Ah, if the side is very long, say from here to America, then even if it is multiplied by $1/4$ it is still very big ...

Miri had some intuitions about the relative changes of size of the rectangles in each family. She used a graphical calculator to support her intuitions and to show that she was right (i.e. that family C will always be last). She interacted with the graphic calculator to display the three graphs on the same coordinate system. This display brought evidence that her intuition was wrong. She did not put into question the correctness of the display, but challenged her intuitions and tried to rethink the relation between the three families by passing to the algebraic representation and rechecking the findings suggested by the computerized graphs.

In this example, Miri started to realize that an interaction with the computerized tool, while solving problems, may lead to unexpected evidence, which confronts former intuitive hypotheses. The evidence (the artifacts/ resources-embedded in the representations) demands reinterpretation. Judging mathematical hypotheses with tools in order to confirm or refute them, becomes normative in the class. It was surprising to find that when displays or actions lead to a conflict with previous beliefs, it was

agreed that the computer displays and actions ‘represent the situation’. We interpret this ‘trust’ in the computer as evidence of intensive practice of ‘translation tasks’ or working in analogy on different presentations, while investigating a problem situation: If the algebraic representation corresponds to the numerical values obtained, and the graph is displayed on the computer screen by entering the algebraic representation, the graph cannot but ‘represent’ the growth of the rectangles. It is worthy of note that for many of the students, this norm is not negotiated in the sense that the teacher takes an active role in its establishment. Rather, ‘translation practices’ in activities around problem-situations with multirepresentational tools, establish tacit agreement that displays from different representations constitute evidence about problem-situations. If, like in the case of the Growing Rectangles, a conflict arises between beliefs and what the computer displays and actions show, the student will try to revise these beliefs. The sociomathematical norm of what constitutes evidence in problem-situations was formed here as a consequence from the interaction with the tool. In this activity, it triggered learning about linear, quadratic and exponential growth. At the stage described in this first example (six weeks into the course), Miri felt uncomfortable to admit in front of her class community that her first intuition was wrong and that even after having ‘the evidence’ from the tool, she did not understand why she was wrong. She chose to turn to an outsider – the silent observer. Admitting one’s own misunderstanding was then not normative. That this social norm was established later, is clear from our second example.

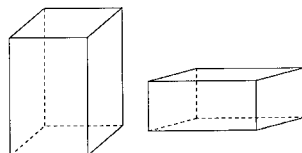
Second example: What constitutes a good hypothesis

Overseas is a problem-situation given towards the end of the course. Students were first given the homework assignment appearing in Figure 3a. They brought their models to class and were given a worksheet (Figure 3b). The students were invited to work in groups of four, each being equipped with their models and two graphic calculators. Each group was asked to write a report on their ideas and steps as they worked on the problem. Then, the teacher initiated a class discussion in which (a) various groups were invited to report on hypotheses and methods for finding the dimensions of the ‘ideal container’, (b) students were invited to express their (dis)agreement with what was reported, (c) students gave evaluative judgments on the nature of making hypotheses, or of general learning strategies, (d) the teacher led the whole class in summing up the mathematical ideas involved.

In sum, the class first experienced collaborative work to hypothesize and find the dimensions of the ideal container. The students manipulated

The Overseas Problem

a. The freight company "Overseas Inc." uses containers to ship goods by sea from country to country. The containers are big boxes made of wood. Their base needs to be a square, and their volume, must be 2.25 m^3 . The containers must also be open at their top (see drawing). Find the dimension of two examples of such containers and construct suitable paper models (1 m of the container – 5 cm of the paper).



Calculate the wood area needed to construct each of the two containers.

b. As wood is expensive, the company is interested in designing ideal containers with as little wood as possible. Try to make hypotheses about the dimensions of the ideal container and explain your steps. Can you help the company find the dimensions of the "ideal container"?

Figure 3. The 'Overseas Problem'.

models of the container, then manipulated computer actions, reported on all previous actions and finally the teacher initiated a discussion that led from concrete and computer actions to the construction of mathematical ideas. To a large extent Overseas was typical, and students often engaged in similar processes of mathematization.

The beginning of the class discussion, is reported here. The teacher opened the discussions by asking students to *describe the processes they went through*, rather than the numerical data of the ideal container. At the beginning the discussion focused on the ways they hypothesized the dimensions of the ideal container. One of the students, Zippi, started to report, the 'cube hypothesis':

Zippi: We took numbers with which we felt comfortable, to avoid fractions. To begin with, we did calculations step by step, and then we did it in a more logical way.

She described the calculations of the height and the surface area of the two models discussed in her group and declared:

Z(ippi): Now, we saw that the second box was smaller than the first one, it's as if the surface is smaller, so we learned that the more the base is close to the height, perhaps it will be smaller.

T: Smaller? OK. Everybody understands Zippi's team's hypothesis? I want you to repeat her hypothesis.

Zivit: The closer the x side is to the h side, the smaller will be the result.

T: What does the ideal container according to Zippi, look like?

Rinat: A cube.

T: So, this is a hypothesis based on two examples. Is it correct or not?

[Zippi remains silent; other students assert 'No, it is not the smallest']

T: (to Z) So, was your hypothesis right or wrong?

Z: Not exactly. It was very close. It was the closest.

T: But as it is, the hypothesis was right or wrong?

Z: [hesitates, then mumbles] Wrong.

T: (To the whole class) OK. But it was a very beautiful hypothesis. Well, if somebody was asking you whether this hypothesis is reasonable or not, what could you say?

Before analyzing this excerpt, Zippi already reported on the right answer and knew that the ideal container is not a cube. When being asked to describe what she went through, she knew that the teacher was not asking for the answer, but for her mathematical hypothesizing. This example shows that Zippi's verbal report and her interaction with the teacher is at two levels. When Zippi solved Overseas jointly with her group, she hypothesized mathematical hypotheses. Hypothesizing is an action, and hypotheses are the objects of hypothesizing. Following Zippi's report on the cube hypothesis, the teacher made the students aware of the above distinction in her response: She first turned to the whole class, to note *So, this is a hypothesis based on two examples*, relating to hypothesizing as an action: She stressed that the hypothesis is based on examples, but only on two examples. It is only then that she asked Zippi *Is it correct or not?* (relating to the cube hypothesis as an object). Zippi hesitated and her answer *Not exactly. It was very close* shows her perplexity about the correctness of what she did, because she related to her hypothesis both as an action and as an object at the same time. The insistent question of the teacher about the correctness of the object-hypothesis, led Zippi to answer that the hypothesis (as an object) was wrong. But the teacher continued to stick to the above distinction and immediately praised her. Although she used the term 'hypothesis' she really meant that Zippi's action of hypothesizing was beautiful.

In sum, the teacher 'collects' a report, and turns it into an artifact shared by the community of learners to encourage further action (verbal reporting/criticizing) on previous actions. The action of hypothesizing becomes the object of the discourse in the class. The new object considered now is more abstract. It is a previous action disconnected from its object. The continuation of the class discussion makes clear this process. Other students in the class reported on their group work, starting by evaluating Zippi's 'cube hypothesis'.

Miriam: First, I think that this hypothesis (the cube hypothesis) fits only a part of the function, but it's not the entire function. First, we also hypothesized this hypothesis, then another one.

[Miriam consults Osnat].

T: Let's see which other hypothesis you raised.

Osnat: We were in a team and first, we saw we had two extreme examples, one with a very small base.

[The students report on the two examples $0.5 * 0.5 * 9$ and $3 * 3 * 0.25$].

T: These are really extreme examples.

Osnat: Then, we saw that the result, the surface, say when the side of the base is bigger, say, the result, the surface is smaller.

Liat and Hanna (give the surface area of the two extreme examples): The first one 18.25, and the second one 12.

T: So you based your hypothesis on the fact that it is smaller?

Osnat: No! We also thought that in fact, it's not a closed box or something like that. Now if the base will be bigger, we don't multiply it, but the height, we multiply by four. So perhaps, it's OK for the height to be small, and we multiply it by four ...

T: Is this hypothesis clear? That is, you told us that the height has more weight ...

Hanna: It's stronger

T: Stronger, because ...

Hanna: We multiply it by four.

T (to the whole class): Do you agree with this hypothesis? That the height has a bigger-weight?

Liat: It's not exactly like that. We know that the two (the base side and the height) are linked.

T (to the group): Do you agree with this hypothesis after you solved the problem?

Miriam: I think that it can be like that for special cases, but generally, it's not like that.

[Mali, a girl from another group, intervenes].

Mali: I think that I have an example that, as if, it's possible to see that for the first box, the height was 2.25, and for the second box the height was 0.25, and in spite of that the first box had a smaller surface, in the first we obtained 10 and for the second 12.

Miriam's initial intervention in this excerpt is based on her critique of Zippi's way of hypothesizing. She admits that her group had a similar hypothesis that *fits only a part of the function*. The teacher invites her group to formulate their hypothesis. Osnat continues Miriam's report and describes on their hypothesis (with the help of Hanna and Liat who act here as if the group is one single person), and characterizes the examples on which this hypothesis is based as being extreme. Then they move to describe an additional hypothesis which is grounded on visual properties (Osnat says that *Now if the base will be bigger, we don't multiply it, but the height, we multiply by four*). The teacher leads the group to contrast this hypothesis with the problem solution. The girls in the group admit that this hypothesis fits only special cases. Mali, a girl from a different group brings in a counterexample. There is here a process initiated by Zippi's report, then the teacher's praising the cube hypothesis as expressing valuable hypothesizing, its appropriation by Miriam followed by the integration of this hypothesis-object by other students to construct a new object, more abstract in the sense that it is characterized by students by its properties (an hypothesis that fits a part of the function and based on

Nurit told me the following story: My son, Gil, was born when I was 24.

Twelve years ago, I was 3 times older than him.

Explore relations between the ages of Nurit and her son.

For example: find, if possible, appropriate ages to the following ratios.

Mother's age	Gil's age	Age Ratio
		25
		9
		1.5
		1.25
		1
		0.75

How does the age ratio change along the years? Explain!

Figure 4. The 'Age Problem'.

extreme cases). At all stages, the teacher repeats the grounds for the different hypotheses, not the hypotheses themselves (e.g., you *told us that the height has more weight.*). Liat rejects it, relating also to the critique on the hypothesis ground (*It's not exactly like that. We know that the two are linked*) and Mali refutes this hypothesis by bringing one counter-example.

The debate leads students to change beliefs, such as Zippi's belief that a hypothesis is good if it is close to the mathematically correct answer. Students learn that hypotheses are not good because of their closeness to the result but because of the grounds on which they are based. They learn that if an hypothesis fits only part from the numerical cases it is not fully grounded. They also learn that opposition to hypotheses is legal (for example by providing one counter-example) when it contradicts the grounds of the hypothesis. The participants are not bashful anymore about wrong or partial correct hypotheses and report in the forum of the class community on wrong results, misunderstandings or false starts.

The idea of what constitutes a good hypothesis seems to form mainly as a consequence of, both the verbal interactions among students and between students and teacher, as well as the interaction with the kinds of tasks and activities (the multiple phases around each task). But, through the interaction with the tool, the students gradually gained confidence that there is time to discuss, plan, hypothesize and to 'act on actions' to lead to mathematization because the finding of the solution given the algebraic representation, can always be left to the computer. The final example will show another aspect of the role of tools in the establishment of socio-mathematical norm, a communicative non-verbal aspect.

Gil's Age	Mother's Age	Age Ratio
0	24	#DIV/0!
1	25	25
2	26	13
3	27	9
4	28	7
5	29	5.8
6	30	5
7	31	4.42857143
8	32	4
9	33	3.66666667
10	34	3.4
11	35	3.18181818
12	36	3
13	37	2.84615385
14	38	2.71428571
15	39	2.6
16	40	2.5
17	41	2.41176471
18	42	2.33333333
19	43	2.26315789
20	44	2.2
21	45	2.14285714
22	46	2.09090909

Figure 5. A Computer Screen produced for the Age Problem.

Third example: What is an acceptable explanation when interacting with a spreadsheet program

The *Age problem* (see Figure 4) was given in a Grade 7 introductory course in algebra. The main goal is to develop 'searching for patterns' abilities. Students are led to express relations in algebraic symbols, in order to produce the sequences of numerical data. They use EXCEL (a spreadsheet program).

Two students, Shula (S) and Matan (M), *organized the information* in the spreadsheet template (see Figure 5) in the following steps: They labeled each column. They then created the first column by using the recursive formula $= A2 + 1$, and dragged it down. For the second column, they used

the formula = A2 + 24 (on the B2 cell). For the ratio (the third column) they wrote the formula = B2/A2. They did all this very skillfully, and the pairs of ages corresponding to the ratios of 25 and 9 were found immediately. They then started looking for the ages for which the ratio is 1.5 (see Figure 5). Immediately Shula declared that when *she gets older, the ratio decreases*. She had a *good global sense* about the changes in the ratio. The pair ‘dragged down’ further the numbers in the template, discussed them and even made their own calculations to be sure that no mistake occurred, or to understand better the ratio calculations, and found that the ratio of 1.5 occurs for the ages of 72 and 48. The two went on discussing:

S: I did not understand it. I don’t want, I can’t, I did not understand it.

M: What?

S: The formula. How can one write the formula?

M: The ratio?

S: I am not like you . I don’t want to copy until I find it.

M: What? there is a formula. We have to do B divided by A.

S: O.K., but I don’t know. Do you know what will be B divided by A for a million?

They continue to argue and after a while:

S: So, why should I, why can’t I think about it by myself? Why should I continue to copy?

M: You can! one second. Let us say that you drag it down ...

S: I don’t want, I want to know how to do it without passing through all of them ... I want to find this formula in order to find it by its place value ...

M: But she does not want us to find it by its preceding place.

S: Matan, this is on its place. You don’t understand what I want to tell you.

M: I don’t rely on numbers

S: Not to rely on predecessors but on the place. If I am in one place, I need to rely on this place. I don’t want to go on copying [Shula types the number 100 in the mother’s age column and gets the ratio directly without dragging].

M: How do you get it?

S: What I want to tell you is that I don’t want simply to copy, especially when we do not reach the answer. It’s totally meaningless

M: I understand

While this problem was meant to focus on producing the different ratios by B/A , and exploring how they change, Shula is bothered by a further problem, finding the numbers for a given ratio without dragging down the template; that is, finding a solution to the equation $(x + 24)/x = 1.5$ where x represents Gil’s age.

As shown here the girls used *formulae as keys for constructing tables* (this feature of formulae was already noticed in the Growing Rectangles, where formulae were the keys for drawing graphs). Shula and Matan have mastered the basic skills linked to the inductive construction of tables through EXCEL syntax (especially dragging down). Implementing those practices to evidence patterns or to search for solution of algebra problems has become normative. Matan is then satisfied by the inductive conclu-

sions, based on numerical data created by inserting ‘formulae’ in columns, and dragging them until relevant data are displayed. However, Shula is not satisfied by this kind of justification. She feels she needs a kind of justification beyond the evidence displayed by the computer inductively (*Why should I continue to copy it?*). The above formula shows only how elements are linked to their predecessor (even if the B/A and the $A + 24$ columns show structural features). She is interested in adopting another standard of justification (*I want to know how to do it without passing through all of them . . . I want to find this formula in order to find it by its place value*). She shows to Matan how she want to reach the link between mother’s age and ratio, in a non-verbal way (by showing how, by entering the number 100 in the mother’s age column, it is possible to find directly the ages ratio). but she means the inverse (finding the mother’s age from the ages ratio). Surprisingly, Matan understands now Shula’s concern through this non-verbal communication. Therefore, through successive interactions with the spreadsheet, Shula challenges the acceptability of explanations reached through automated actions. It is possible in fact that this challenge has been made possible because Shula has been freed from computation burdens and does not value actions that the tool turns to automated. She then seeks for a more acceptable way to explain algebraic evidence. She interacts with Matan to renegotiate this new norm. This happens through discussion, reference to spreadsheet displays and the use of a spreadsheet display and action to communicate a new sociomathematical norm, what is an acceptable explanation when working with the spreadsheet tool.

We turn now to the intervention of the teacher (T) who is ‘wandering’ among the different pairs of students, passes by our two girls to find herself in the middle of the concern that bothers Shula:

T: What are you doing there?

S: I don’t get it.

T: What happens? 1.5 you succeeded?

S: Yes but we go on copying.

T: Now what do you need to find?

M: The problem is that we copy and that we do not lean on . . . [pause] We rely on predecessors

T: What’s the problem?

M: It’s that we don’t have the formula

T: Each number . . . [The teacher points on the computer screen] all these, you entered them by hand?

M: No, but you know

T: The numbers, you entered them one by one, 90, 91, 92?

M: We copied . . .

Two phenomena are salient in this excerpt. First, the two students, who are used to collaborating on the computer, act as a single person in their

interaction with the teacher. Matan explains the problem to the teacher better than Shula! Secondly, the teacher is not part of the debate (and in this particular situation she cannot even be aware of what she had missed). She tries to coinvestigate the problem with the students, but *fails to detect the student's 'trouble'*. Typically with work in small groups, *the teacher has difficulties in reconstituting the missing links*. This is a general phenomenon also linked to computers: many representations, and symbols are produced to reinforce meaningfulness, but at the same time they create 'opportunities for disconnection' because the interpretation of an outsider is problematic.

5. DISCUSSION AND CONCLUSIONS

In the three examples studied in this article, we used the emergent perspective to describe classroom activities in rich environments. We identified the negotiation of socio-mathematical norms and the change in mathematical beliefs. The conditions under which the norms were negotiated significantly departed from the studies of Cobb, Yackel and their colleagues. So, in addition to the verbal interactions among students and between students and teacher we considered interactions between students and tool and students and the multi-phase activity.

All problem-situations are rooted in translation practice; computer displays and actions are designed to show parallelism between representations. In fact, after a short sequence of activities, students believe that the three representations (algebraic, tabular, graphical) are three ways to organize and display the same entity. In the Growing Rectangles, this belief is not challenged, but instead, the intuitions about linear, quadratic and exponential growth are reconstructed. The interactions with computer displays and actions in multi-representational software in problem-situations, allow the emergence of the norm 'what count as evidence for a phenomenon', and provides an opportunity to rethink hypotheses in light of that evidence.

The Growing Rectangles activity illustrates another phenomenon: the teacher invites students to practice hypothesizing. In this early activity she does not try to discuss the value of the hypotheses, but by voting in favor of different hypotheses she gives legitimization to the possibility of having wrong hypotheses, and respect for the process of hypothesizing. When, several months later, she discusses the hypotheses raised while solving Overseas, this discussion is rooted in practice and in students' experience. She can then negotiate with students what constitutes a good hypothesis.

The discussion with the teacher in the Overseas activity, occurs at the end of a series of phases including homework, collaborative work in small

groups with software and report. Thus, the above social constructs form, not only as a result of interactions with the teacher or the tool, but as a result of the engagement in activities within the rich environment. The multiple phases were occasions for the formation of new norms and of new beliefs about mathematical objects.

The Age problem shows that, through recurrent interaction with a spreadsheet program, Shula feels that she can't accept findings and explanations obtained in an inductive mode only. She convinces her peer, Matan, through the presentation of an action on the spreadsheet (a non-verbal communication with the tool). They feel that they need a more algebraic form of explanation. We can see that the interaction with tools and with activities organized in multiple phases had another impact. Students freed from technical burdens, could concentrate on higher level activities.

This study raises a number of additional issues. For example, the Age problem raises a dilemma concerning the role of the teacher and its limitations in this learning situation. The teacher who was not party to the debate between the pair of students, does not grasp that the students are negotiating a norm of acceptability for inductive findings and explanations. The teacher who misses links, among the plethora of signs, displays, and actions that are present on their computer screen, is unable to continue the negotiation initiated by the two students. There are, of course additional dilemmas and questions to be asked that were not discussed here, such as: If we believe that a creation of the above kind of norms is desirable and can be planned by teachers, how to make the teachers aware to this way of teaching? To what extent the above classroom research, in which we documented a few episodes of what we saw as an emergence of norms show that they are becoming normative for the whole class? Another issue is: Do rich environments have the potential to enable the formation of non-desirable norms? This crucial question is not discussed here. We hope to deal with such issues in our research program that accompanies the implementation of the CompuMath project in a variety of classrooms and mathematical topics.

NOTES

- ¹ The terms 'represent' and 'representation' used in this article designate tools with the help of which it is possible to display embodiments of mathematical objects. These terms are not used here to mean mental constructs.

ACKNOWLEDGEMENTS

We would like to thank Maxim Bruckheimer for his thorough reading of an early draft of this paper. We are very grateful to Alan Bishop and Paul Cobb

for their challenging comments that made us think on future directions in this research.

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