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An analytical model for studying the development of learners' mathematical ideas and reasoning using videotape data[☆]

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Abstract

We review the literature on videotape methodology for observational research in mathematics education. We organize the review by presenting issues related to data collection, ethical concerns, data analysis, tapes as data versus transcripts as data, and research presentation. To address a gap we perceive in the literature, we propose a model for analyzing data in the context of investigations into the mathematical work and growth of thinking of students engaged in mathematical inquiry. The model we propose is based on nearly two decades of research experiences in the Robert B. Davis Institute for Learning, Graduate School of Education, Rutgers University, New Brunswick, NJ. © 2003 Elsevier Inc. All rights reserved.

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This paper describes the theoretical basis of a model for analyzing videotape data, outlines each phase of the model, and provides examples of the model in action. The model is based on longitudinal, cross-sectional study, now in its sixteenth year and sponsored mainly by the National Science Foundation,¹ on the development of mathematical ideas of a focus group of students (Davis & Maher, 1990, 1997; Maher, 2002; Maher & Martino, 1996a; Maher & Speiser, 1997). Through teaching experiments designed to create classroom environments in which sense making is a cultural norm, researchers

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engaged students in coherent strands of mathematics, including algebra, combinatorics, probability, and mathematical modeling. In the course of these experiments, a particularly striking outcome of the culture of sense-making has been the emergence of argumentation, justification, and proof-making in the students' discourse.

Mathematicians and educators who have viewed video-recordings of classroom interactions from this project have commented that they had previously not believed that children so young could reason mathematically with such sophistication. Throughout the 16-year span of research, the Robert B. Davis Institute for Learning has built up a rich archive of photographic, audio, and video recordings. We have progressively refined our methodological and interpretive approaches and have developed an evolving model for analysis of video data. These refinements, moreover, have occasioned revisits of archived data to make deeper analyses. Consequently, we have detailed accounts of the development of mathematical ideas in individual students over many years, which are enhanced as we gather new data.

Our longitudinal research project has several goals: (1) to study in detail the development of mathematical ideas in students over several years; (2) to provide in-depth case studies of the development of justification and proof making in students; (3) to investigate the relationship of students' earlier ideas and insights to later justifications and proof building; (4) to trace the origin, development, and use of representations of student ideas, explorations, and insights relating to explanation, justification, and proof building. Also, within the context of the learning community formed through our project, we have additional goals: (5) to investigate the nature of researcher intervention in the growth of student mathematical ideas; and (6) to study individual cognition in the context of the movement of ideas within the community of learners. Furthermore, we investigate questions that emerge from working with data at various phases of analysis.

To accomplish these sets of goals and address emergent questions, we have produced videotapes of students in classrooms and in after-school sessions as well as in a summer institute and individual or small-group clinical interviews. Before examining and commenting on our analytical model for investigating the mathematical work and thinking of learners engaged in mathematical inquiry, we review briefly some of the literature on videotape methodology for research in mathematics education.

1. Brief review of literature on videotape methodology

For decades, researchers in mathematics education have been using technology to capture and study audio and then audio coupled with visual images of teachers and students engaged in mathematical activity. According to Erickson (1992) inside and outside of educational research, the use of imaging technologies for studying interactions has intellectual antecedents in several analytic approaches. The approach of *context analysis*, which emerged at the beginning of the 1950s, involved the study of detailed transcripts of "cinema film of naturally occurring interactions" (p. 201). Roughly a decade later, the approach of *ethnography of communication* used audio and video recordings to examine the "moment-by-moment organization of the conduct of interaction" (p. 203). In the late 1950s, the sociologist Goffman, as Erickson reports, studied the presentation of self and, in part, used "still photography to glean insights on significant moments of interaction" (p. 203). In recent times, the capability of videotaping to record the moment-by-moment unfolding of sounds and sights of a phenomenon has made it a powerful and widespread tool in the mathematics education research community. Employing video as data, researchers have contributed fascinating descriptions of teachers and students in both clinical and classroom settings

involved in an array of mathematical tasks. Some descriptions have emerged from large-scale, video-based international surveys of classroom instruction, such as ones from the Videotape Classroom Study, a component of the Third International Mathematics and Science Study (TIMSS) (Stigler, Gonzales, Kawanaka, Knoll, & Serrano, 1999).

In the literature on the use of videotape data to inquire into students' mathematical activity, some authors discuss video-related methodological issues implicitly when reporting results of their research, while others do so explicitly, raising important methodological and theoretical issues concerning the use of videorecording in data collection, analysis and interpretation, as well as presentation and ethical issues. However, even though there is an increased and extensive use of audio and videotape data, it has been only recently that mathematics education researchers have begun to articulate explicit methodological and theoretical issues and questions pertaining to videotape in research (see, for instance, Cobb & Whitenack, 1996; Davis, 1989; Davis, Maher, & Martino, 1992; Hall, 2000; Lesh & Lehrer, 2000; Pirie, 1996, 2001; Roschelle, 2000). Despite the prevalence of video data, Hall (2000) claims that little is known and written about the use of videotape for "collecting, watching, and interpreting video as a stable source of data for research and presentation purposes" (p. 647).

In education, medical and social sciences, and other disciplines, videotape has become a popular medium for capturing and archiving data for both quantitative and qualitative researchers (Bottorff, 1994; Roschelle, 2000). Methodologically, video technology lends itself to a strict application or a mixture of qualitative and quantitative approaches in both data collection and analyses. A salient reason for this is, as Pirie (1996) observes in discussing video recording in mathematics education, that videotaping a classroom phenomenon is likely to be "the least intrusive, yet most inclusive, way of studying the phenomenon" (p. 554). In the next section, we present briefly ways in which researchers of mathematics teaching and learning have begun to theorize about ways of collecting, watching, and interpreting videotape data.

1.1. Data collection

Video is an important, flexible instrument for collecting aural and visual information. It can capture rich behavior and complex interactions and it allows investigators to reexamine data again and again (Clement, 2000, p. 577). It extends and enhances the possibilities of observational research by capturing moment-by-moment unfolding, subtle nuances in speech and non-verbal behavior (Martin, 1999, p. 79). It overcomes a human limitation of observation by being able to capture not just "part of the whole picture" (Martin, 1999, p. 76) and is better than observer notes since it does not involve automatic editing (Martin, 1999, p. 81). Quoting Grimshaw (1982), Bottorff (1994) notes two main potentials of videorecordings as a resource for research: density and permanence (p. 245). Density reflects the advantage of videorecording over an observer who, even with access to all that a camera sees, has difficulty monitoring different, simultaneous details of ongoing behavior (p. 246). Furthermore, from the perspective of density, videorecordings capture two data streams — audio and visual — in real time. Bottorff's notion of permanence will be discussed later in the data analysis section.

It is worthwhile highlighting a unique use and collection of video data in the TIMSS Videotape Classroom Study. Its goal was to understand how teachers construct and implement eighth-grade mathematics lessons in Germany, Japan, and the United States. As has been claimed, the study was the first time anyone had used video to collect national probability samples of anything and in this case of teaching (Stigler & Hiebert, 1999, p. 17; Stigler et al., 1999, pp. v and 2). Besides videotapes, other data types included teacher responses to questionnaires as well as textbook pages and worksheets corresponding to

lessons videotaped (Stigler & Hiebert, 1999, p. 18). In the end, the study produced a video survey of 231 eighth-grade mathematics lessons in Germany, Japan, and the United States of America.

Though video is a valuable methodological instrument for gathering data, it is not unproblematic. Davis (1989) discusses practical methodological issues when videotaping interviews that probe children's understanding and thought processes including scripting, pilot trials, group size and dynamics, and video technicalities. Bottorff (1994) lists three reasons why, like human observations, video data are incomplete: capable of selectivity because of mechanical limitations; incapable of discerning the subjective content of behavior being recorded; and usually unable to convey historical context of captured behavior (p. 246). Along similar lines, Hall (2000) cautions "against taking this new media as relatively complete, direct, or veridical" (p. 663) and argues that video data is technology and theory laden. That is, during data collection, selections are made from ongoing phenomena on the basis of technology used and theoretical interests. In turn, these realities both constrain and shape later analyses and presentation of results. Furthermore, video cannot capture everything. In aiming a video camera, researchers implicitly or explicitly edit and make sampling choices by focusing or not on particular events (Martin, 1999, p. 81). Pirie (1996) underscores this issue in the following way: "Who we are, where we place the cameras, even the type of microphone that we use governs which data we get and which we will lose" (p. 553).

The question arises whether researchers can ameliorate human and technological biases of video-recordings. Roschelle (2000) warns researchers that there are no simple ways to overcome biases of the medium and that "videos are a constructed record" (p. 726). Nevertheless, to acquire data amenable to rigorous research methods, Roschelle (2000) points out the importance of selecting appropriate video equipment; developing competent videography techniques, and planning and documenting systematic recording strategies consistent with clearly-defined research purposes. Further, Roschelle (2000) discusses technical and practical details in recording research-quality videos, including the use of pilot studies for improving videographic techniques and examining the effect of the camera on participants' behavior (pp. 726–727).

Recognizing that videorecording is both technology and theory laden, it is important to recognize that videorecording does not necessarily guarantee quality data collection and analysis. Researchers have suggested ways of ameliorating the necessarily narrow window into phenomena that videos offer by augmenting data sources. Pirie (1996) recommends coupling videorecording with students' written work in order to have a more inclusive examination of students' mathematical activity (p. 554). Lesh and Lehrer (2000) suggest that video data be combined with other data sources such as ethnographic observations, clinical interviews, and teaching experiments (p. 670).

Some researchers (for instance, Hall, 2000; Suchman and Trigg, 1991) recognize video as a construction rather than a representation of relationships. Hall (2000) problematizes data collection, encouraging caution in thinking of videorecordings as objective and theory-neutral data (p. 649). Both the possibilities and limitations of the video and audio technology as well as a researcher's theoretical perspective constrain and shape data records. He states that devices for capturing visual and aural images can be "deployed to record human activity in ways that make selections from ongoing interaction . . . creating data records that show just those parts of interaction we already find interesting and little more" (p. 659).

1.2. Ethical issues

Capturing information pertaining to individuals or groups of individuals with videos in both quantitative and qualitative research introduces a variety of ethical issues, including and beyond informed consent.

Essentially, the principle of informed consent implies that researchers insure that participants in an activity to be videotaped are fully informed and understand what it means to participate, that they realize the intended implications of having their voice and body images captured on video, and that they consent to the intended uses of the taped images. Roschelle (2000) suggests obtaining “progressive levels of consent” as they are needed. These include consent for “small research group use only,” “scientific conferences and meetings,” and “general broadcast via TV, CD-ROM, or computer networks” (p. 726). Whatever level of informed consent is required, it should be in a formal and written form, specifying who has access to the data and the use of the data.

However, informed consent does not necessarily protect consenters against a number of problematic situations. Consent is typically given before videorecording commences. Nevertheless, consenters and researchers may find themselves in a predicament. For instance, videographers may inadvertently record a participant performing an unbecoming behavior or, as Roschelle (2000) states it, “damaging material can be acquired accidentally” (p. 726). In his turn, Hall (2000) points out a further problem that researchers confront concerning appropriate uses of video as data obtained with support of public funds: the data as a public resource may be used in unanticipated ways by the public (p. 648). As he indicates, once accessible on the World Wide Web, stretches of videorecordings can be “repurposed in ways that undermine the entire research undertaking, regardless of the kind of surrounding details we attach to the records” (p. 662).

Ethical questions related to repurposing intersect with issues of validity. Some researchers attempt to provide readers of their reports access to the data upon which their reports are based. Specifically, within many research traditions, researchers are concerned with adequately describing both behaviors and their contexts so that readers can form their own judgment concerning a researcher’s analysis of what people are doing (McDermott, Gospodinoff, & Aron, 1978). To address this concern, researchers usually provide readers with a complete rendition of the transcript upon which an analysis is based. Moreover, since the advent of digitized videos and Web-based video transfer protocols, readers can also obtain digitized video segment corresponding to the transcript (see, for example, Koschmann, Glenn, & Conlee, 2000, p. 57, footnote 5). In this situation, even when videotaped participants have consented to public access to their recorded voice and body images, they often cannot be fully aware of and, therefore, consent in fully informed ways to how readers might repurpose a video segment that contains participants’ images. In general, repurposing raises serious issues for the extent to which informed consenters can be fully informed before consenting, especially given global access to streamlined video on the World Wide Web.

Who has access to captured information on video as well as in transcripts is an issue of confidentiality. The storage and disposition of the videotapes are also key issues of confidentiality. For Bottorff (1994), confidentiality and strategies to maintain it are as important as the usual concern for informed consent. Moreover, she insists that researchers must inform participants when videotaping is occurring and give them the option to interrupt or discontinue a taping session (p. 252). She even suggests that researchers should consider modifying identifying features to protect the identity of participants if otherwise research could not take place (p. 253).

1.3. Data analysis

Videorecordings offer many advantages for data analysis. As we mentioned earlier, Bottorff (1994) argues that a main potential of video data is its permanence. Stigler et al. (1999) note that live observations introduce significant problems to ensure that different observers record behavior in comparable ways (p. 3). Unlike the ephemeral nature of live observations, with videotapes, researchers can view recorded

events as frequently as necessary and in flexible ways such as “real time, slow motion, frame by frame, forward, backward,” and attend to their different features (Bottorff, 1994, p. 246). Similarly, Roschelle (2000) observes that video supports interpretations from multiple perspectives and offers the possibility for participants to assist in providing interpretations (p. 727). When examining the use of video to study the growth of mathematical understanding, Martin (1999) notes that videorecordings provide researchers with the possibility to make considered judgments and revisitings of the learning scene (p. 79) and borrowing from Erickson (1992), “reduce the dependence of the observer on premature interpretation” (p. 80). Roschelle (2000) indicates other potentials of video data for analysis such as interpretations from different, multidisciplinary frames of analyses and opportunities for participants to share their viewpoint concerning their behavior (p. 727). On this last point, viewing the other side of the coin, Martin (1999) observes that videotaping enables the researcher to interact with learners as they work and consequently test nascent theories (pp. 85–86). Roschelle (2000) refers to data reduction and sampling challenges with video data and mentions that computer software can be helpful in addressing these challenges. Significantly, for instance, videorecordings allow for in-depth examination of the developing mathematical work and thinking of the same students over several years as well as for the study and analysis of the cognitive growth of individual students in the setting of a social group (Davis et al., 1992; Maher & Alston, 1991).

Detailed analyses of longitudinal as well as short-term video data are made efficacious by multiple viewings. Video not only allows for multiple viewings but also for viewing from multiple points of view. According to Lesh and Lehrer (2000), a productive analysis of videotapes involves viewing through multiple windows or aspects, including theoretical aspects such as mathematical, psychological and teaching; physical aspects such as observers’ notes, transcripts, and videotapes from different cameras; and temporal aspects that include analyses of isolated sessions, analyses of group sessions, and analyses of similar sessions across several groups (p. 677). Further, they suggest going through “a series of triangulation and consensus-building cycles” to test and refine interpretations (pp. 677–678).

Repeated viewing has the potential to enhance triangulation in data analysis. Despite this ability, Alston and Maher (1990) note a limitation of data obtained from such episodes in that some children may be more verbal than others. They write, “follow-up interviews could provide insight into the nature of the uncertainty category as well as an opportunity to probe for meanings that are unclear or inconsistent in written statements” (p. 9). Other researchers have also constructed methodological procedures to boost triangulation in data analysis. In this regard, Maher and Martino (1996a, 1999) advance the notion of a “video portfolio” as a collection of different kinds of data centered at an episode or a series of episodes of interest. For them, a video portfolio contains (a) videotape “cuts” of the episodes, (b) documented episodes from videotapes that emerged from the analysis, (c) associated written work of students, and (d) researcher notes documenting the mathematical activity that researchers deem as a trace of the development of a mathematical ideas (Maher & Martino, 1996a, p. 202). Importantly, a video portfolio can provide a visual, aural, and written account of learners thinking about mathematical situations and the development of their ideas over time. A documentary example of the use of longitudinal videotape data to trace a student’s cognitive evolution from pattern recognizing to theory posing is contained in Maher and Martino (2000).

Some researchers transcribe video data, and their resulting transcripts constitute their analytic medium. This movement from video data to transcription is not without associated difficulties. Transcribing video data involves representing interactions. Researchers attempt to produce as veridical a representation of interactions by including representations of not just verbal but also of gestic interactions. However, a transcript is not a simple, universal category. Even though, it is impossible to render an exact, genuine

transcript of verbal and gestic interactions captured on videotape, it is possible to produce transcripts that, on the one hand, are “necessarily selective” (Atkinson & Heritage, 1984, p. 12) and “theoretically guided” (Erickson, 1992, p. 219) and, on the other hand, are nonetheless close approximations to being exact and genuine for particular research purposes. That is, transcripts can be more or less valid representations of interactions and their conventions depend on researchers’ analytic purposes (Erickson, 1992). Furthermore, representation systems for transcribing interactions are not uniform. For instance, the transcription system evolved by Jefferson (1984) has as its purpose to transfer to the page the sound and sequential positioning of talk. Whereas, the transcription convention used in the Video Classroom Study of TIMSS is designed to record speech only and not other behaviors that surround speech (Stigler et al., 1999, p. 161).

1.4. Tapes as data versus transcripts as data

A critical methodological issue with the use of videorecordings concerns whether either the tapes or the transcripts of recordings are the data upon which to base analysis. Pirie (1998) describes this interestingly as there are those for whom “*the data are the tapes*” and those for whom “*the data are the transcripts*” (p. 160). Each position has its merits and demerits. Digital video cameras certainly are the currently best available tool for capturing and preserving the moment-by-moment unfolding of phenomena, revealing as Davis et al. (1992) state a “great complexity within what was once thought of as the ‘simple’ world of ‘doing mathematics’ ” (p. 187). Superior to ethnographers’ field notes, tapes can make visible subtle nuances in speech as well as non-verbal behaviors. Although both require electronic equipment to review them, compact discs or DVDs of digitized video are less bulky than tapes. In contrast, transcripts are more portable than tapes and, unlike tapes, CDs, and DVDs, require no special equipment to access them. Indeed, with transcripts, Maher and Alston (1991) advance the idea that “careful analysis of videotape transcripts of children doing mathematics enables a detailed study of how children deal with mathematical ideas that arise from the problem situation” (pp. 71–72).

Nevertheless, many things are potentially missed in the movement from tape or CD to transcript. Citing Hammersley and Atkinson (1983), Martin (1999) notes that videotaping ironically can produce too much data and that transcribing makes it difficult to maintain contact with one’s theoretical perspective while sampling. On this point of data saturation, in a discussion of clinical interview methodologies, Clement (2000) observes that “difficulty with rich source data like videotape is that there is too much data to analyze in a meaningful way!” (p. 572). He remarks that an investigator must decide “what aspect of such a continuous stream of behavior are most relevant to the purpose and context of the study” and “what is relevant depending on the level of the research question in which he or she is interested” (p. 572). For meaningful analyses of extensive or non-extensive video data, Martin (1999) notes that Pirie (1996) advocates transcribing no tapes and instead working exclusively on the tapes (p. 82), a position held by other investigators, as well.

1.5. Presentation

Apart from analytical issues, both tapes and transcripts are useful for presentation purposes. From the position that Pirie (1996) advances where analyses ought to be based exclusively on tapes, transcripts of episodes of videorecordings are useful for presenting evidence for interpretations. A limitation of a written format for presenting analyses of video data is that video segments are usually unavailable to the reader. Researchers attempt to make relevant features of visual material accessible through transcription

and description but, often, these provide results that are not wholly satisfying. Research reports made available on the Web can circumvent this limitation. With advances in video-streaming technologies on the World Wide Web and hyperlinks, researchers can integrate clips of videorecordings into research reports, and readers with appropriate software resident in their computer can view these clips.

Videorecordings are appealing and helpful not only for written but also for oral communications. As Roschelle (2000) notes videorecordings allow “a researcher to make more direct connections between observable behaviors and interpretations” (p. 728). Yet the use of videorecordings in research presentations can be problematic given the temptation for researchers to showcase their “best case” instead of the “more typical performance” (p. 728). Video images can be powerful and persuasive. Stigler and Hiebert (1999) point out that “images produced by video can be too powerful, because they can focus attention on one striking example, even when the example is not typical” (p. 22). To ameliorate the tendency of video images to falsely portray typicality, they combine impressionistic images with coded, quantified video data (Stigler et al., 1999). Roschelle (2000) warns researchers against presenting video clips without first explaining to their audience contextual information and the criteria for selecting the clips. Without doing so, audiences may develop conflicting interpretations outside of the research context. To avoid such problems, he argues that the research community needs “to establish guidelines for presenting video clips” (p. 728).

2. An evolving analytical model in practice: various examples

We have just presented a review of issues related to the use of video data identified by researchers in education, in general, and in mathematics education, in particular. In our reading of the literature, despite the almost ubiquitous use of video technology in mathematics education research, we noticed essential voids in methodological discussions. Except for Erickson (1992), we found no discussion of criteria indicating research situations for which video data collection is useful. Furthermore, we have encountered little explicit elaboration of analytical methods for using video data for observational studies of the development of mathematical thinking. Though not explicitly working in mathematics education, some researchers have proposed models for audiovisual data collection and analysis coupled with participant observation (Erickson, 1992, and the Santa Barbara Classroom Discourse Group at the University of California at Santa Barbara).

In mathematics education, specific discussions of methods for analyzing video data are sparse. For inquiring into the teaching and learning of mathematics and science in classrooms, Clarke and his collaborators (Clarke, 2001b) have developed a qualitative analytic approach, called Complementary Accounts Methodology. A distinguishing feature of this approach is that a common body of videotape and interview data, which include the retrospective construal of events by participants, is analyzed from multiple theoretical frameworks (Clarke, 2001a), with no common methodological approach to the analysis of the video data. Pirie (2001) discusses how she and her collaborators in the “ μ -group” at the University of British Columbia undertake a common method for video analysis from multi-perspectives. In their work, they examine a set of data from several different theoretical perspectives that consider the growth of mathematical understanding. Cobb and Whitenack (1996) present their four-phase method for analyzing videorecordings and transcripts for case studies that they claim is consistent with the constant comparative method that Glaser and Strauss (1967) advocate.

In contrast to the paucity of models for video data analysis for studying mathematical cognitive development, reports (journal articles, conference presentations, doctoral dissertations, and course activities)

that have emerged from the Robert B. Davis Institute for Learning (RBDIL) at Rutgers University contain explicit and implicit pointers to a general analytical approach that can accommodate different theoretical frameworks. Our approach has developed over nearly two decades in an attempt to understand the development of mathematical ideas (Davis et al., 1992). It rests upon a longitudinal study, currently in its sixteenth year, on the development of mathematical ideas of a focus group of students (Davis & Maher, 1990, 1997; Maher & Martino, 1996a; Maher & Speiser, 1997). To understand how students think and reason about a collection of mathematical ideas, the research and data analysis typically lead analyses of individual learners either in the context of clinical interviews or working in groups, constructing mathematical knowledge (Davis et al., 1992; Maher & Speiser, 1997; Speiser & Walter, 2000).

Through the longitudinal study, our research group at the RBDIL attempts to understand the growth of mathematical understanding by examining temporally the discourse and inscriptions of students as they engage in mathematical inquiry. The theoretical underpinnings of this study come from three sources: research on the development of mathematical ideas (Davis, 1984; Davis & Maher, 1990, 1997; Speiser & Walter, 2000), models of the growth of understanding (Pirie, 1988; Pirie & Kieren, 1989, 1994), and theories concerning the generation of meaning (Dörfler, 2000).

A critical prerequisite for using video to capture data is to have clear criteria for employing this data collection and analytical device. In agreement with Erickson's (1992) general criteria for investing resources of time and energy into analyses of interaction within an educational study, we consider ethnographic analysis of video particularly useful for research in mathematics education

when . . . events are rare or fleeting in duration or when the distinctive shape and character of . . . events unfolds moment by moment, during which it is important to have accurate information on the speech and nonverbal behavior of particular participants in the scene . . . when one wishes to identify subtle nuances of meaning that occur in speech and nonverbal action — subtleties that may be shifting over the course of activity that takes place. (pp. 204–205)

Using these criteria and particular ways of examining and analyzing video data can yield insights into explicit and implicit meanings of participants in an educational setting. Our analytical model for studying the development of mathematical thinking employs a sequence of seven interacting, non-linear phases:

1. Viewing attentively the video data
2. Describing the video data
3. Identifying critical events
4. Transcribing
5. Coding
6. Constructing storyline
7. Composing narrative.

This taxonomy of analytic phases that we propose is not meant to describe the way any particular researcher might or should proceed with analysis of video data but rather to put forth our hypotheses about appropriate phases of analysis.

Similar attempts to propose a research method using video data have tried to characterize the phenomenon under study (Erickson, 1992). In our analytical model, we view the development of mathematical ideas and reasoning as complex and non-linear processes. We inquire into particular manifestations of these processes such as learners' presentations of their mathematical ideas and reasoning in talk, inscriptions, and gestures. Nevertheless, our experience persuades us that it is ultimately a research issue

to determine the nature and contours of what constitutes mathematical ideas and reasoning. Our position invites researchers to decide in the context of their research the *important* aspects of ideas and reasoning to focus upon as well as the implementation sequence of the proposed phases.

Before discussing the phases of our model, we present a general sense-making, research tool: analytical memoranda. Our use of memoing extends beyond that described by some analysts (Charmaz, 1983; Miles & Huberman, 1994). In our model, as researchers watch, describe, code, and otherwise attend to their video data they continually write in a notebook or an electronic PDA or computer-based file — called an analytic notebook — about their emerging and evolving theoretic, analytic, and interpretive ideas; about annotative commentary of transcripts; about hypotheses concerning mathematical ideas and reasoning revealed in participants' discourse; about participants' use of inscriptions to communicate ideas among themselves and with others; about connections between and among codes; about themes exiting across codes; about larger divisions of categories; about an emergent central phenomena; about assemblages of narrative components; and so forth. In sum, as Creswell (1998) notes, these memoranda form preliminary hypotheses, jottings about emerging categories and connections between them (p. 241). These memoranda also produce an intermediate bridge between coding of the data and constructing a storyline as well as composing drafts of a narrative report. Moreover, during the describing phase, analytical memoranda can serve as a repository of interpretive, inferential commentaries that sometimes creep into descriptive accounts of video data.

We will illustrate each of our analytic phases with examples based largely on one video portfolio from our longitudinal research project. The video portfolio includes approximately one hour and a half of video recording of four students engaged in resolving a deliberately open-ended task, typical of our project tasks, and two researchers interviewing them about their work as well as the students' written work. The task comes from a strand of tasks in combinatorics that forms part of a larger, multi-strand collection of mathematical tasks that we have developed in the course of our longitudinal research project. How we engage students with the tasks reflects our perspective on learning and teaching. Key to this perspective is that knowledge and competence develop most effectively in situations where students work together on challenging problems, discuss various strategies, argue about conflicting ideas, and regularly present justifications for their solutions to each other and to the entire class. The role of the teacher–researcher, in our research perspective, includes selecting and posing problems, then questioning, listening, and facilitating discourse, usually without direct procedural instruction (Maher, 1998; Martino and Maher, 1999).

The task of our video portfolio — the Taxicab Problem — is presented in Fig. 1 in the exact form given to the four students.² The task employs the non-Euclidian context of taxicab geometry (Krause, 1986; Menger, 1952, 1979) to provide a landscape and mathematical structure to the combinatorial situation.

To work on the Taxicab Problem, we invited four students — Brian, Jeff, Michael, and Romina — seniors at the David Brearley High School in the working-class town of Kenilworth, NJ. These students have participated in the longitudinal study since its inception and, in the course of it, have worked on problems that lead them to build mathematical ideas that are similar to the underlying mathematical structure of the Taxicab Problem (see Powell, 2003). Consequently, in context of our research project, this problem was proposed as a culminating task with the following central guiding research questions:

1. How do learners come to understand the problem?
2. What mathematical ideas do learners generate?

² In the context of this problem, for detailed analyses of the students' mathematical ideas and forms of reasoning through their discourse and inscriptions, see Powell and Maher (2002, 2003) and Powell (2003).

The Taxicab Problem

A taxi driver is given a specific territory of a town, shown below. All trips originate at the taxi stand. One very slow night, the driver is dispatched only three times; each time, she picks up passengers at one of the intersections indicated on the map. To pass time, she considers all the possible routes she could have taken to each pick-up point and wonders if she could have chosen a shorter route.

What is the shortest route from the taxi stand to each point? How do you know it is the shortest? Is there more than one shortest route to each point? If not, why not? If so, how many? Justify your answer.

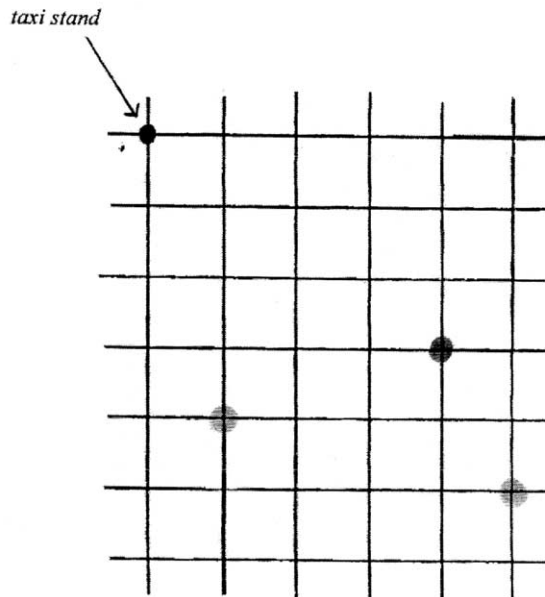


Fig. 1. The problem task. In the original, from left to right, the four identified intersections are colored black, blue, red and green.

3. How do learners generalize the problem and their solution?
4. How do learners construct isomorphisms between this problem and other problems on which they have worked?

In what follows, we use as an analytic lens for examining these questions the theoretical ideas of Dörfler (2000) concerning prototypes and protocols as tools for constructing meaning. In the subsection on coding, we present operational definitions of his three categories of prototypes and of his notion of protocol.

2.1. Viewing attentively the video data

To become familiar with the content of the video data, researchers watch and listen to the videotapes several times. In this phase, researchers watch and listen without intentionally imposing a specific analyt-

ical lens on their viewing. The goal is to become familiar with the research session in full. Depending on a researcher's general data collection and analytic framework, this phase may suggest, as in the case of, say, grounded theory, further data that ought to be collected (Charmaz & Mitchell, 2001; Corbin & Strauss, 1990). Similarly, in the case of stimulated recall, specific episodes may be selected for participants to observe and to reflect verbally (Davis, 1989) or, as in the circumstance of clinical interview, screening the video data may inform subsequent tasks in which to engage participants (Ginsburg, 1997; Haydar, 2002).

2.2. Describing the video data

Owing to the density of the medium, the use of video data often results in an enormous amount of information. For analytic purposes, this poses the challenge of not only familiarizing oneself with the content of the videotape data but also knowing it in fine detail. The previous phase is one, preliminary way of addressing this challenge. In that or a separate phase, as our model proposes, researchers note in an ethnographic-like fashion particular time-coded transitions of situations, activities, or meanings. For instance, with Pirie's "time activity trace" (Pirie, 2001, p. 348), researchers write brief, time-coded descriptions of a video's content. These could be descriptions of 2- to 3- or even 5-min intervals. It is important, however, that, at this phase of work, the descriptions are indeed descriptive and not interpretative or inferential. Researchers state what corporal actions and other movements can be seen as well as what utterances and other noises that can be heard. Pirie (2001) notes that instead of "inferential remarks such as: 'He is trying to ...' or 'She seems to have ...' or even 'A confusing diagram on the board ...'" that simple, factual descriptions are best: "'He writes ...', 'She says ...', 'The teacher draws ...'" (p. 349). In general, the idea is to map out the video data so that someone reading the descriptions would have an objective idea of the content of the videotapes. Indispensably, descriptions help the researcher become ever more familiar with the data set than one does by attentively watching and listening to the video record. The time-coded descriptions also allow the researcher to locate quickly particular vignettes and episodes. Indicating time codes, with the aide of mechanical timers or software devices, is especially useful for later locating particular video content. Software programs such as vPrism make it possible for researchers to bring together electronically video content, text, and time code.

In Fig. 2, the first 4 min and 44 s of videotape are described. The three time intervals are chosen to be small and thematic. In these intervals, the four students are seated around a trapezoidal table. After a researcher distributes the task, the students ask questions about the task and among themselves wonder why all efficient routes to a particular destination point have the same length.

2.3. Identifying critical events

By viewing and describing video data, researchers acquire a fairly in-depth knowledge of the content of their videos. Afterward, researchers move to the next phase of data analysis, which consists of carefully reviewing their tapes and identifying significant moments or, as we term them, *critical events* (Maher, 2002; Maher & Martino, 1996a, 1996b, 2000). Within our theoretical framework, to study the history, development, and use of learners' thinking over time, in agreement with Maher and Speiser (2001), we identify *events* as connected sequences of utterances and actions that, within the context of our *a priori* or *a posteriori* research questions, require explanation by us, by the learners, or by both. An event is called *critical* when it demonstrates a significant or contrasting change from previous understanding, a conceptual leap from earlier understanding (Kiczek, 2000; Maher, 2002; Maher & Martino, 1996a; Maher, Pantozzi, Mar-

Time Interval	Description
00:00:00-00:02:06	Researcher 1 pulls up a chair, sits down between two students on the right side of the table, thanks the group of four students (from left to right: Michael, Romina, Jeff, and Brian) for coming, distributes the Taxicab Problem, and asks them to read and see whether they understand it. Afterward, Researcher 1 stands up and, as she backs away from the table, she removes her chair. With his head bent downward, facing the problem statement, Jeff asks aloud whether one has to stay on the grid lines and whether they represent streets. Researcher 1 responds, “Exactly.” Romina, Brian, and Jeff discuss that 5 is the number of blocks it takes to reach the blue destination point and that different routes to blue are the same length as long as one doesn’t go beyond it. Brian says that they should prove it.
00:02:06-00:02:42	Researcher 1 walks back over to the table and asks the students for their understanding of the problem. Jeff says that the task is to find the shortest route while “staying on the streets.” Researcher 1 adds that it is about finding whether there is more than one shortest route. Both Brian and Romina agree. Researcher 1 goes on to say that if there is more than one, they have to determine how many shortest routes. Jeff inquires with Researcher 1 whether she is asking how many different shortest routes? She says that not only do they have to find the number of shortest routes but also that they will “have to convince us” that they have found all of them. Researcher 1 then walks away from the table.
00:02:42-00:04:44	Jeff asks for colored markers. Jeff, Romina, and Brian choose to each work on different destination points. Romina says that it is five blocks to the blue point. Brian suggests counting them and being sure. Jeff asks why the length of each route to blue is the same. Michael explains that to get the blue point one has to go “four down and right one” since one cannot go backward or diagonally. Romina asks how to devise an area for that. Jeff and Michael tells her that it’s not area, it’s perimeter with each segment of the grid considered as one unit.

Fig. 2. Content description without interpretation of the nearly the first 5 min of video data.

tino, Steencken, & Deming, 1996; Steencken, 2001), or, as Bruner (1960/1977) calls an intuitive mistake, “an interestingly wrong leap” (p. 68). Significant contrasting moments may be events that either confirm or disaffirm research hypotheses; they may be instances of cognitive victories, conflicting schemes, or naïve generalizations; they may represent correct leaps in logic or erroneous application of logic; they may be any event that is somehow significant to a study’s research agenda. By connecting sequences of critical events and further analyzing them, for example, using constant comparisons (Glaser & Strauss, 1967), researchers build narratives that initially are amalgams of hypotheses and interpretations and that in turn influence subsequent identification and analyses of critical events. In this sense, critical events and narratives co-emerge.

Critical events are contextual. An event is critical in its relation to particular research questions pursued. Thus, an instance in which learners present a mathematical explanation or argument may be significant for a research question concerned with students' building of mathematical justification or proof and, as such, be identified as a critical event. In contrast, a researcher concerned with the impact of teacher interventions on students' reflective abstraction or mathematical understanding might deem as critical those events that connect teacher interventions and associated student articulations of their thinking. However, the relation between critical events and research questions pursued also implies that researchers might identify events as critical that include negative instances of a hypothesis, instances of wrong leaps, and somehow significant to the study's research question. It is also interesting to note that critical events are similar to what Gattegno (1970, 1974, 1987, 1988), observing learners doing mathematics, calls moments of awareness and that these events or moments often compel researchers to reflect on their antecedent and consequent events.

Critical events are not only identified in the video record. Researchers may find critical events in non-video, artifactual material such as in students' inscriptions or in the written statements of a student journal. Afterward, researchers may review the video record to locate antecedent events that can be used to explain the identified critical event (for examples of this point, see Powell, 2003; Speiser & Walter, 2003).

Videorecordings greatly enhance the search and identification of critical events. Repeated watching, for instance, allows researchers to view the data as many times as necessary before deciding to flag a particular video episode as a critical event or to discard a previously chosen critical event. Shared viewing and collaboration with other researchers greatly enhance the quality and validity of identified critical events. The density of video can provide researchers with extensive data from which to select critical events. When studying, for instance, the development of mathematical ideas or the growth of mathematical understanding, a critical event is associated with a time line, and researchers may then search for related events in the past and in the future. If the related events are critical and lead to growth in understanding, then the set of critical events form what Kiczek (2000) defines as a *pivotal strand* or a *pivotal mathematical strand* (Steencken, 2001). Within a narrative such strands may emerge and point to, for example, the mathematical ideas and forms of reasoning learners develop that are key in building learners' mathematical understanding. Analytically, it is important to name the pivotal event, thereby indicating the narrative theme to which it belongs (Powell, 2003). Moreover, when analyzing the discursive interaction of learners, a critical event that qualitatively changes their inquiry trajectory, we call a *watershed critical event* (Powell, 2003). Such an event is often preceded by a series of related critical events that can be collected together as pivotal strand to indicate a discursive thread that begets the watershed critical event. In turn, the watershed event initiates a cascade of events some of which may be critical. The antecedent and consequent data may already exist and further data may need to be collected through subsequent investigations such as clinical or stimulated recall interviews. Digitized video on CDs or DVDs viewed with appropriate software such as vPrism help researchers to navigate through the data while looking for and pointing to related critical events. Some software allow researchers to juxtapose related critical events so as to highlight their relationships. Moreover, critical events are indispensable for a research narrative that discusses particular research questions in light of data. Critical events identified at this phase may provide evidence for findings described in the narrative itself.

Fig. 3 contains commentary about and the transcript of an identified critical event that relates to the second guiding research question (What mathematical ideas do learners generate?) and fourth question (How do learners construct isomorphisms between this problem and other problems on which they have worked?).

Time	Commentary on and Transcript of a Critical Event
00:55:31	Before this episode, Jeff had suggested to Romina that they count the number of shortest routes by starting with ones easier than the red and green destination points. Between them, they spent time establishing a method for counting. Starting with a point that is 2 blocks east and 2 blocks south of the taxi stand (forming a 2 by 2 sub-grid), they count the number of shortest routes of several nearby points and record their results in the taxicab grid. After working with their 2 by 2 sub-grid for which they found 6 shortest routes, they worked on a 3 by 3, finding 15 shortest routes. They also worked on 2 by 4, 2 by 3, and 4 by 3 sub-grids. In this way, they were controlling for variables, a heuristic that they had developed and employed in several other tasks in our longitudinal study. At the start of this episode, Michael is double-checking that the number of shortest routes for the 3 by 3 sub-grid is 20. Meanwhile, Brian announces to Romina that he and Jeff have verified that 15 not 12 is the number of shortest routes in a 4 by 2 sub-grid. When Romina notes that 15 must also be the number of shortest routes for a 2 by 4 sub-grid, she voices her implicit awareness of a symmetrical property of the numerical pattern of shortest routes that she and Jeff have developed. Moreover, she observes that the pattern corresponds to Pascal's triangle. Nevertheless, Romina is concerned about their datum for the 3 by 3 sub-grid. Brian offers to recount the routes for that one, using his method of "down and over." However, Michael had been counting the shortest routes for the 3 by 3 sub-grid and now states that he found 20. Suspecting that this confirms that they have Pascal's triangle, Jeff states, "why does Pascal's triangle work for this is the question." This event is critical since it points to key mathematical ideas that the students generate as well as heuristic and content connections they make to other problems. This event illustrates that the students seek to understand and explain reasons why Pascal's triangle underlies the mathematical structure of the Taxicab Problem. In addition, at the end of this episode, Romina suggests that they relate (find an isomorphism between) this problem to the Towers Problem, a problem they have already met and resolved. BRIAN: Did you figure out the five by five? MICHAEL: Five by five? I'm doing three by three right now.

Fig. 3. This episode is identified as a critical event in relation to several guiding research questions.

01:00:16	ROMINA:	Did it again. You got twelve for this one? Fifteen I mean? [Romina changes the 9 in the second row to a 6.] You got twelve for this one? Fifteen I mean? [She rewrites the numbers on the grid and adds a 15 to the right of the 10 and under the 10.]
	BRIAN:	Which one are you expecting to be twenty? Three by three? [Romina nods, yes.]
	MICHAEL:	What are you guys all doing?
	BRIAN:	Checking.
	ROMINA:	I don't think- here- he has- He was just doing three by three wasn't he? [Romina looks through her papers.]
	BRIAN:	Yeah. It's no big deal.
	ROMINA:	I'm already stuck. [Brian draws a 3 by 3 rectangle on his paper. Romina draws in shortest routes for the "imaginary" 3 by 3 on her grid. Romina's pen stops when drawing a route.]
	JEFF:	You shouldn't be. Where you going?
	ROMINA:	Three by three. [She shows the paper to Jeff.]
	JEFF:	You said F making the- the [Inaudible].
	MICHAEL:	Yeah I got twenty for that one.
	JEFF:	For three by three?
	MICHAEL:	Yeah.
	JEFF:	Alright well then- I mean can't we explain why we think- well- alright. [Jeff waves his hand.]
	MICHAEL:	//They're going to ask us-
	JEFF:	//Alright then the next question is why- //why-
	ROMINA:	//Now-
	MICHAEL:	//How do you know-
	ROMINA:	//Just relate this back to the //blocks. [Jeff points to the grid on the transparency with his marker.]

Fig. 3. (Continued)

00:56:56	BRIAN:	Let's just agree. If we already know what it is then we have to figure out-
	MICHAEL:	I just want to make sure that's twenty. So- [Michael counts routes with his pen on his grid.]
	MICHAEL:	I'm missing two. That's probably right though.
	BRIAN:	Did you get the, uh, staircase one?
	MICHAEL:	Which one? For the three by three?
	BRIAN:	Yeah. [Inaudible]. [Romina returns.]
	ROMINA:	Oh, you guys went and wrote on this didn't you?
	MICHAEL:	I didn't do it.
	BRIAN:	Did Jeff tell you?
	ROMINA:	What?
	BRIAN:	That this one-
	ROMINA:	For which one?
	MICHAEL:	//For-
	BRIAN:	//Four by two.
	ROMINA:	So you did get fifteen? So now it's working? [Meaning that the pattern of shortest routes corresponds to Pascal's triangle.] And then the two by four has to be fifteen too. Now if we do three by three and that's twenty, then we're done.
	BRIAN:	Which are you doing?
	ROMINA:	What?
	BRIAN:	He said he was off by two. [Inaudible]. [Romina begins to erase the numbers on the grid transparency then takes a new transparency with a grid on it.]
	ROMINA:	I'll just turn this around.
	BRIAN:	It's only a couple of numbers. [Romina is writing numbers in the grid; first row is 2 3 4 5; second is 3 9; third row is 4; and fourth row has a 5.]

Fig. 3. (Continued)

	JEFF: //Wait- Why is this- why does the Pascal's Triangle work for this is the question.
	ROMINA: //Exactly. Relate it to the blocks. [The word "blocks" here refers to the Towers Problem.]

Fig. 3. (Continued).

2.4. Transcribing

One of the decisions researchers have to make when analyzing research data that includes recordings of participants' utterances and actions is whether to transcribe or not. The reasons to transcribe vary. Some researchers transcribe to provide evidence of students' assertions in a research report. Others transcribe in order to use a particular analytic approach that relies heavily on transcript data. There are also researchers who maintain that transcripts reveal important things not always visible otherwise. As [Atkinson and Heritage \(1984\)](#) note, transcripts are "necessarily selective" (p. 12) and, according to [Erickson \(1992\)](#), are "theoretically guided" (p. 219). It is therefore important to realize that it is impossible to render an exact, genuine transcript of verbal and gestic interactions captured on videotape. Nonetheless, it is possible to produce transcripts that are close approximations to being exact and genuine for particular research purposes. Transcripts can be more or less valid representations of interactions and their conventions depending on researchers' analytic purposes ([Erickson, 1992](#), p. 219). A useful transcription system is one based on the one evolved by [Jefferson \(1984\)](#) that has as its purpose to transfer to the page the sound and sequential positioning of talk. Such a transcript of discourse is from a hearer's perspective and presents tied sequences of utterances that constitute speakers' turns at talk and at holding the floor.

There are essential reasons to transcribe videotapes. First, following procedures within data collection and analytic traditions, researchers may implement an open-coding process on data to discover themes that are above, beyond, and beside those suggested by specific, *a priori* guiding research questions and deductive codes. The production of the transcript and the physical, static rendering of a research session affords researchers opportunities for extended, considered deliberations of talk and noted gestures. Second, researchers analyzing participants' discursive practices, especially their dialogue, find it useful to view the printed, sequential rendering of speech to see what it reveals about the mathematical meanings and understandings participants construct. Since discursive practices include actions that are not only utterances, researchers indicate in their transcripts relevant body movements as well as inscriptions (writings, drawings, sketches, and so on) that participants create. Third, transcripts are, for practical purposes, a permanent record and can reveal important categories that are not always capable of being discerned by viewing videotapes since, notwithstanding the technology of replay, the visual and aural video images that the viewer's mind eye and ear captures are essentially ephemeral. Instead with transcript data, one can consider more than momentarily the meaning of specific utterances. Fourth, researchers transcribe so that later, if and where appropriate in narrative reports, they can provide evidence of findings in the participants own words.

Transcripts allow researchers to perform synchronous coding with videotapes and other artifacts. As researchers code transcripts, they continually review corresponding episodes in the video record to perceive subtle nuances in speech and non-verbal behaviors as well as visible influences on patterns of behavior. The importance of transcripts notwithstanding, examining the video record is indispensable to analyze certain artifacts such as inscriptions since they are built in a layered fashion over time.

In our analytic model, researchers transcribe critical events to closely analyze elements such as language and flow of ideas as well as for presentation purposes (see Fig. 3). We also transcribe portions of video data, vignettes or episodes, that provide evidence for important theoretic or analytic matters relative to our guiding research questions (see Fig. 4). For whatever purpose a transcript has been produced, several viewers check it for accuracy. However, our analyses are not based solely on inspection of transcripts independent of direct reference to original video recordings. Some researchers find voice recognition equipment helpful and software such as vPrism, that allows researchers to attend simultaneously to the content of speech, the speakers' gestures, and the time frame of episodes.

2.5. Coding

With or without transcripts, coding is crucial to analysis of video data. This activity is aimed at identifying themes that help a researcher interpret data. In our model, this activity is similar to identifying critical events in that both require watching videos intensively and closely for long periods of time. At this phase of analysis, the difference is that researchers focus attention on the content of the critical events. Therefore, videorecording is helpful in this activity in much the same way as it is in enhancing the identification of critical events. Importantly, employing observational coding schemes decided upon prior to observations or videotape viewing may blind researchers and make it difficult to notice unanticipated behaviors. Nevertheless, as Alasuutari (1996) argues, coding is not theoretically "innocent" (p. 373). Like identifying critical events, coding is directed by researchers' theoretical perspective and research questions. Repeated and shared viewing, made possible by video technology, and the density of video data enhance researchers' ability to search and identify codes, whether these are predetermined or emergent.

Just as with critical events, codes are defined in relation to the research question pursued or emergent themes. Over the years of our investigation into the development of mathematical ideas, we have developed coding schemes informed by our assumptions about mathematical thinking and our research practices into the development of mathematical ideas and forms of reasoning. We have found it particularly useful and important to code for learners' mathematical ideas, mathematical explanations or arguments, mathematical presentations (symbolic, pictorial, and gestic), and features and functions of discourse. We have also refined codes related to several constructs such as critical events, trace, and the flow of ideas among learners in a group.

Inquiring into the development of probabilistic thinking, through coding data sets for critical events, Kiczek (2000) notices how these events are connected. In turn, she traces how particular probability ideas are built among her participants. The connected sequence of critical events leads to growth in understanding of particular probability ideas, and Kiczek (2000) develops the construct of pivotal strand to describe this phenomenon. In tracing the growth of understanding of fractions among a class of fourth-graders, Steencken (2001) codes for learners' representation of fractions with Cuisenaire rods. When in comparing fractions, Meredith builds two different models to support her reasoning, Steencken documents the emergence of the idea of equivalent fractions, traces retrospectively the origins of Meredith's ideas, and then follows the flow of Meredith's ideas among her peers (Steencken, 2001; Steencken & Maher, 2003). Analyzing the contribution of participants, discursive practice on the mathematical ideas they build as they resolve the Taxicab Problem, Powell (2003) distinguishes connected sequences of critical events in which participants, whose actions form a pivotal strand, implement a new agenda for action that qualitatively changes their problem-solving activity, and thus he develops the construct of a watershed critical

Example 1 : Figurative and Relational Prototypes

	Code	Commentary and Excerpt of Transcript
0:01:30	FP	During the first few minutes, students are focusing on geometrical features of the taxicab grid that they deem relevant. The grid stands for an idealized system of perpendicular, two-way streets. Jeff notices that one has to stay on the lines and that they represent streets. Brian recognizes that to reach a destination point, one should travel either down (south) or over (east) and traces different paths to a particular destination point. Here students predominately functioning with figurative prototypes. Romina observes that to determine a shortest route to the red destination point that one must not travel beyond the vertical or horizontal projection of the point in question in order to achieve what we can call an efficient route. She also perceives a relationship among efficient routes to the red destination point: all five such routes have length five. At this point, she is engaging a relational prototype. Interestingly, Brian takes Romina's observation as a conjecture and suggests that they justify it. At this moment, they briefly engage an idea that is potentially an operative prototype. It appears that justification is a habit of mind that is part of the mathematical culture of these students. This engagement is interrupted by Researcher 1's inquire, which is not part of this excerpt, into what they understand the problem to be asking. Nevertheless, after they respond to Research 1, in the form of a question, Jeff reintroduces Brian suggestion. JEFF: You have to stay on the lines, right? Those would be streets? [Jeff waves his hand.] RESEARCHER 1: Exactly. JEFF: Alright. ROMINA: Like anyway you go- BRIAN: Pretty much, because look- ROMINA: As long as you don't go like past it. [They are tapping with their pens and tracing routes.]
		RP
		BRIAN: The first one- No.
		MICHAEL: Well what if you go to the last one-
		BRIAN: You can go all the way down and go over and go down

Fig. 4. Coding and constructing a storyline. Coding for Dörfler's prototypes and protocol. The commentary provides elements of a storyline.

		three and go over two.
	ROMINA:	Isn't it- Don't they all come out to be the same amount of blocks? [Jeff begins to draw.]
	BRIAN:	Five.
	JEFF:	Five?
	ROMINA:	Five? I got seven.
	JEFF:	Uh, which one- Yeah, we were both looking at the red one.
	BRIAN:	I'm looking at blue. [Michael counts by tapping his pen on the grid.]
	JEFF:	Yeah. [Romina and Brian count by tapping, while they talk about the problem.]
	ROMINA:	Oh, okay.
	JEFF:	Alright. I mean pretty much.
	ROMINA:	As long as you don't go like past it you're fine. So it's the same thing.
	BRIAN:	So let's prove it.

Example 2 : Operative and its associated Relational Prototype

Time	Code	Commentary and Excerpt of Transcript
		Coupled with Romina's earlier observation (see Example 1), the students also note that the shortest routes from the taxi stand to a destination point do not involve moving up (traveling north) or to the left (traveling west) and are of the same length. Taking up Brian's suggestion (see the end of Example 1), Jeff asks for someone to explain why the routes they believe to be shortest have the same length. Michael responds and explains that to reach any particular destination point, the shortest distances will always require one to move a fixed number of units down (south) and a fixed number across (east). He observes that one cannot travel diagonally. He generalizes this awareness for all paths, and Jeff becomes convinced. In the conversation, Michael engages an operational prototype. The number of units down and the number of units across are objects related by addition to produce the length of a shortest path.

Fig. 4. (Continued)

0:03:44	OP & RP	Interestingly, when Michael observes that in the context of this problem, one cannot travel diagonally, he touches upon the fundamental distinction between the metric of Euclidean geometry and that of taxicab geometry.	
		JEFF:	So why-why is it the same every time?
		MICHAEL:	You're going left and right. [Michael motions to her grid.]
		ROMINA:	Or four by one, right?
		MICHAEL:	Yeah, four by one, unless you go backwards a couple of times.
		MICHAEL:	You can't go diagonal so you have to go up and down. You can't //get around doing that.
		JEFF:	//You can't get it- Yeah.
		ROMINA:	What if I were to go like to the red when I go one, two, three, four- [pointing at paper]
		MICHAEL:	But they're not asking for like a //[Inaudible].
		ROMINA:	//Five, //six, seven.
		JEFF:	//Five, six, seven. //It's the same thing.
		ROMINA:	//Like //how- how am I going to- like //how would I-
		JEFF:	//It's the same thing.
		MICHAEL:	//It's the same.
		ROMINA:	-devise an area for that? Like this- this area up here? [Romina motions with her pen on her grid.]
		BRIAN:	Like plus and [Inaudible].
		JEFF:	Well, it's not area.
		MICHAEL:	It's not area. It's //just a-
		JEFF:	//It's the perimeter. It's like //each one being one.
		MICHAEL:	//One, two, three, four, five, six, seven. [pointing at paper] [Jeff scratches his head.]
		ROMINA:	Alright.

Fig. 4. (Continued)

		MICHAEL: There's no way you can get around going-
		JEFF: //Going seven blocks.
		ROMINA: //No, yeah, I understand.
		MICHAEL: And down that many because you can't go diagonally. Can't-
		JEFF: Yeah.
		MICHAEL: Can't get around it, so-
		JEFF: I mean, that's the most sensible way I think to say that. Right? And they want to know how many though.

Example 3 : Protocol

Time	Code	Commentary and Excerpt of Transcript
1:24:16		Researcher 2 told the students that their explanation did not make clear to him how and why the number of shortest routes to intersection points on the grid corresponds to the entries in Pascal's Triangle. Researcher 1 suggested that they take a couple of minutes to think about Researcher 2's question. There is evidence that Michael understood Researcher 2's question by his statement to the others: "He [Researcher 2] wants to know why...without saying 'oh it just follows a pattern'." As the students discussed why Pascal's triangle worked for the Taxicab Problem, Romina drew on Michael's representation of Pascal's triangle and filled in a transparency grid with numbers representing the number of shortest routes to intersection points. With assistance from Michael, Romina builds a protocol for an isomorphism between the Towers Problem and the Taxicab Problem. And so doing, she constructs a protocol for the Taxicab Problem, specifying why Pascal's Triangle relates to it. Romina informs Researcher 1 that they are ready to respond to Researcher 2's question and asks him to restate it. RESEARCHER 2: Uh, my question was you said that you found Pascal's Triangle and um, it wasn't clear to me that if you go, let's take- MICHAEL: You want a like reason why- how it relates? RESEARCHER 2: Yeah. ROMINA: Okay.

Fig. 4. (Continued)

1:24:40	P	MICHAEL: Not because it looks like it? You want to know why.
		ROMINA: Now we just picked any point. Let's say we picked this point. No matter how you get to this point-
		MICHAEL: Do the six one. The six one-
		ROMINA: Well we'll do the six and the four.
		MICHAEL: Alright.
		ROMINA: Okay, to this point you know you need to take at least- you have to take four moves. That's the shortest amount of moves because just like a simple one, two, three, four. So that means it's- let's say you we're relating back to this four moves equals four blocks. So I'd have to go down to the four-block area. So that's one, two, three, four. And now here you're going three across and one down. Or- so- [She is illustrating the moves on the taxi grid and pointing to the numbers on the grid and redrawn triangle.]
		MICHAEL: There's no possible way you could-
		ROMINA: //Do anything else.
		MICHAEL: //You have to- no matter how or which way you go you have to go three and then one.
		ROMINA: Right. In any move you're going one down and three across no matter- in any direction you take. So the three across and one down, that relates to three colors and then-
		MICHAEL: Of one-
		ROMINA: Three of one color and one of another. So you go and you look in here. Say- Okay, here's with all one color. This is with one of one color-
		MICHAEL: That's- that's nothing.
		ROMINA: No that's all one color but we're not using that because you can't all go all in the same direction. That's all one color. That's with one of one color and three of the other. So that's four and that's what we have and if you go down to here this is two and two and this is three and one which is the same thing. So there's your other four. And if you go to the sixth, the only way you can get there again is by four moves. It goes one- one, two, three, four. So you're in the four block again but this time you have to take, no matter what you do, you go two across and two down

Fig. 4. (Continued)

1:25:56		anyway you do it. So that would be two and two which is your six but you still make the four block [Inaudible]. [relating the taxi grid to Pascal's triangle]
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Fig. 4. (Continued).

event. Computer-related hardware and software devices can make flagging for codes easier and also help in documenting, storing, and managing codes and their content.

We have indicated that researchers are guided in the codes that they develop by their theoretical framework, their research questions, and the nexus of what they observe. Below is an example of specific codes, developed by [Walter and Maher \(2001, 2002\)](#). These inductive or emergent codes focus on identifying themes and patterns of student-to-student discursive interactions.

- (QA) Question that checks attunement between participants' understandings and seeks demonstrated mutual agreement;
- (QI) Interrogative question for information that is not procedural;
- (QP) Procedural question;
- (QC) Confirmation request by participant regarding participant's own conceptual understanding, differs from attunement by not demonstrating concern for "the other's" understanding;
- (QS) Speculative question that posits potential; and
- (QR) Rhetorical question.

In particular, these codes focus on the nature of student-to-student questions.

Our analytic model is also compatible with the implementation of deductive or *a priori* codes. The theoretical perspective of [Dörfler \(2000\)](#) for understanding the constructing of mathematical meaning offers an analytic lens for examining our video data. We illustrate this with the work of four participants' work on the Taxicab Problem (see [Fig. 1](#)). The use of Dörfler's constructs — prototypes and protocols — provides a useful tool for inquiring into our central research question (What mathematical ideas do learners generate?). As an illustration, we present below operational definitions of these constructs for our coding and analytic purposes.

1. Prototypes:

- (a) *Figurative prototype (FP)*: Learners are engaged with a figurative prototype when they focus their attention and exhibit interest exclusively in the physical or geometric aspect of an object as an instantiation of a particular idea. The object is the carrier of a figurative prototype for the particular idea.
- (b) *Relational prototype (RP)*: Learners are engaged with a relational prototype when they focus their attention on, exhibit interest in, or constitute (conceive) relationships among elements of an object as instantiations of particular mathematical relations. Relations are read *into* the elements of an object. The object is the carrier of a relational prototype for the particular idea.
- (c) *Operative prototype (OP)*: Learners are engaged with an operative prototype when they focus their attention on, exhibit interest in, or execute actions that use, transform, or produce an object. The object is the carrier of an operative prototype for the particular idea. The actions are connected to relational prototypes.

2. *Protocol (P)*: Learners build a protocol when observing, reflecting on, or describing the essential stages, phases, results, and products of activities, constructions, actions or flow of actions, including speech acts. Learners engaged with a protocol create inscriptions when they describe verbally or in writing (symbols, systems of symbols, including diagrams) their actions. The symbolization or carrier of a protocol makes visible aspects of the results of learners' cognitive process that may otherwise not be perceptible. Researchers have evidence of protocols in the carriers that learners produce. A particular protocol arises from a specific situation but might be used to describe other, more general situations.

Learners can establish protocols by observing others carrying out actions and, according to their interpretations, noting those actions. The resulting protocol evidenced by its carrier expresses their focus of interest and attention.³

In Fig. 4, we present analyses of an episode from the video data of the research session with participants working on the Taxicab Problem. Next to excerpts of the transcript, researchers code for the instances of the participants engaged with prototypes and protocol. Above the excerpt, researchers write commentary that discusses and justifies the identified material. At this phase, such commentary often manifests analytic threads of a narrative or a storyline.

2.6. Constructing storyline

A phase that often follows coding in our data analysis is that of identifying or constructing a storyline. The storyline is the result of making sense of the data with particular attention to identified codes (see Fig. 4). In our model, researchers examine closely and intensively identified codes and their respective critical events, trying to discern an emerging and evolving narrative about the data. This process may be as time consuming as identifying critical events or codes. At this analytical phase, data interpretation and inferences play important roles. Constructing a storyline requires the researcher to come up with insightful and coherent organizations of the critical events, often involving complex flowcharting. This process often involves discerning *traces*, which are a collection of events, first coded and then interpreted, to provide insight into a student's cognitive development (Maher & Davis, 1996; Maher & Speiser, 1997). The trace contributes to the narrative of a student's personal intellectual history as well as to the collective history of a group of students who collaborate.

The process of making sense of the critical events and codes is complex and, more often than not, nonlinear. Researchers may have to go back and forth examining critical events, codes and other non-video data such as participants' inscriptions and researchers' field notes. Some critical events or codes may be dropped and new ones searched for, as more evidence may be needed. Some researchers may wish to include participants or other researchers in data interpretations. Once again, much of the way in which video recording enhances the identification of critical events and coding applies here. Repeated viewing allows researchers the opportunity to continually refine their interpretations of a particular episode of video data. Shared viewing involving participants or other researchers, as well as the great detail that often accompanies video data can enhance the quality of the interpretations. Navigational tools of video technology allow researchers to search and juxtapose critical events in ways that highlight important

³ These coding categories are based on the work of Dörfler (2000): Means for meaning. In: P. Cobb, E. Yackel, & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms: perspectives on discourse, tools, and instructional design* (pp. 99–131). Mahwah, NJ: Lawrence Erlbaum Associates, and on personal communications with Willi Dörfler.

insights on students' mathematical thinking and understanding. This is particularly useful for longitudinal studies, where important evidence may be lost over time.

2.7. *Composing narrative*

Although in our model a narrative phase appears last, narrative and other interpretive actions actually begin from the inception of research. Researchers' questions as well as data-gathering procedures and media all imply explicit or implicit choices informed by open or hidden, conscious or unexamined theoretical perspectives. It is in this sense that the construction of a narrative begins at the initiation of research and accounts for why somewhere within a research report, researchers outline their theoretical biases. Notwithstanding sampling decisions that occur before and while gathering data, writing occurs in all phases of our research model even though our model announces that interpretive discussions and results occur after data coding. Obviously, the writing of analytic memoranda is an obvious example of researchers engaging in interpretation before this phase. Nevertheless, at this phase, researchers view the whole of whatever portion of recorded material from the data set to which research questions are being addressed. They decompose this whole into smaller segments, interpreting the smaller segments in light of the whole and then recompose the whole in light of a storyline and explore a particular interpretation of the whole using data as evidence, thereby producing a written narrative (Erickson, 1992). Even in the process of writing, the researcher is engaged in some form of data analysis, constantly revisiting the data and refining earlier interpretations. Furthermore, it is important to note that, as mentioned earlier, advances in video-streaming technologies on the World Wide Web and hyperlinks enable researchers to integrate clips of video recordings into reports accessible through the Web.⁴

2.8. *Conclusion*

We have reviewed some literature on the use of videotape in research on mathematics learning and teaching and we have outlined an evolving analytical model for use with video data when investigating the development of mathematical thinking. While reflecting on the model, the reader may recognize familiar stages of data analysis, only here referred to differently. While this may be true, our intent is to emphasize how research on the development of mathematical thinking is enhanced by video data and how our model takes advantage of this enhancement.

Another issue in the presentation of our model concerns its sequential form. We do not intend to suggest a fixed, immutable order in which analysis of video data must proceed. We do recognize that video-data analysis can follow a different sequence and that certain analytic phases may not be emphasized or included in the analysis at all. Some researchers may decide not to write descriptions of the content of videotapes as they watch them to familiarize themselves with the data. Instead, they might skip the describing phase and proceed directly to watching a tape and identifying critical events. Even when all phases occur in analysis, researchers may cycle back and forth revisiting other phases of the model in their attempt to generate insightful and coherent narratives of students' mathematical thinking.

It is important to note that our model rests on particular theoretical assumptions concerning mathematical thinking and the practice of research. Over the years, theories on the development of mental representations (Davis et al., 1992) on the growth of mathematical understanding (Pirie, 1996; Pirie &

⁴ This can be accomplished, for example, with the application, VideoPaper Builder 2, <http://vpb.concord.org/>.

Kieren, 1994) and even more recent ideas on mathematical meaning (Dörfler, 2000) have contributed to the development of this model, especially on issues related to deciding how to go about studying mathematical thinking. For instance, coding for mathematical representations rests on the assumption that “central to doing mathematics is the construction of external and internal (re)presentations” (Dörfler & Maher, under review). This implies examining carefully students’ exteriorizations of thinking, be it through gestures, speech, or written work.

Our analytic model also builds on assumptions about the practice of research. For instance, in contradistinction to research that focuses on investigating students’ misconceptions or what they otherwise do not do correctly, a distinguishing feature of questions entertained in our research program concerns understanding, as students are socially engaged in mathematical tasks, *what* mathematical ideas do individual students build and *how* do they employ their ideas so as to attain growth in their mathematical thinking. In our research program, as (Maher & Speiser, 2001) specify, the “what” is descriptive and the “how” narrative. Moreover, the “what” and the “how” perspective of our research program, and consequently of our analytical model, informs the codes we develop to flag critical events as well as even which events we consider critical. Furthermore, we recognize that research on such a complex phenomenon like the development of mathematical thinking is not a linear, unidirectional process. Rather, it is a complex, cyclic, and recursive process that requires multiple visits with data as well as collaborative exchanges among researchers with different analytical and theoretical foci so as to provide rich descriptions and narratives of the phenomenon studied (Maher & Davis, 1996; Pirie & Kieren, 1994). As already indicated, these research practices are greatly enhanced by the evolving analytic model we have presented.

In this article, we have focused on the use of video data in small-scale studies. Obviously, large-scale studies are also important and allow researchers to generalize results to larger populations. Nevertheless, we contend that they too must be based on careful, detailed analyses that are attentive to the complexity of the variables that affect learning and teaching of mathematics. For this reason we maintain that careful observational as well as other ethnographic studies must precede and inform large-scale studies. The model we propose is a step forward in developing an analytic approach using video data for observational investigations into the development of learners’ mathematical ideas and forms of reasoning.

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