

## **Variation, Covariation, and Functions: Foundational Ways of Thinking Mathematically\***

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In beginning this chapter we immediately faced a dilemma. There is nothing that can be called “*the* concept of function.” The phrase “concept of function,” regardless of its meaning, immediately calls into question whom we envision having it. Is it a mathematician, a teacher, a student, or a researcher in mathematics education? A student’s conception of function will not be as developed as that held by a mathematician, and a mathematician’s conception of function may not include detailed information that a math education researcher has about how students’ function understanding develops. Another dilemma in writing this chapter is that different researchers in mathematics education have had different conceptions of function and therefore held different norms for “students’ understanding of function.” Because there are many meanings and ways of thinking that various individuals and groups hold that could fit under the heading concept of function, we avoid speaking as if there is a standard, generally accepted meaning of function against which others should be compared. Instead, we specify the meanings and ways of thinking that we envision a person having a concept of function holds.

We organized this chapter into six parts to capture the broad swath of issues surrounding the idea of covariation as a foundation for function in mathematics; the ways that covariation can be conceived among students, teachers, and researchers; and implications of various forms of reasoning covariationally. Specifically, we (1) provide a brief overview of how conceptions of function evolved historically and the central role that covariation played; (2) clarify what we mean by variational and covariational reasoning and where these meanings came from; (3) examine research on students’ and teachers’ variational and covariational reasoning in selected areas; (4) comment briefly, with a covariational lens, on past research on students’ and teachers’ conceptions of function; (5) discuss curricular treatments of function, again with a covariational lens; and (6) outline possible directions for future research.

### **The Evolving Idea of Function in Mathematics**

In mathematics, different ways of thinking that encompassed the various meanings of function emerged beginning around 1000 CE and continue to evolve today. Boyer (1946), in line

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with Kleiner (1989), identified four broad eras in the development of mathematicians' conceptions of function: the eras of proportion, equation, and function (with the era of function itself spanning two eras). The era of proportion was characterized by nascent attention to motion, but people represented relationships between quantities geometrically, capturing generality by way of similarity. However, since representations were geometric, they could not represent motion explicitly, and thus they conveyed relationships statically. Boyer described the era of equation, enabled by Vieta's and Descartes' creation of algebraic notation, as being characterized by the use of equations to represent constrained variation in related quantities' values. For example, in writing  $2x + y = 1$ , mathematicians could envision variation in the value of  $x$  being constrained in relation to variation in the value of  $y$ —however much the value of  $y$  varies, the value of  $x$  must vary  $-1/2$  times as much for their sum to remain 1. People presumed tacitly that variables varied continuously. Boyer described the third era as characterized by explicit representations of a relationship between values of two quantities so that values of one determined values of another. Values of variables varied continuously, and relationships between them were defined by a formula or a graph. Function notation emerged during the third era. The fourth era, initiated by Dirichlet and continuing to today, is characterized by values of one variable being determined uniquely by values of another with “a precise law of correspondence between  $x$  and  $y$  [that] can be stated clearly” (Dirichlet, as quoted in Boyer, 1946, p. 13). Dirichlet offered the function  $f$ , defined over the real numbers by the rule  $f(x) = 0$  if  $x$  is irrational and  $f(x) = 1$  if  $x$  is rational, to clarify that a law of correspondence could be arbitrary and that functions could be highly discontinuous. The mathematical definition of function today follows Dirichlet's definition, but is stated in terms of Cartesian products and ordered pairs. Ideas of variation and covariation in variables' values no longer fit within today's mathematical definition of function.

Kaput (1994) argued that emerging conceptions of quantities' values varying continuously were central to the emergence of calculus as a body of thought. We see Kaput's argument as also supporting the claim that emerging conceptions of continuous covariation were central to the development of the mathematical idea of function. They certainly were explicit in mathematicians' reasoning during the first three Kleiner eras. The idea of continuous variation was at the heart of Newton's mathematics. He spoke specifically of quantities flowing from one value to another and of quantities becoming particular values, and he stated his image of continuous variation explicitly in his formulations. For example, he introduced his demonstration of finding the rate of change of the product of two varying quantities by saying,

Let  $X$  and  $Y$  be two variable Lines, or Quantities, which at different periods of time acquire different values, by *flowing* [emphasis added] or *increasing continually* [emphasis added], either equably or alike inequably. For instance, let there be three periods of time, at which  $X$  *becomes* [emphasis added]  
 $A - \frac{1}{2}a$ ,  $A$ , and  $A + \frac{1}{2}a$  . . . . (Newton, 1736, p. xiii)

Similarly, Euler and Leibniz spoke of changes in one variable producing changes in another variable. They also presumed tacitly that all variation is continuous. However, due to advances in functional analysis and the emergence of abstract algebra, wherein Dirichlet's correspondence conception of function became central, variational ways of thinking became less prominent, to the point where it was considered a legacy way of thinking and something that should not be used. Young, Denton, and Mitchell's position on the meaning of “variable” reflected the movement away from variational thinking in modern mathematics:

A variable is a symbol which represents any one of a class of elements. This definition is very general; the elements of the class may or may not be numbers. *The usual definition of a variable as being a number which varies is not only unnecessarily restrictive, but has the defect of introducing an extraneous and rather vague notion, that of varying or changing* [emphasis added]. (Young, Denton, & Mitchell, 1911, pp. 192–193)

To a large extent we agree with Young et al. that thinking of a variable as having a value that varies is not productive for thinking about mathematical structures like groups, rings, fields, graphs (as in graph theory),  $L^p$  spaces, and so forth. At the same time, we see Young et al. (1911) as having “thrown the baby out with the bathwater” when they advocated for the meaning of variable only as a symbol that stands for an element of a set. Unfortunately, the set theoretic meaning of variable traveled throughout advanced mathematics, into calculus, and into school mathematics—becoming the accepted foundation for school definitions of function in many countries (Cooney & Wilson, 1993). Although the correspondence meaning of function solved problems that arose for mathematicians, introducing it in school mathematics made it nearly impossible for school students to see any intellectual need for it. We use “intellectual need” in the sense of Harel (2013). Students experience intellectual need for an idea when their meanings are inadequate for a situation they have conceptualized, and a new meaning solves their problem by bringing coherence to their understanding of the situation.

We emphasize that continuous covariational reasoning, or reasoning about values of two or more quantities varying simultaneously, played a crucial role in mathematicians’ invention of concepts that led to the modern definition of function, from the use of equations to represent constrained variation to explicit representations of deterministic relationships among quantities. The crucial role that continuous covariational reasoning played in the development of mathematics does not imply that variational and covariational reasoning are important for students’ mathematical learning. That stance requires a separate argument, which we provide. We argue that ideas of continuous variation and continuous covariation are epistemologically necessary for students and teachers to develop useful and robust conceptions of functions. Put another way, we argue that variational and covariational reasoning are fundamental to students’ mathematical development. We ground this claim in research that highlights difficulties students experience regarding function relationships by not having the ability to reason variationally or covariationally and in research that shows productive shifts in teachers’ and students’ conceptions and uses of function that result from reasoning covariationally.

### **Variational and Covariational Reasoning as Theoretical Constructs**

As we outlined in the introduction, covariation became an explicit form of reasoning in mathematics beginning circa 1000 CE. Although covariational reasoning became explicit in mathematicians’ thinking, it was not an explicit mathematical concept. Rather, it was what Harel and Thompson (as explained in Thompson, Carlson, Byerley, & Hatfield, 2014) called a way of thinking—the “habitual anticipation of using specific meanings or ways of thinking in reasoning” (p. 13). As a way of thinking, covariational reasoning was always tacit among mathematicians, and thus was never defined. Therefore, this section discusses various works that led to variational and covariational reasoning as theoretical constructs. It will end with a statement of the theoretical meanings of variational and covariational reasoning that we use throughout the remainder of the chapter.

Covariational reasoning as a theoretical construct appeared in the late 1980s and early 1990s in the works of Jere Confrey and Pat Thompson. Confrey characterized covariation in terms of coordinating two variables' values as they change. Thompson characterized covariation in terms of conceptualizing individual quantities' values as varying and then conceptualizing two or more quantities as varying simultaneously.<sup>1</sup> The two characterizations seem quite similar, but, as we shall explain, they have very different entailments regarding students' conceptions of variables and very different implications for characterizing students' conceptions of functional relationships.

Confrey's focus on coordinating changes in variable's values stemmed from her attention to the history of meanings for ratios and for exponential growth, which crystalized in her and her colleagues' characterizations of pedagogically powerful approaches to supporting students' learning about exponential functions (Confrey, 1991, 1992; Confrey & Smith, 1995; E. Smith, Dennis, & Confrey, 1992). Confrey and Smith (1994) distinguished a covariation conception of function from a correspondence (modern mathematical) conception:

A covariation approach, on the other hand, entails being able to move operationally from  $y_m$  to  $y_{m+1}$  coordinating with movement from  $x_m$  to  $x_{m+1}$ . For tables, it involves the coordination of the variation in two or more columns as one moves down (or up) the table. (Confrey & Smith, 1994, p. 137)

They emphasized the centrality of sequences in their meaning of covariation: "In a covariation approach, a function is understood as the juxtaposition of two sequences, each of which is generated independently through a pattern of data values" (Confrey & Smith, 1995, p. 67). Confrey and Smith clarified, however, that they were not entertaining the idea that students should scan through columns of a table independently of the other columns: "The elements and structure of the domain and range are cogenerated through simultaneous but independent actions, creating a covariation model of function" (E. Smith & Confrey, 1994, p. 337). Figure 13.1 illustrates an operationalization of covariational reasoning that Confrey and Smith mentioned. They envisioned that changes in one variable (e.g., increasing the value of  $x$  by 1) are coordinated with changes in another variable (e.g., increasing the value of  $y$  by 2, or changing the value of  $y$  by a factor of 3).

| x   | y   |
|-----|-----|
| 1   | 3   |
| 2   | 5   |
| 3   | 7   |
| ... | ... |
| 10  | 21  |

| x   | y     |
|-----|-------|
| 1   | 3     |
| 2   | 9     |
| 3   | 27    |
| ... | ...   |
| 10  | 59049 |

Figure 13.1. Two examples of Confrey covariation: Changes in one variable are coordinated with changes in another.

Confrey and Smith (1994) noted that students who successfully take a covariation approach to building a functional relationship are positioned to develop a rule of correspondence (e.g., by looking across rows vertically in a table to search for an invariant relationship between columns). But they did not address an important issue: how one might think about what happens between entries in a table.

Thompson's meaning for covariation stemmed from a different source than Confrey's. Thompson's primary concern was to understand ways students conceive situations as composed

of quantities and relationships among quantities whose values vary and ways students conceive rate of change (Thompson, 1988, 1990, 1993, 1994a; Thompson & Thompson, 1992). However, “quantity” to Thompson was not the same as number. He defined a quantity as someone’s conceptualization of an object such that it has an attribute that could be measured. Numbers can be measures of quantities, or they can be abstracted from operations of measurement and seem bare. But numbers are rooted epigenetically in reasoning about quantities’ measures.

Quantitative reasoning, in Thompson’s theory, is someone conceptualizing a situation in terms of quantities and relationships among quantities (Thompson, 1990, 1993, 1994a, 2011). Although quantitative reasoning is about conceptualizing situations, arithmetic is about evaluating quantities that one has conceptualized, and algebra is about representing the calculations one would do to evaluate quantities within a quantitative structure when there is insufficient information to calculate them arithmetically (Thompson, 1988, 1990, 1993, 2011). Formulas, in Thompson’s system, result from a person propagating a calculation (represented numerically or algebraically) that she has inferred about a quantity in a quantitative relationship to calculations one would make to evaluate quantities to which it is related (Thompson, 1988, 1990, 1993, 2011). Variation and covariation became necessary in Thompson’s theory of quantitative reasoning to explain the reasoning of students who conceptualized a situation quantitatively and at the same time took it as dynamic—they envisioned quantities in their conceptualized situation as having values that varied.

In Thompson’s system, a person could use a symbol to represent a quantity’s value with three meanings. If the person envisions the quantity as having a value that does not vary ever, the symbol, to that person, has the meaning of a constant. If the person envisions the quantity as having a value that can change from setting to setting but does not vary within a setting, the symbol, to that person, has the meaning of a parameter. If the person envisions that the quantity’s value varies within a setting, then, to that person, the symbol has the meaning of a variable. Thompson’s meaning of parameter is related to Freudenthal’s (1983) and Drijver’s (2001) distinction between “sleeping” parameters and “dynamic” (slider) parameters. The difference is that Thompson spoke of how someone might conceive the quantities’ values that his symbol represents while Freudenthal and Drijvers spoke of parameters as symbols that represent numbers in function and equation definitions.

When someone conceptualizes a runner’s location in terms of a measure of the distance from a reference point, and envisions that distance varying, then, to that person, the runner’s measured distance varies. If the person lets  $d$  stand for the runner’s distance from the start, then, to this person, the value of  $d$  varies because he envisions the runner moving away from her start. Therefore, a variable’s variation comes from a person thinking, either concretely or abstractly, that the quantity whose value the letter represents has a value that varies. Different people can conceptualize the “same” situation (same in the eye of an observer) in terms of different quantitative structures, and different people can envision different ways in which a quantity’s value varies (e.g., discretely or continuously).

In Thompson’s theory of quantitative reasoning, a person reasons covariationally when she envisions two quantities’ values varying and envisions them varying simultaneously. Figure 13.2 depicts a person’s conceptualization of a runner who moves through space, always having a distance from a reference point, and who envisions elapsed time being measured as the runner moves.

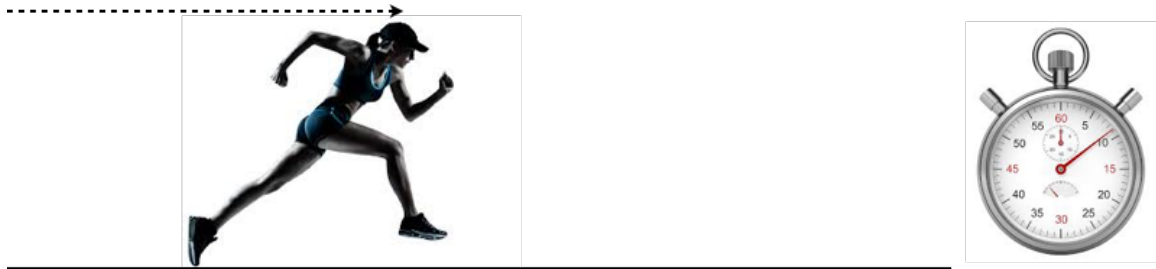


Figure 13.2. A person conceives of a runner's distance from a reference point as varying and of the elapsed time measured on a stopwatch as varying. Uniting the two in thought so that they vary simultaneously constitutes covariational reasoning.

We hasten to add that we do not intend that Figure 13.2 be seen as implying that someone is actually thinking of a stopwatch. Rather, it implies that she is consciously aware that elapsed time is being measured, so that the runner is some distance from her start at every moment of elapsed time. Saldanha and Thompson (1998) made this clear when they said,

Thinking of covariation as the coordination of sequences fits well with employing tables to present successive states of a variation. We find it useful to extend this idea, to consider possible imagistic foundations for someone's ability to "see" covariation. In this regard, our notion of covariation is of someone holding in mind a sustained image of two quantities' values (magnitudes) simultaneously. It entails coupling the two quantities, so that, in one's understanding, a multiplicative object is formed of the two. As a multiplicative object, one tracks either quantity's value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value. (p. 299)

Saldanha and Thompson (1998) also conjectured that covariational reasoning is developmental:

In early development one coordinates two quantities' values—think of one, then the other, then the first, then the second, and so on. Later images of covariation entail understanding time as a continuous quantity, so that, in one's image, the two quantities' values persist. An operative image of covariation is one in which a person imagines both quantities having been tracked for some duration, with the entailing correspondence being an emergent property of the image. (p. 299)

Marilyn Carlson also contributed to covariation as a theoretical construct. Her entrée was different from either Confrey's or Thompson's. As part of her interest in studying students' conceptions of functions, Carlson (1998) conducted a cross-sectional investigation of college algebra and calculus 2 with graduate mathematics students that included tasks prompting them to construct a graph to represent how the values of two quantities change together in a dynamically changing event. One item (Figure 13.3), similar to the well known flask task (Bell & Janvier, 1981; Janvier, 1981), asked students to construct a graph of the water's height in a bottle in terms of the amount of water in the bottle. Carlson reported that most high-performing students (grade of A) in calculus 2 were unable to construct an appropriate graph (Carlson, 1998, pp. 138–139). She further documented that students' static view of dynamic situations contributed to their inability to construct meaningful formulas to represent one quantity as a function of another.

Imagine this bottle filling with water.  
Sketch a graph of the height as a function  
of the amount of water that's in the  
bottle.

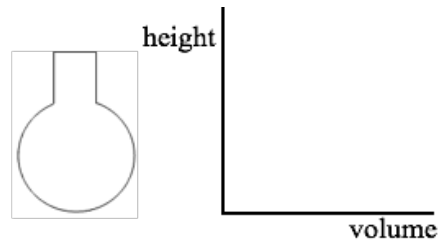


Figure 13.3. The Bottle Problem. From “A Cross-Sectional Investigation of the Development of the Function Concept” by M. P. Carlson, in J. J. Kaput, A. H. Schoenfeld, and E. Dubinsky (Eds.), *Research in Collegiate Mathematics Education*, 3, CBMS Issues in Mathematics Education (Vol 7, pp. p. 123), 1998, Washington, DC: Mathematical Association of America. Used with permission.

Puzzled as to how to characterize the various ways of thinking that students exhibited, Carlson drew from Confrey’s and Thompson’s earlier work to create a framework for analyzing the covariational reasoning that calculus 2 students employed in her 1998 study (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). Carlson et al.’s (2002) framework elaborated Saldanha and Thompson’s (1998) conjecture that covariational reasoning is developmental. They specified levels of mental actions and levels of competence that become more sophisticated in regard to the nature of a student’s coordination of quantities’ values, such as by attending to amounts or direction of changes in quantities’ values. Their framework extended Saldanha and Thompson’s meaning of covariation to include students’ coordination of the average and instantaneous rates of change of one quantity with respect to another quantity, as the two quantities’ values vary. Carlson et al. included rate of change in their covariation framework because of their interest in ways that students understood a function’s rate of change on successive intervals of a function’s domain. Because their data was collected in the context of calculus tasks, the authors were also interested in describing how a student might justify changing the concavity of a graph and constructing a smooth curve. They described students’ anticipating that for any fixed interval of change one might consider refinements of the function’s average rate of change on smaller and smaller intervals.

### Ways of Reasoning Variationally

The great majority of studies that cite Confrey, Thompson, or Carlson employ the construct of covariational reasoning to frame their investigation of an idea having covariational reasoning as its foundation. They do not contribute directly to defining the construct. Several publications by Castillo-Garsow, however, fleshed out the construct of covariational reasoning by focusing on the idea of variation itself. According to Castillo-Garsow (2010, 2012), students might conceive of a quantity’s value varying discretely (e.g., a car has traveled one mile, then two miles, and so forth, with no thought that the car moved between those markers) or continuously. Castillo-Garsow also distinguished between two conceptions of continuous variation, which he called chunky and smooth. Chunky continuous variation is a way of thinking that is similar to thinking that values vary discretely, except that the student has a tacit image of a continuum between successive values. What Castillo-Garsow called chunky continuous reasoning could also be called discrete continuous reasoning.

When thinking of variation happening chunkily, a student’s image entails intermediate values, but it does not entail an image of the quantity actually having those values. This image of variation is like laying rulers end to end and marking the endpoints. Castillo-Garsow illustrated

chunky continuous reasoning with this example from a teaching experiment on exponential functions (Castillo-Garsow, 2010). Tiffany (age 15) was an algebra 2 student.

*Pat:* If I'm going sixty-five miles per hour, what does that mean?

*Tiffany:* That in one hour you've gone, you should have gone sixty-five miles.

*Pat:* Can I travel for just one second at sixty-five miles per hour?

*Tiffany:* No. You have to do . . . You have to do, um . . . Well, yeah, you could.

(Castillo-Garsow, 2012, p. 60)

Castillo-Garsow pointed out Tiffany's mention of completed chunks of time and distance, and attributed her hesitation about whether Pat could travel for one second at 65 mph to her chunky continuous reasoning. She knew that cars move continuously (i.e., without disappearing from one place and appearing at another) and that time passes continuously, having no breaks. But Tiffany envisioned measured time and measured distance in chunks. Thus, her hesitation was to revise her time chunk from one hour to one second. Tiffany seemed to think of an hour as a chunk of time that contained intermediate times, although she did not conceptualize that traveling at 65 mph for one hour as meaning that one passes through all moments in time within an hour.

Castillo-Garsow (2012) contrasted Tiffany's chunky continuous reasoning with that of Derek, another algebra 2 student. Derek reasoned about variation in terms of change-in-progress, which Castillo-Garsow called smooth continuous reasoning. He gave the following example, embedded in his discussion with Derek about a bank's policy for reporting the value of Patricia's account. The account gains 8% interest per year, compounded annually, and the bank's reporting policy is to compute current account values by including the simple interest gained up to that moment within a compounding period. The account opened with an initial deposit of \$500. Derek is explaining how to think about the value of the account over time.

*Derek:* It's growing constantly, but once it gets to one year, it's a total of eight percent higher. And then it grows by still eight percent higher than the five hundred, but just takes that value and it gets up to there each year.

*CG:* Okay, so what do you mean by it's growing constantly?

*Derek:* It's always more money is being put in, because (pause) and it keeps going.

(Castillo-Garsow, 2012, p. 61)

Castillo-Garsow described Derek's imagery as involving change-in-progress:

Derek imagines that the account is growing, in the present tense. That is, Derek imagines that there is a mapping from his own current experiential time to a year in Patricia's world, and that as he is talking, time "gets to one year."

(Castillo-Garsow, 2012, p. 61)

Another way that Derek revealed the power of his thinking was in the way he could envision the growth of an account for which interest compounded continuously. Derek understood quickly the scenario offered by Castillo-Garsow that a bank pays 8% per year simple interest on every dollar or part of a dollar in the account beginning the moment it enters the account. At Castillo-Garsow's request, Derek sketched a graph (Figure 13.4) of the account's rate of change in dollars per year in relation to the number of dollars in the account. It was continuous and linear. Derek explained the graph's behavior (but not its linearity) by saying, "As long as your (pause) the money in your account is growing, then so will the rate of growth grow. So then it will just keep going up" (Castillo-Garsow, 2010, p. 169). Derek had conceptualized a relationship between account value and the rate of change of account value per year. His

relationship had the property that the account's rate of change, in dollars per year, is proportional to the number of dollars in the account. Derek went on to sketch a qualitatively accurate graph (not shown here) that displayed how the account's value would change with the passage of time. He guided his graphing activity by always keeping in mind that as the account value grows with time, its rate of change grows with respect to time so that it remains proportional to the account's value.

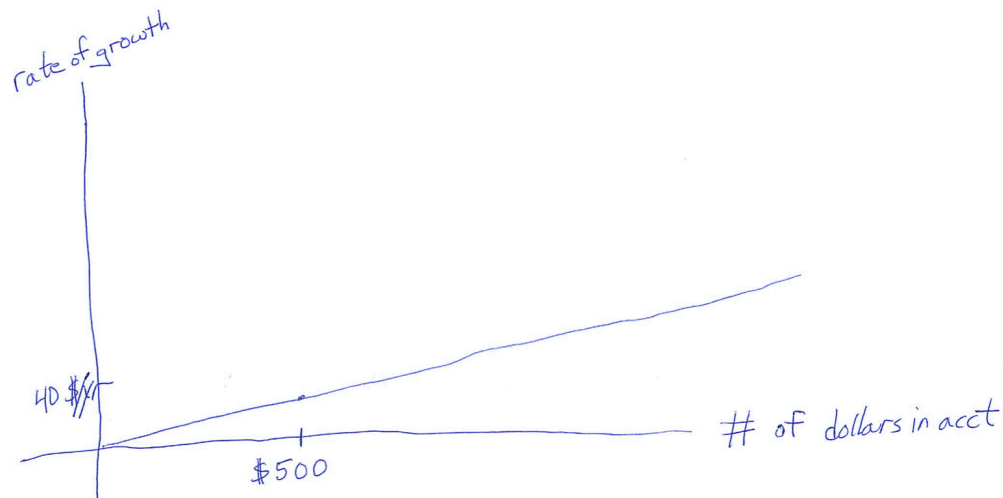


Figure 13.4. Derek's graph of the account's rate of change in dollars per year in relation to the value of the account in dollars. From "Teaching the Verhulst Model: A Teaching Experiment in Covariational Reasoning and Exponential Growth" (Unpublished doctoral dissertation, p. 169), by C. C. Castillo-Garsow, 2010, Arizona State University, Tempe.

Tiffany drew a phase plane graph similar to Derek's, but hers was a graph of  $y = 1.08x$ . She did not speak of the rate growing as the value of the account grew. When Tiffany attempted to sketch a graph of the account's value relative to elapsed time based on her phase plane graph, she plotted points, performing pseudo-calculations to estimate where she should place them.

Although there is much more to tell about Castillo-Garsow's accounts of Tiffany's and Derek's thinking, the essential point for our purpose is this: Tiffany's ability to think of continuous growth was constantly hampered by her chunky thinking, while Derek's smooth continuous thinking supported his rich and connected thinking about the behavior of exponential growth relative to time.

It is appropriate here that we comment on the role of number lines in students' variational thinking. Derek's graphs, as reported by Castillo-Garsow (2010, 2012), rarely had numbers on either axis. In a sense, they were not number lines as we normally think of them. But he used them as if they were number lines. In this regard, there is relatively little research on students' meanings and understandings of number lines. Psychological research in this regard portrays number lines as nebulous objects on which researchers presume that people do informal arithmetic (Geary, Hoard, Nugent, & Byrd-Craven, 2008; McCrink, Dehaene, & Dehaene-Lambertz, 2007; Nuerk, Kaufmann, Zoppoth, & Willmes, 2004), the main interest being by what method people use it to determine sums, products, and so forth. Mathematics education research seems to see the target idea of a number line as being relatively unproblematic and focuses on using it as an instructional aid, helping students understand how to locate numbers on it, or using it as a tool in reasoning (Bright, Behr, Post, & Wachsmuth, 1988; Earnest, 2015; Ernest, 1985; Izsák, Tillema, & Tunc-Pekkan, 2008; Saxe et al., 2010). In both cases, number lines seem to be taken by researchers as lines full of numbers.

Bass (2015) offers an interesting distinction that we believe will clarify ways that students might conceive of number lines profitably. He proposed a distinction between the way number lines are conceived in Western math education and the way they are conceived in Russian curricula, especially the El'konin-Davydov (E-D) curriculum. Bass characterized Western conceptions of a number line as a line that is already full of numbers, which leads to educators thinking that children's quest is to come to understand those numbers. Bass characterizes the E-D conception of a number line as starting with no numbers, but full of positions, such as the positions one would pass over were one to walk along it. In other words, Bass characterizes the E-D number line as starting with the concept of a continuum, but not a numerical continuum. As children develop conceptions of numbers as magnitudes, they can then consider where to place them on the continuum. This is similar to the realistic mathematics education (RME) notion of an empty number line (Gravemeijer & Doorman, 1999; Stephan & Akyuz, 2012; Torbeyns, Schneider, Xin, & Siegler, 2014), except that the RME empty number line really is empty. Students are encouraged to use it to show jumps to other places on it, where the "jumps" are the important feature. Also, RME portrays jumps as actually leaving the line—there is no image of passing through positions on the line as one leaves one place and lands at another. We see Bass's proposal for educators to focus early on having students conceptualize a line as a continuum of positions as being highly propitious for students to conceptualize smooth variation. Of course, this is speculative. We return to this idea in the Future Research section.

### **Time and Variational Reasoning**

In our work with students and teachers over previous decades, we noticed their persistent predilection to speak about values changing by whole number amounts with no attention to quantities' values between successive amounts. This behavior often expressed itself in the action "connect the points" when constructing a graph. Teachers and students would plot points, then swipe sideways to draw a segment or curve that connects successive points—again, with no discussion of the meaning of points on those segments or curves. Indeed, students often said that a function's graph is made of lines that pass through plotted points, or segments that connect points, but the only points on the graph were the ones they plotted (Bell & Janvier, 1981; Goldenberg, Lewis, & O'Keefe, 1992; McDermott, Rosenquist, & vanZee, 1987; Stein, Baxter, & Leinhardt, 1990). Images of variation like these are insufficient for conceiving of functions as models of what curriculum or instructional designers envisioned as continuously changing phenomena. Unfortunately, saying that students and teachers are not thinking in ways that support thinking of variables or quantities changing continuously does not explain the conceptual operations that might make such thinking possible, nor does it say how instruction might help students envision relationships between continuously changing quantities.

One approach that seems promising at first glance would be for teachers and texts to have students think about small changes in quantities as a way to promote their reasoning with continuous variation. However, Castillo-Garsow (2012) and Castillo-Garsow, Johnson, and Moore (2013) argue compellingly that this will not work. As they say, thinking in "chunks" of change is chunky thinking, no matter the chunks' sizes. They argue that thinking with continuous variation necessarily involves thinking about motion—something moving. Their argument is reminiscent of Newton's description of fluents—the flowing quantities that were at the root of his calculus.

Castillo-Garsow and colleagues' claim gains backing from cognitive linguistics through the construct of fictive motion (Talmy, 1996). Fictive motion is expressed in a sentence that

contains a motion verb but does not have a subject that actually moves—as in “Interstate 5 goes from the Mexican border to the Canadian border.” Nothing actually moves in this statement. Yet, we speak as if something is moving. Matlock (2001, 2004) conducted a sequence of latency-response studies to establish quite convincingly that humans think of fictive motion by actively imagining something moving. Langacker (1986), Núñez (2006), and Matlock, Holmes, Srinivasan, and Ramscar (2011) argued that even abstract motion—as in “values of  $x$  go from 3 to 5”—involves fictive motion. We speak of values of  $x$  “going” from one place to another, but nothing actually moves. We might add that as one develops a reflected image of something moving fictively, the image remains present but tacit, like a sweep of attention that is involved in thinking about all elements of a set.

Matlock et al. (2011) argued that conceptions of time both entail and build upon fictive motion. We agree. However, we also point out that conceptions of fictive motion or passage of time, by themselves, cannot support conceptions of continuous variation of a quantity’s value. We suspect that even young children can imagine “going” from a place called “three” to a place called “five.” But we suspect that they do not imagine a quantity’s value going from a measure of three to a measure of five with an image of passing through all intermediate measures.

Thompson (2008b, 2011) addressed the problem of characterizing conceptual operations that constitute thinking with continuous variation of a quantity’s value. He claimed, first, that a person must be thinking of quantities’ measures or magnitudes. Second, he hypothesized that a person always conceives of variation happening over an interval of measures. To conceive of continuous variation in a quantity’s value over an interval, he hypothesized that we

think of a value that varies by infinitesimal amounts, and varies within those amounts as well. In other words, variation, as a way of thinking, is recursive. We define variation within a  $\varepsilon$ -sized interval in the same way we define variation within intervals it contains and within any interval that contains it. . . . Even when thinking of variation happening in bits, we want students to imagine that variation happens within bits as well. (Thompson, 2011, p. 47)

We know of no model of students’ development of smooth continuous variation in relation to their development of their conceptions of time. However, there are hints of a possible model when we look across several studies of covariational reasoning and several studies of children’s construction of time as measured duration.

With respect to the role of time in quantitative covariation, Thompson (1994a) and Thompson and Thompson (1994) described children thinking of measured time in their concept of speed as the number of speed-lengths (the distance traveled in one time unit) contained in a traveled distance. They could not conceive of covariation of distance traveled simultaneously with elapsed time when elapsed time was embedded within their image of distance traveled. Measured time, at first, was not a quantity for them when thinking about distances traveled. Rather, measured time was a number determined by how many speed-lengths were contained in the traveled distance. The children later abstracted measured time from their activities of measuring distances traveled in units of speed-length so that, in their understanding, distance and time covaried. We should note that the measured time that these students abstracted varied in ways that Castillo-Garsow today would call chunky continuous. Lobato, Hohensee, Rhodehamel, and Diamond (2012) reported a similar phenomenon in describing pivotal intermediate conceptions in eighth-grade students’ construction of what they called quadratic covariation—the covariation of elapsed time and the distance a body moved when accelerated uniformly. Their

first pivotal intermediate conception was that students construct motion as measured distance traveled from a reference point. Their second pivotal intermediate conception was that students construct elapsed time as a quantity that is separate from the body's motion, abstracting elapsed time from elapsed distance. Ellis and colleagues (Ellis, Özgür, Kulow, Williams, & Amidon, 2015) studied three eighth graders' construction of a cactus's growth as varying exponentially with elapsed time. Ellis et al. (2015) reported that children initially envisioned the cactus's height as a result of repeated doubling or repeated tripling, where the number of doublings or triplings corresponded with a number of time intervals—irrespective of whether the time intervals were the same duration. Their image of the passage of time was embedded in their image of iterating the growth-in-height pattern. The students eventually began to attend to the number of time units within time intervals, so that they were thinking of growth in the cactus's height with respect to the passage of measured time. Keene (2007) also pointed to the need for students to abstract time as a quantity in the context of modeling dynamic situations with differential equations and employed the idea of fictive motion as a mechanism by which they might do so.

The general pattern is that students' initial image of variation was (from our perspective, not the students') of one quantity varying that contained a tacit image of another quantity embedded within it. The conceptual shift that all three studies described is to disembed the image of the tacitly conceived quantity from the image of the explicitly conceived quantity. When they achieved this, students' conception of distance (height, area, volume) was new because time was no longer tacit in it, and they also had a quantity that is tantamount to conceptual time (imagined measured duration). Piaget's (2001) notion of reflecting abstraction gives an excellent account of this disembedding—students differentiate not only the imagined act of iterating from the product of iterating, they differentiate the imagined act of iterating into two actions: the act of increasing distance (height, area, volume) and the act of increasing the amount of elapsed time. We suspect that this same operation, disembedding an image of one quantity varying from an image of another quantity varying, is at the root of the construction of intensive quantities (Johnson, 2015; Kaput, 1985; Kaput & Pattison-Gordon, 1987; Nunes, Desli, & Bell, 2003).

With respect to studies of students' conceptions of time, we distinguish between research on how someone experiences time (e.g., Brown, 1990; Levin, 1977; Levin, Wilkening, & Dembo, 1984) from research on how someone conceptualizes measured time. With regard to measured time, we find great clarity in von Glasersfeld's (1984, 1996) interpretation of Piaget's (1970) account of children's construction of time as a quantity. Von Glasersfeld describes a concept of time as the coordination of two streams of successive events, at least one of which the experiencing subject takes as occurring rhythmically. Kamii and colleagues (Kamii & Russell, 2010; Long & Kamii, 2001) devised tasks to see at what grade level children would iterate repeatable durations to time an event and whether their judgments of time were conserved across different activities. For iterating units of duration, Kamii and colleagues examined whether children chose to repeat one event (e.g., draining water from a flask using a small tube) to see which of two other events (e.g., playing two recorded songs) lasted longer. For conservation of time judgments, they examined whether the speed at which children accomplished one activity, (e.g., counting marbles quickly or slowly) interfered with their judgment of how much time another event took (water drained from a cylinder) while performing that activity. They found that not until sixth grade were children sufficiently proficient at all tasks that they could be said to have conceptualized time as a quantity (i.e., as a measurable duration). In another study, Kamii and Russell (2012), using Steffe's theory of unit hierarchies (Steffe & Olive, 2010; Steffe, von Glasersfeld, Richards, & Cobb, 1983) examined 126 second-through-fifth-grade children's

ability to quantify elapsed time. Kamii and Russell described a number of events to children, including start times and end times given in hours and minutes and asked students how long the events lasted. The questions varied from easy (whole-number hours) to difficult (e.g., 6:40 to 9:15). Although success rates increased by grade level for all items, only questions about whole numbers of hours were easy for children. Even by fifth grade, only 60% of children answered questions involving a half-hour correctly, and only 31% of fifth graders answered 6:40 to 9:15 correctly. Kamii and Russell claimed that children's main difficulty was coordinating hours and minutes as a hierarchy of units.

The reason that we digressed to discuss research on children's conception of time is to point out that students' conceptions of continuous variation can be affected by both their ability to conceive of time as a quantity and, when they are thinking about specific measures of quantified time, by their concept of number. Although fictive motion can supply an image of smooth variation when students are not thinking of quantifying the variation, Kamii and Russell's (2012) results suggest that fictive motion is insufficient for students' conceptualization of quantified variation. Students' concepts of number will be an important factor in their ability to quantify variation once quantification enters their judgments, and their concepts of number will affect their image of a quantity's value having varied through all values in an interval when they are thinking of those values as numbers instead of as magnitudes (Thompson et al., 2014).

## Multiplicative Objects

Though the next major section focuses on research on variational and covariational reasoning, we share results from two studies in particular to highlight the importance of what Saldanha and Thompson (1998) said about multiplicative objects: "our notion of covariation is of someone holding in mind a sustained image of two quantities' values (magnitudes) simultaneously. It entails coupling the two quantities, so that, in one's understanding, a multiplicative object is formed of the two" (p. 299). Saldanha and Thompson's idea of multiplicative object derived from Piaget's notion of "and" as a multiplicative operator—an operation that Piaget described as underlying operative classification and seriation in children's thinking. A person forms a multiplicative object from two quantities when she mentally unites their attributes to make a new conceptual object that is, simultaneously, one and the other. Thompson and Saldanha (2003) illustrated how the idea of multiplicative object was at the root of conceptualizing and quantifying torque. They described someone conceiving of "amount of twist" as being made simultaneously from a force and the distance from a fulcrum to the force's point of application and how this conceptualization supported the quantification of torque.

We must clarify that our meaning of object in "multiplicative object" is not identical with the meaning of object in APOS (action, process, object, schema) theory (Arnon et al., 2014; Dubinsky & McDonald, 2001) or in Sfard's theory of reification (Sfard, 1991, 1994). Their meaning of object is more like a highly schematized mathematical concept that a person can operate upon mentally, such as a function as an object or a vector subspace as an object. We are speaking of a specific cognitive act, such as understanding each cell in a three-dimensional table as a container for the number of items in a collection that have three specific values on three quantities simultaneously. Figure 13.5, for example, contains a three-dimensional table with dimensions speaker's conception of an event, speaker's description of the event, and listener's understanding of speaker's description. A person who has conceptualized cells in this table as multiplicative objects will understand that a number in the rear left top cell gives the number of occasions where a speaker conceptualized an event as stochastic, described it using nonstochastic

terms, and the listener understood the speaker as describing a stochastic event. This person would also understand that the sum of entries in the bottom layer of the table would be the number of occasions that a listener understood a speaker's description as describing a nonstochastic event irrespective of how the speaker conceptualized or spoke of the event.

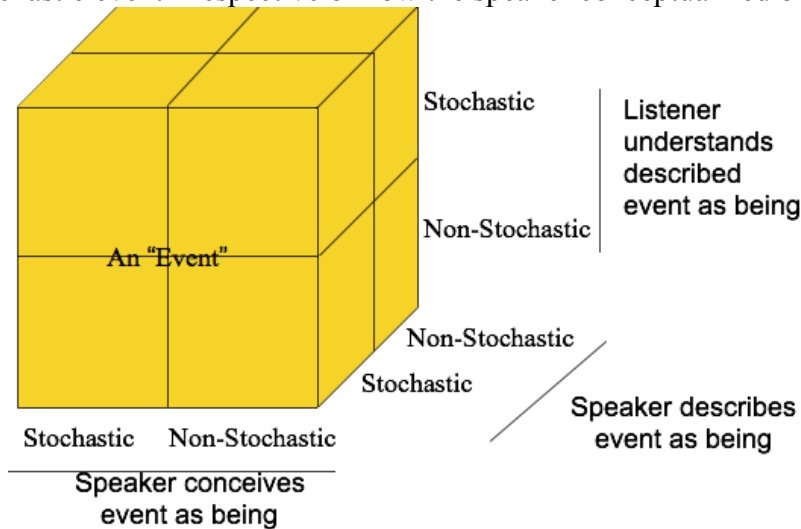
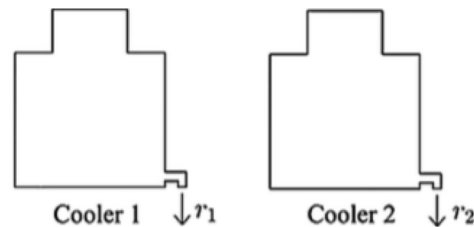


Figure 13.5. Coordinating three perspectives on an event.

Stalvey and Vidakovic (2015) reported a study of 15 calculus 2 students' responses to the task in Figure 13.6. Stalvey and Vidakovic asked students to do these activities: (a) graph a relationship between time and the volume of water for both coolers on the same coordinate plane, (b) graph a relationship between time and the height of the water for both coolers on the same coordinate plane, (c) graph a relationship between the volume of the water and the height of the water for both coolers on the same coordinate plane, and (d) indicate the orientation of your graph in (c).



Assume that Coolers 1 and 2 are the same size. Imagine that they are full of water and being emptied at constant rates  $r_1$  and  $r_2$ , respectively. Assume that  $|r_1| < |r_2|$ .

Figure 13.6. Bottle task. From "Students' Reasoning About Relationships Between Variables in a Real-World Problem" by H. E. Stalvey and D. Vidakovic, 2015, *The Journal of Mathematical Behavior*, 40, p. 196.

Students could understand questions (a) and (b) in terms of constant rate of change with respect to time and linearity and imagine either quantity varying with the passage of time. But questions (c) and (d) demanded that students covary height and volume as each varied with respect to time. Students needed to create volume versus height as a multiplicative object, so that even when they focused on one of the quantities, they had a persistent awareness that the other

quantity varied also. Stalvey and Vidakovic shared an insight by one student, Bailey, that enabled him to conceptualize the covariation of volume and height: “So, yeah, the volume and the height, they move as one” (Stalvey & Vidakovic, 2015, p. 206). They also described another student, Oliver, who exemplified a difficulty had by many students:

Notice that Oliver never described height as decreasing while he described the volume in the above excerpt. Instead, he considered only that the volume would decrease. This provides evidence that Oliver had difficulty coordinating independent processes of height and volume decreasing with respect to time. (Stalvey & Vidakovic, 2015, p. 207)

Bailey had constructed volume-versus-height as a multiplicative object; Oliver had not. Stalvey and Vidakovic drew two primary conclusions, (1) that covariational reasoning is linked to the APOS notion of process conception of function (which we discuss later), and (2) that for students to understand the covariation of volume and height when they envisioned each as varying separately with time required them to create volume-versus-height as a multiplicative object. We note that Stalvey and Vidakovic’s (2015) bottle task differs in important ways from Carlson et al.’s (2002) bottle task. Stalvey and Vidakovic first asked students to think explicitly of volume as a function of time and height as a function of time before requesting that they think of volume and height as covarying. Carlson et al. did not force students to attend to volume as a function of time separately from height as a function of time, and therefore students could coordinate volume and height by imagining the bottle being filled in experiential time.

In another study, Thompson, Joshua, Yoon, Byerley, and Hatfield (2016) reported results from their investigations of teachers’ mathematical meanings for teaching (Thompson, 2015) on the task on the left in Figure 13.7. The task is somewhat unusual in that it asked teachers to respond to an animation. The authors presented the animation to secondary mathematics teachers from the United States ( $n = 132$ ) and South Korea ( $n = 368$ ).

The authors projected an image (3.05 m by 2.29 m) of two bars, one on each axis within a rectangular coordinate system (Figure 13.7, left). The bars’ lengths varied simultaneously, each with one end fixed at the origin. The horizontal bar’s unfixed end varied at a steady pace from left to right while the vertical bar’s unfixed end varied nonuniformly. The animation played for two minutes, showing the complete variation six times. A voiceover spoke the item’s text as the animation began and announced when one minute remained and when 30 seconds remained.

Each teacher had a response sheet (Figure 13.7, right) on which he or she sketched a graph. The response sheet contained the presented item, the request to sketch a graph of the value of  $u$  relative to the value of  $v$ , and the statement that the diagram showed the initial values of  $u$  and  $v$ . Figure 13.8 shows an accurate graph.

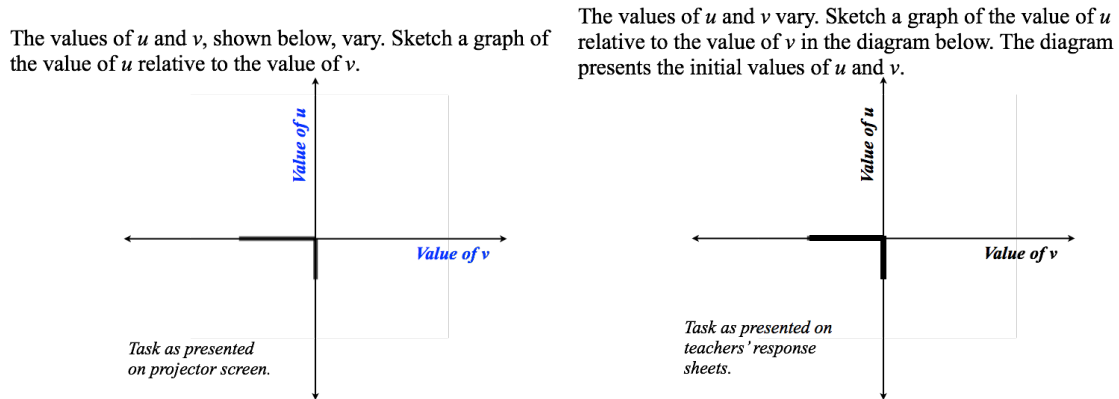


Figure 13.7. Animated item and teachers' response sheet to assess teachers' covariational reasoning. From "The Role of Multiplicative Objects in Teachers' Covariational Reasoning" by P. W. Thompson, S. Joshua, H. Yoon, C. Byerley, and N. Hatfield, 2016, Manuscript submitted for publication, p. 10. Copyright 2015 by the Arizona Board of Regents. Used with permission.

The values of  $u$  and  $v$ , shown below, vary. Sketch a graph of the value of  $u$  relative to the value of  $v$ .

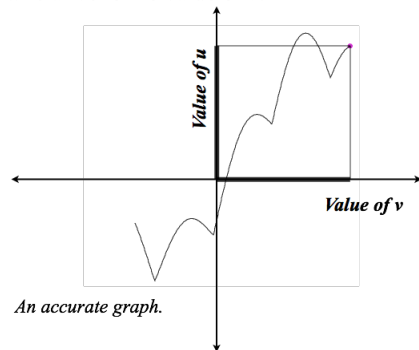


Figure 13.8. Accurate graph for animated task presented in Figure 13.7. Copyright 2015 Arizona Board of Regents. Used with permission.

The task was scored according to a rubric that was constructed and validated as described by Thompson (2015). The rubric for this task focused on two dimensions of teachers' graphs: placement of initial point, and placements of local extrema (not counting endpoints).

Placement of initial point was scored as

- Level A2: Teacher placed the initial point such that both the  $u$ - and  $v$ -coordinate were within 1 cm of the accurate location.
- Level A1: Teacher placed the initial point outside the region for Level A2 and either the  $u$ - or  $v$ -coordinate was within 1 cm of the accurate value.
- Level A0: Initial point placement does not fit any of the above levels.

Placements of local extrema (not counting endpoints) were scored as

- Level B4: The graph had four local minima in ascending order and three local maxima in ascending order. Minima or maxima in ascending order means, e.g., if  $u_i$  and  $u_{i+1}$  were successive local minima, then  $u_i < u_{i+1}$ .
- Level B3: The graph had 6 or 8 total extrema with minima in ascending order and maxima in ascending order.
- Level B2: The graph had 4–5 or 9–10 total extrema with minima in

ascending order and maxima in ascending order.

Level B1: The graph is monotonically increasing.

Level B0: The graph did not fit any of the above levels.

Thompson et al. (2016) scored teachers' placement of the initial point separately from the graph's shape to see whether teachers had the realization that any point on their graph represented both values simultaneously. The response sheet contained the bars' initial values, so if a teacher saw points on his graph as capturing the two values simultaneously, it would be necessary for him to use the bars' initial values to plot the graph's first point.

Table 13.1 and Table 13.2 give results separately for South Korean and U.S. teachers. Thompson et al. did this to highlight similarities across cultures in regard to the importance of constructing a multiplicative object in thinking covariationally. Responses at Levels B4, B3, or B2 contained a graph that had an accurate or semi-accurate shape. Responses at Level A0 contained a wildly inaccurate placement of the graph's initial point. Table 13.1 shows that, for South Korea, only 18% of graphs that had a badly misplaced initial point (Level A0) also had an accurate or semi-accurate shape (Levels B2 to B4), while 67% of graphs that had a well-placed initial point (Level A2) had an accurate or semi-accurate shape. Similarly, Table 13.2 shows that, for the United States, only 12% of graphs that had a badly misplaced initial point also had an accurate or semi-accurate shape, while 52% of graphs that had a well-placed initial point had an accurate or semi-accurate shape.

Table 13.1. Initial Placement vs. Number of Extrema (S. Korea)

|        |       | Level for graph's shape (S. Korea) |                |               |               |               |               |                 |
|--------|-------|------------------------------------|----------------|---------------|---------------|---------------|---------------|-----------------|
|        |       | IDK                                | B0             | B1            | B2            | B3            | B4            | total           |
| Korea) | IDK   | 3<br>(100.0%)                      | 0<br>(0.0%)    | 0<br>(0.0%)   | 0<br>(0.0%)   | 0<br>(0.0%)   | 0<br>(0.0%)   | 3<br>(100.0%)   |
|        | A0    | 0<br>(0.0%)                        | 83<br>(69.7%)  | 14<br>(11.8%) | 11<br>(9.2%)  | 5<br>(4.2%)   | 6<br>(5.0%)   | 119<br>(100.0%) |
|        | A1    | 0<br>(0.0%)                        | 39<br>(41.9%)  | 12<br>(12.9%) | 16<br>(17.2%) | 10<br>(10.8%) | 16<br>(17.2%) | 93<br>(100.0%)  |
|        | A2    | 0<br>(0.0%)                        | 44<br>(31.0%)  | 3<br>(2.1%)   | 32<br>(22.5%) | 24<br>(16.9%) | 39<br>(27.5%) | 142<br>(100.0%) |
|        | total | 3<br>(0.8%)                        | 166<br>(46.5%) | 29<br>(8.1%)  | 59<br>(16.5%) | 39<br>(10.9%) | 61<br>(17.1%) | 357<br>(100.0%) |

Note. Cells contain number of respondents and percent of row total. "IDK" is short for "I don't know." From "The Role of Multiplicative Objects in Teachers' Covariational Reasoning" by P. W. Thompson, S. Joshua, H. Yoon, C. Byerley, and N. Hatfield, 2016, Manuscript submitted for publication,

Table 13.2. Initial Placement vs. Number of Extrema (USA)

|        |     | Level for graph's shape (USA) |        |        |        |        |        |          |
|--------|-----|-------------------------------|--------|--------|--------|--------|--------|----------|
|        |     | IDK                           | B0     | B1     | B2     | B3     | B4     | total    |
| t(USA) | IDK | 7                             | 0      | 0      | 0      | 0      | 0      | 7        |
|        |     | (100.0%)                      | (0.0%) | (0.0%) | (0.0%) | (0.0%) | (0.0%) | (100.0%) |

|       |             |               |               |              |              |              |                 |
|-------|-------------|---------------|---------------|--------------|--------------|--------------|-----------------|
| A0    | 0<br>(0.0%) | 56<br>(67.5%) | 17<br>(20.5%) | 4<br>(4.8%)  | 6<br>(7.2%)  | 0<br>(0.0%)  | 83<br>(100.0%)  |
| A1    | 0<br>(0.0%) | 9<br>(47.4%)  | 4<br>(21.1%)  | 2<br>(10.5%) | 2<br>(10.5%) | 2<br>(10.5%) | 19<br>(100.0%)  |
| A2    | 0<br>(0.0%) | 8<br>(34.8%)  | 3<br>(13.0%)  | 2<br>(8.7%)  | 4<br>(17.4%) | 6<br>(26.1%) | 23<br>(100.0%)  |
| total | 7<br>(5.3%) | 73<br>(55.3%) | 24<br>(18.2%) | 8<br>(6.1%)  | 12<br>(9.1%) | 8<br>(6.1%)  | 132<br>(100.0%) |

*Note.* Cells contain number of respondents and percent of row total. “IDK” is short for “I don’t know.” From “The Role of Multiplicative Objects in Teachers’ Covariational Reasoning” by P. W. Thompson, S. Joshua, H. Yoon, C. Byerley, and N. Hatfield, 2016, Manuscript submitted for publication.

Thompson et al.’s (2016) primary interest was the conceptual operations teachers would use in a context that experts commonly take as underlying the construction of a graph of quantities’ covariation in a dynamic situation. It was not just whether teachers could construct an accurate graph. Thompson et al. hypothesized that teachers who gave evidence of not having constructed pairs of values as multiplicative objects would have trouble tracking the values of the two quantities simultaneously. Their results supported their hypothesis.

Thompson et al. (2016) contended that past research has frequently overinterpreted students’ and teachers’ conceptual operations when they sketch a graph that the researcher takes as appropriate or when they interpret a graph as reflecting a situation holistically. Thompson et al. turned the question around, asking whether teachers would understand that what they were being asked to do was to use the conventions of graphing to sketch a graph in the Cartesian plane when presented with what the authors took to be two varying quantities whose values are the coordinates of points on the graph.

Thompson et al. commented that when sharing the results for placing the graph’s initial point, they often heard the comment, “But this is just plotting a point. Surely there is something wrong with your task.” They replied by pointing out that the common image of “plot a point” confounds three different cognitive acts. The following three scenarios highlight the differences among them:

1. You are given the coordinate pair (2, 3) and are asked to plot it in Cartesian coordinates.
2. You are given a specific point in a Cartesian plane with the question, “What are its coordinates?”
3. You are given two quantities’ values, and are asked to represent those two values simultaneously.

The first two scenarios commonly trigger a convention that one has learned—how to plot points when given a coordinate pair or how to estimate a point’s coordinates within a coordinate system. The third scenario is quite different. The idea of coordinates or a coordinate system is not mentioned. The person must decide that placing a point in a coordinate system does what she desires—to represent the values of two quantities simultaneously.

Like Stalvey and Vidakovic (2015), Thompson et al. (2016) argued that their results highlight the importance of creating a multiplicative object from two quantities’ values for holding a covariational meaning for graphs. They noted that Tables 13.3 and 13.4 suggest that many teachers did not create a point as a multiplicative object whose coordinates were the values

of the animation's two quantities. In the South Korean sample, 33% of teachers had a badly misplaced initial point, while 63% of teachers in the U.S. sample had a badly misplaced initial point. The teachers in Thompson et al.'s (2016) study were all secondary school mathematics teachers. This suggests, to us, that to construct a coordinate pair as a multiplicative object is nontrivial and that its difficulty is underappreciated in mathematics education broadly. We suggest that many of students' difficulties with graphs as records of covarying quantities are grounded in their not having conceived points on a graph as multiplicative objects that represent two measurements simultaneously.

### **Revised Covariational Reasoning Framework**

Based on Castillo-Garsow's distinctions among students' discrete, chunky continuous, and smooth continuous thinking about how a quantity's value varies; research on the students' conceptions of time as a quantity; and a new understanding that conceptualizing multiplicative objects is essential to reason covariationally, we can revise prior covariational frameworks in two ways:

1. By attending to students' variational reasoning separately from their covariational reasoning
2. By attending to how students coordinate their images of quantities' values varying, taking into account their way of reasoning variationally and taking into account the ways they construct multiplicative objects of quantities' values

Table 13.3 presents a framework for the construct of variational reasoning. It combines Castillo-Garsow's (2010, 2012) distinctions between smooth and chunky images of change and Thompson's (2008b, 2011) construct of a recursive image of continuous change. We offer it in this chapter as a construct to be used and refined in further research, not as a definitive description of variational reasoning. We suspect that the levels are developmental, but we leave that to future research to investigate empirically. Ways that covariational reasoning develop in students' thinking is a major theoretical problem that requires further research from many perspectives.

Table 13.3. Major Levels of Variational Reasoning

| Level                       | Description                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 |
|-----------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Smooth continuous variation | The person thinks of variation of a quantity's or variable's (hereafter, variable's) value as increasing or decreasing (hereafter, changing) by intervals while anticipating that within each interval the variable's value varies smoothly and continuously. The person might think of same-size intervals of variation, but not necessarily.                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              |
| Chunky continuous variation | The person thinks of variation of a variable's value as changing by intervals of a fixed size. The intervals might be the same size, but not necessarily. The person imagines, for example, the variable's value varying from 0 to 1, from 1 to 2, from 2 to 3 (and so on), like laying a ruler. Values between 0 and 1, between 1 and 2, between 2 and 3, and so on, "come along" by virtue of each being part of a chunk—like numbers on a ruler—but the person does not envision that the quantity has these values in the same way it has 0, 1, 2, and so on, as values.<br>Chunky continuous variation is not just a person thinking that changes happen in whole number amounts. Thinking of a variable's value going from 0 to 0.25, 0.25 to 0.5, 0.5 to 0.75, and so on (while thinking that entailed intervals "come along") is just as much thinking with chunky continuous variation as is thinking of increases from 0 to 1, 1 to 2, and so on. |
| Gross variation             | The person envisions that the value of a variable increases or decreases, but gives little or no thought that it might have values while changing.                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          |
| Discrete variation          | The person envisions a variable as taking specific values. The person sees the variable's value changing from $a$ to $b$ by taking values $a_1, a_2, \dots, a_n$ but does not envision the variable taking any value between $a_i$ and $a_{i+1}$ .                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          |
| No variation                | The person envisions a variable as having a fixed value. It could have a different fixed value, but that would be simply to envision another scenario.                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                      |
| Variable as symbol          | The person understands a variable as being just a symbol that has nothing to do with variation.                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             |

We wish to point out again the recursion in our descriptions of smooth continuous variation. We are not saying that a person who thinks with smooth continuous variation actively engages in recursion "all the way down." Rather, the person, although reasoning variationally, is alert to the potential need to think about smaller intervals in precisely the same way as they are thinking about the interval that is currently in their reasoning—with smooth and continuous variation. Also, we do not wish to imply that Table 13.3 provides a learning progression in the sense that one level should be targeted instructionally before the next higher level. As Castillo-Garsow et al. (2013) point out, teachers should emphasize smooth variation in their talk and actions whenever they can. Students will reason at the level they will, and if at some point in time they reason variationally at the highest level, they get all other levels for free.

Table 13.4 presents our current view of covariational reasoning as a theoretical construct.

It retains emphases on quantitative reasoning and multiplicative objects (Thompson) and coordination of changes in quantities' values (Confrey, Carlson) and adds ways in which an individual conceives quantities to vary (Castillo-Garsow). It also removes rate of change as part of a covariational framework. For students to conceptualize rate of change requires that they reason covariationally, but it also requires conceptualizations that go beyond covariational reasoning, such as conceptualizations of ratio, quotient, accumulation, and proportionality. We address ways in which covariational reasoning is foundational for rate of change concepts in our later discussion of research that investigates students' and teachers' covariational reasoning.

Table 13.4. Major Levels of Covariational Reasoning

| Level                         | Description                                                                                                                                                                                                                                                                                                                                 |
|-------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Smooth continuous covariation | The person envisions increases or decreases (hereafter, changes) in one quantity's or variable's value (hereafter, variable) as happening simultaneously with changes in another variable's value, and the person envisions both variables varying smoothly and continuously.                                                               |
| Chunky continuous covariation | The person envisions changes in one variable's value as happening simultaneously with changes in another variable's value, and they envision both variables varying with chunky continuous variation.                                                                                                                                       |
| Coordination of values        | The person coordinates the values of one variable ( $x$ ) with values of another variable ( $y$ ) with the anticipation of creating a discrete collection of pairs $(x, y)$ .                                                                                                                                                               |
| Gross coordination of values  | The person forms a gross image of quantities' values varying together, such as "this quantity increases while that quantity decreases." The person does not envision that individual values of quantities go together. Instead, the person envisions a loose, nonmultiplicative link between the overall changes in two quantities' values. |
| Precoordination of values     | The person envisions two variables' values varying, but asynchronously—one variable changes, then the second variable changes, then the first, and so on. The person does not anticipate creating pairs of values as multiplicative objects.                                                                                                |
| No coordination               | The person has no image of variables varying together. The person focuses on one or another variable's variation with no coordination of values.                                                                                                                                                                                            |

Like Steffe et al. (1983) and Carlson et al. (2002), we intend that readers think of each level in Table 13.3 and Table 13.4 in two ways. A researcher could use it to describe a class of behaviors, or she could use it as a characteristic of a person's capacity to reason variationally or covariationally. As a descriptor of a class of behaviors, individuals at various levels of sophistication could exhibit behavior that a framework level describes. A research mathematician, for example, could exhibit gross coordination of values in a particular setting because that is all she needs for dealing with that situation at that moment. A framework level used as a characteristic of a person's capacity to reason variationally or covariationally, however, would mean that one is convinced that the person can reason reliably across settings in ways that this level implies, but he cannot reason reliably at higher levels. Researchers using this framework should take care to make their usage of a level clear.

We elaborate the thinking for each level using the bottle problem discussed in Carlson et al. (2002). A student at the no coordination level would recognize that the water's top is going up in the bottle, or that more water is being added to the bottle, but would make no attempt to coordinate the height of the water in the bottle with the amount of water added to the bottle. A student at the precoordination level would notice that after some amount of water is poured into the bottle, the water level on the bottle rises. A student at the gross coordination level would describe the covariation as "the height increases as the volume increases." A student at the coordination of values level would focus on the water's height in the bottle and the number of cups of water added to the bottle with no thought given to intermediate values of volume or height. A student at the chunky continuous covariation level would imagine the water level rising for each increment of water added, including all values of volume and height between successive values, but without envisioning height and volume passing through those values. Finally, a student at the smooth continuous level would imagine both the water's volume and height varying smoothly through intervals simultaneously, while anticipating that within each interval the amount of water and height of water vary smoothly and continuously.

Another example of reasoning covariationally at different levels comes from Castillo-Garsow (2012). He shared ninth-grade algebra 1 students' responses to this problem:

I'm trying to save for a big screen TV. I make the decision to have \$55 of my monthly paycheck go towards my TV savings fund. After four paychecks I have a total of \$540. Sketch a graph that shows how much money I have saved at each moment in time during the first 8 months after [I start to save for the TV]. Be sure to think about how much money I have saved between paychecks. (Castillo-Garsow, 2012, p. 56)

Castillo-Garsow reported three genres of student answers. One showed points plotted above successive months. This exemplifies, at most, discrete covariation. Another shows the same points plotted, but connected with line segments. This exemplifies, at most, chunky continuous coordination. We say "at most" because students might have connected points simply because they thought that this is what one does when sketching a graph. The third genre is illustrated by the student answer in Figure 13.9. Though the graph is discontinuous, we see it as consistent with the student having conceptualized the TV account's balance in relation to time in terms of smooth continuous covariation. Castillo-Garsow pointed out that Carol, the student who drew this graph, envisioned the TV account's balance as having a value at each moment of elapsed time. He also observed that, "Carol's solution showed that although the deposit events were discrete, the time that they were occurring in was continuous" (Castillo-Garsow, 2012, p. 58). Carol thought about the account's balance at all moments between deposits as well as at months' ends.

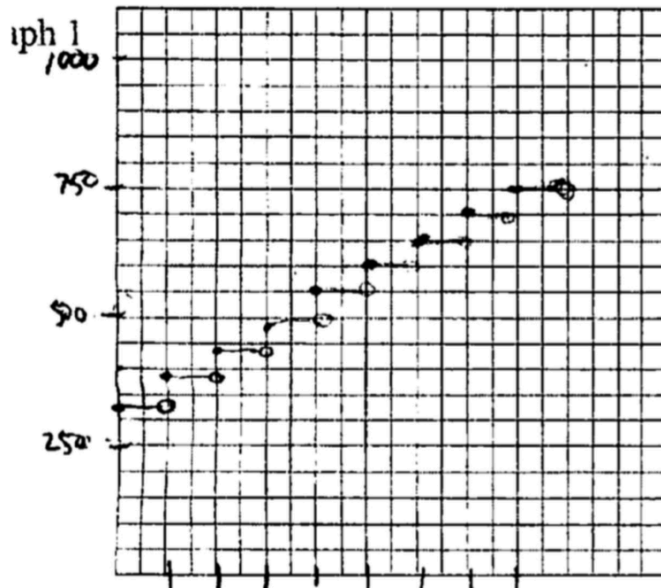


Figure 13.9. Graph of TV savings account that shows smooth continuous covariation. From “Continuous Quantitative Reasoning” by C. C. Castillo-Garsow, in R. Mayes, R. Bonillia, L. L. Hatfield, and S. Belbase (Eds.), *Quantitative Reasoning: Current State of Understanding*, WISDOMe Monographs (Vol. 2, p. 59), 2012, Laramie: University of Wyoming. Used with permission.

The framework offered here focuses on variation, covariation, and multiplicative objects. It is therefore essential to attend to how students are thinking that quantities’ values vary and how they are uniting quantities’ values when considering their meanings for covariation. We also emphasize that variational and covariational reasoning are complex cognitive acts that students, teachers, and researchers can engage in at multiple levels of sophistication and with multiple meanings. When reading a research article that uses “vary,” “variation,” or “covariation,” ask immediately what the author means by it and what cognitive competencies they are attributing to subjects they claim do it. Ask also whether the author uses “vary” or “covary” as a descriptor of mental behavior or as a characteristic of the reasoner. Moreover, when reading research that describes a situation as involving variation or covariation, ask, “in whose eyes does this situation involve variation or covariation and with what meaning?” Is it in the eyes of the researcher that the situation involves variation or covariation, or is the researcher claiming that it is in the eyes of the student?

### Functions, Covariationally

We turn now to the matter of concepts of function from a covariational perspective. We examined 129 articles in our Endnote database that used both “covariation” and “function.” We found many instances of phrases like “covariational approach to functions” and “covariational meaning for functions”—too many to cite. We did not find one instance of a definition of function from a covariational perspective. Most used the terms “function” and “covariation” without explication. Several stated a version of Euler’s description, “If . . . some quantities depend on others in such a way that if the latter are changed the former undergo changes themselves, then the former quantities are called functions of the latter quantities” (quoted in Rütting, 1984, pp. 72–73).

We therefore offer a meaning of function that is based on covariational reasoning. We do

this with the hope that future research will be conducted from a common basis of meaning.

A function, covariationally, is a conception of two quantities varying simultaneously such that there is an invariant relationship between their values that has the property that, in the person's conception, every value of one quantity determines exactly one value of the other.

The above meaning of function avoids using the customary terms independent variable and dependent variable in any causal sense. What is independent and what is dependent will depend entirely on the person's conception of the situation and which way they envision dependence, if they envision dependence at all. We say, "if they envision dependence at all" to include cases where they do not think of one quantity's value as caused by the other quantity's value, but instead that they simply co-occur. At the same time, however, we agree with Piaget, Blaise-Grize, Szmecinska, and Bang (1977, pp. 167–196) that to a person conceiving a function, the function expresses at least a minimal sense of dependency of one quantity on another, even if only because she thinks of one before the other. We simply add that it is through covariation that the dependency becomes crystalized in her thinking as being invariant across quantities' values.

It is important to note that we said that a function is a conception. A function resides in someone's thinking, so the nature of a conceived function is relative to the person conceiving it. Notice also that we did not specify a particular way in which a person conceives that quantities' values vary, nor did we specify the way in which a person conceptualizes covariation. The researcher or teacher who claims that someone has conceived a relationship between quantities as a function must describe the way this person has conceived that values vary and the way in which they covary, otherwise the claim is vague. Also, a researcher must describe the person's conceived domain and range (the values that the person envisions quantities or variables having), which will depend on the individual conceiving the function.

We also clarify that by "invariant relationship" we mean that, in the mind of a person who has conceived a function, the same relationship can be used to determine, in principle, a value of what he takes as the dependent quantity from any value of what he takes as the independent quantity. The relationship might be qualitative, with no rule of assignment, as in "Jane's height at every moment in time since her birth," or it might be in parts, as in "The tree grew at the rate of 3 m/yr for its first two years, and then it grew at the rate of 2 m/yr for the succeeding 5 years." The invariant relationship regarding the tree's growth rate and time is, "The tree's growth rate is 3 m/yr if . . . , or the tree's rate is 2 m/yr if . . ." The whole statement is the relationship between time and the tree's growth rate. Of course, constructing the idea of a piecewise relationship as one relationship is nontrivial for students.

The idea of function covariationally is, in modern language, tantamount to a function defined parametrically. Here, the conventional use of "parameter" differs from the idea of a value that is constant within one situation but can differ across situations. Instead, it is used in the sense of a variable, but one that is not assigned to an axis in a coordinate system. It is like what Freudenthal (1983), Drijvers (2001, 2002), and Keene (2007) called a "slider," but one that a person envisions as sliding on its own, without the need of human intervention, like a quantity whose value varies. In terms of relationships between quantities, each quantity has a value that exists in conceptual time, with conceptual time made explicit in the person's awareness. Thus, the coordination of quantities' values is like forming the pair  $[x(t), y(t)]$ , where  $t$  stands for a value of conceptual time. We distinguish between experiential time and conceptual time as follows: Experiential time is the experience of time passing, whereas conceptual time is an image

of measured duration. We say *image* of measured duration to dispel interpretations that someone must think he is actually timing an event. Rather, we are speaking of someone imagining a quantity as having different values at different moments, and envisioning that those moments happen continuously and rhythmically.

As argued by Keene (2007) and Stalvey and Vidakovic (2015), thinking of covariation parametrically, as described above, is more general than standard meanings of function. On the one hand, if someone imagines the quantities being coordinated as having values  $[x(u), y(u)]$  where  $x(u) = u$ , then this is the standard meaning of function as change-dependent variation in two quantities' values (Johnson, 2012a, 2012b, 2015). However, the more general meaning of covariation supports thinking of covarying quantities in nonstandard ways, such as thinking of a circle as the graph of a function, which Euler in fact did—the function being  $[x(t), y(t)] = [\cos(t), \sin(t)]$ ,  $0 \leq t < 2\pi$ . One can have an interesting discussion with students over the question, “Do we get all the points on a circle when we say  $0 \leq t < 2\pi$ ?”

The covariational meaning of function in terms of multiplicative objects can generate surprising insights for students. For example, Figure 13.10 shows two views of the graph of the function  $t \mapsto (\sin(4\pi t), \cos(3\pi t))$   $0 \leq t \leq 1$ . The left side of Figure 13.10 displays the graph of  $(\sin(4\pi t), \cos(3\pi t))$  in a two-dimensional Cartesian coordinate system. Values of  $t$  are imagined separately from the graph. The right side of Figure 13.10 displays the graph of  $(\sin(4\pi t), \cos(3\pi t), t)$  in a three-dimensional Cartesian coordinate system, with values of  $t$  represented explicitly on an axis. In both cases, as  $t$  varies from 0 to 1, each value of  $t$  produces the coordinates of exactly one point on the respective graph.

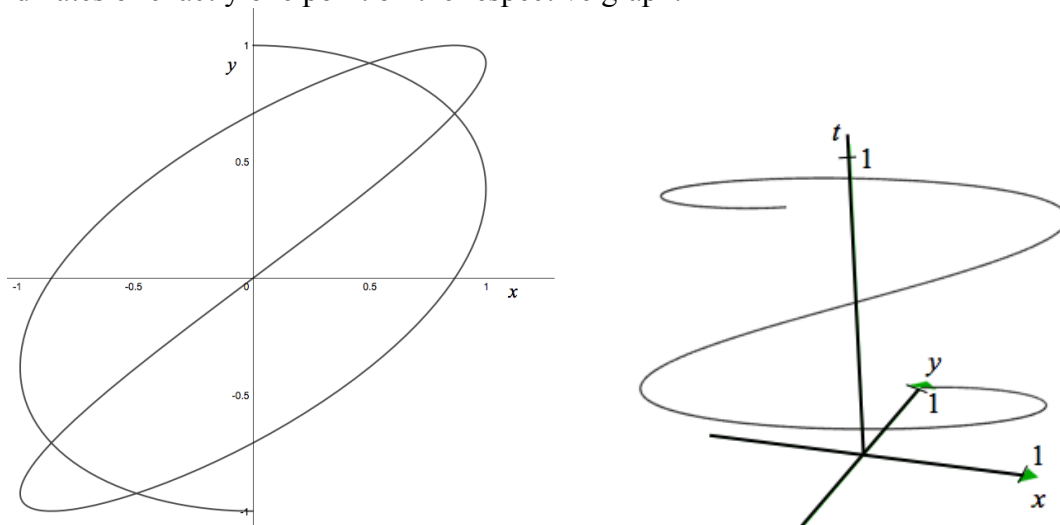


Figure 13.10. Two views of the graph of  $[x(t), y(t)] = [\sin(4\pi t), \cos(3\pi t)]$ ,  $0 \leq t \leq 1$ . The left side displays the graph of  $[x(t), y(t)]$  in a two-dimensional Cartesian coordinate system. The right side displays the graph of  $[x(t), y(t), t]$  in a three-dimensional Cartesian coordinate system.

When students understand the idea of function as invariant relationship, they are less likely to think of a function's graph as a shape. Students following the “vertical line rule” would say of Figure 13.10's left display, “This is not the graph of a function,” or more likely, “This graph is not a function.” This example is highly related to what Moore and colleagues call static and emergent shape thinking (Moore, Paoletti, & Musgrave, 2013; Moore & Silverman, 2015; Moore & Thompson, 2015). Static shape thinking means to make inferences about a function's

behavior strictly by having built associations between graphs' shapes and function properties. Emergent shape thinking is to interpret a graph as an emergent trace of variables having covaried.

We end this discussion of covariational reasoning by emphasizing when covariational reasoning enters one's thoughts most strongly. Like quantitative reasoning, covariational reasoning is about comprehending situations. Covariational reasoning happens most strongly when a person is strategizing how to keep track of quantities' values simultaneously. The execution of such strategies, both flawless and problematic, is an expression of covariational reasoning. Covariational reasoning happens in a person's thinking prior to expressing it in behavior.

### **Research on Variational and Covariational Reasoning in Contexts**

Research related to variational and covariational reasoning has occurred largely on students' and teachers' understandings of concepts other than covariational reasoning. Covariational reasoning provided the background for the investigated ideas, but the focus was on other matters. We organize this literature within domains of variable, exponential growth, algebra, calculus, and trigonometry. We discuss some studies because they shed light on ways that reasoning covariationally supports students' or teachers' mathematical thinking, meaning that the study contributes positively to our understanding of ways that people reason about the investigated idea. We discuss other studies because they shed light on difficulties that students or teachers have with an idea, and for which we think that the theory of covariational reasoning offers potential reasons for their difficulties.

### **Variables and Variation**

Freudenthal (1982) described the quest in mathematics to eliminate what he called kinesthetic meanings of variable—the idea that variables vary:

Locutions like . . . the point  $P$  runs on the surface  $S$ , the element  $x$  runs through the set  $S$  . . . witness this kinematic aspect of the “variable.” It is true that in the course of, say, the past half century, such locutions have been outlawed by purists. . . . Well, one can dispense with that kind of kinematics provided one has once been in its possession, learned to use it and then to eliminate it. (pp. 7–8)

Many studies of students' understandings of variables bear out Freudenthal's concern, that students cannot develop rich meanings of variables without first having the idea that variables vary. Janvier (1996) and Kaput (1994) were early advocates of greater attention in both research and practice to having students think that values of variables vary. However, the culture of school mathematics focused on static variables. White and Mitchelmore (1996), for example, reported their effort to enrich first-year calculus students' understandings of derivatives. Their intervention failed, they explained, because students conceived of variables statically—as symbols to be acted upon procedurally or as representations of unknown values. Students did not think of variables as symbols that represented quantities whose values varied. When students viewed variables statically, they could not envision that expressions containing them represented relationships among varying quantities, which eliminated conceiving symbolic expressions as representing a rate of change of one quantity with respect to another. White and Mitchelmore hypothesized that students' static conception of variables was a byproduct of schooling in which

variables are always proposed statically. Similarly, López-Gay, Martínez Sáez, and Martínez-Torregrosa (2015) pointed to static conceptions of variables as being the root of many difficulties in physics students' use of calculus in modeling physical phenomena.

Trigueros and Ursini (1999) examined a cross section of Mexican students (37 middle school, 30 high school, 31 college freshmen) to examine “how the concept of variable evolves through schooling” (p. 273). They used three types of questions—variable as unknown number (U questions), variable as general number (G questions), and variable as varying (V questions; they called this type “functional relationship”). The average number of correct answers across grades showed different trends depending on the type of question. The number of U questions answered correctly showed a slight upward trend across grade levels; G questions answered correctly showed a downward trend across middle grades, an upward trend across high school grades, and a dip at college level; and V questions answered correctly showed an overall downward trend from beginning of middle school to college. We should point out that average scores ranged between 20% and 40% correct regardless of question type or trend. The trend in V questions was most striking. Students who had yet to take algebra had the highest average, and the average went down with each year of schooling—even within the grades in which they took analytic geometry or calculus.

Trigueros and Ursini's types of questions align nicely with our earlier description of Thompson's three meanings for the quantitative use of symbols. Variable as unknown number reflects cognitive processes that are similar to symbol as representing a constant, variable as general number reflects cognitive processes that are similar to symbol having the meaning of a parameter, and variable in a functional relationship reflects cognitive processes that are similar to symbol as having the meaning of the value of a quantity that varies. Ursini and Trigueros (2001) reported the positive effect of an intervention in which they focused on having introductory algebra students recognize the different meanings of symbols used representationally in different settings. One aspect of their intervention seemed to help students avoid thinking with two meanings of variable that were at cross-purposes, such as variable as constant in an equation but also variable as varying in a function. We also see their intervention being very much aligned with emphases on having students use symbols as aids to their quantitative reasoning, which we describe more fully next.

### **Quantitative Reasoning and Covariation in Algebra**

Moore and Carlson (2012) explained the importance of variational and covariational reasoning for students' abilities to model dynamic situations. They reported a clinical study of nine precalculus students' quantitative imagery while solving word problems. One problem was common in precalculus and calculus textbooks:

Starting with an 11 in.  $\times$  13 in. sheet of paper, a box is formed by cutting equal-sized squares from each corner of the paper and folding the sides up.  
Write a formula that predicts the volume of the box from the length of the side of the cutout.

Moore and Carlson reported that all nine students had difficulty constructing an appropriate formula and attributed their difficulty to poorly formed images of the quantities and relationships involved in making the box as the situation describes and their static meanings for symbols they used. Figure 13.11 depicts the reasoning of one student, Travis. Travis drew an appropriate diagram that showed the squares that would be cut out, labeled the side length of one

square “ $x$ ,” and then wrote  $V = 13 \cdot 11 \cdot x$ . Travis explained that the length of the cutout could vary, but did not speak of the box’s length, width, or base varying as a result of varying  $x$ . Moore and Carlson explained that each student’s difficulty was resolved after probing questions led him or her to conceptualize the quantities height of box, width of base, and length of base and to imagine ways their values varied in relation to variations in the cutout’s width.

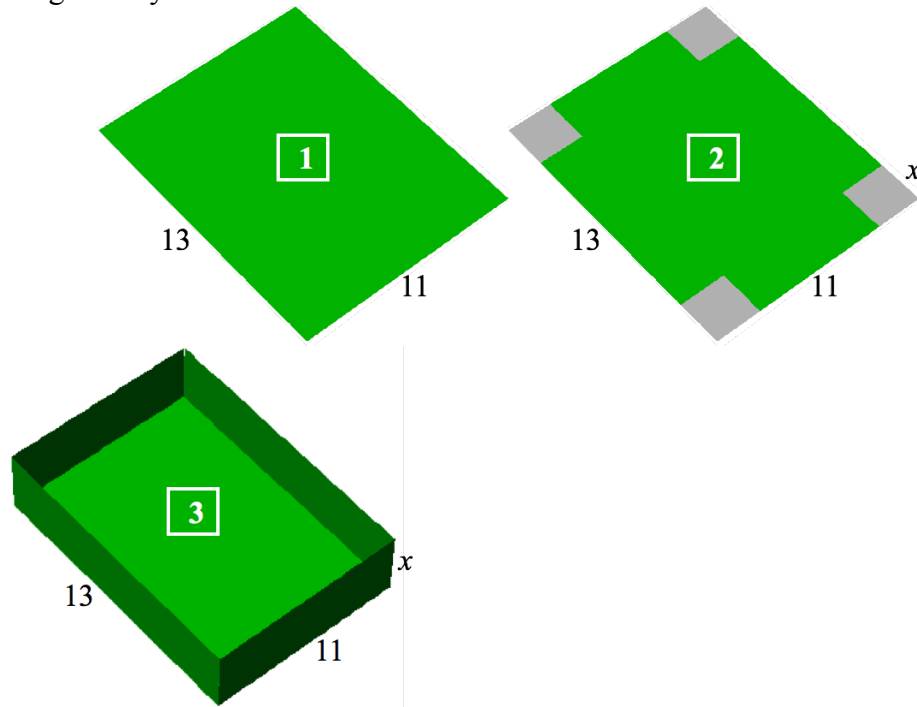


Figure 13.11. Travis’s thinking about the box problem.

The example from Moore and Carlson illustrates the role of quantitative reasoning in creating symbolic expressions that model relationships among quantity’s values. When students conceptualize a situation as having a quantitative structure, they then have a basis to imagine constrained variation—their conceived relationship between quantities’ values both constrains and supports the ways they can imagine those values varying.

In regard to the box problem, Moore and Carlson’s students came to envision the base’s width as being constrained by the fact that to form the base, the sheet’s width is diminished by the lengths of two squares and the sheet’s length is diminished by the lengths of two squares. They also saw that, for the end construction to be a box, all squares must have the same side length. As Thompson (2011) pointed out, once a student envisions a situation in terms of a quantitative structure, she is positioned to propagate information about how to calculate values of quantities in the structure in terms of arithmetic or algebraic expressions that are implied by the structure. For example, students understood the box’s base length as being made by diminishing the paper’s length by the lengths of the cutout squares (Figure 13.12), and they represented the square’s length with “ $x$ .” Students then saw that they could therefore propagate this information to conclude that the base length is  $13 - 2x$ , that the base width is  $11 - 2x$ , that the box’s base area is  $(13 - 2x)(11 - 2x)$ , and therefore that they could calculate the box’s volume with the expression  $x(13 - 2x)(11 - 2x)$ .

Figure 13.12 illustrates an image of the constrained variation among the quantities that constitute the box. As the value of  $x$  varies (i.e., as the length of each square varies), the box’s

height and the base's width and length vary accordingly, as reflected in the statement

$$V = x \overbrace{(13 - 2x)}^{\text{Base Length}} \overbrace{(11 - 2x)}^{\text{Base Width}}.$$

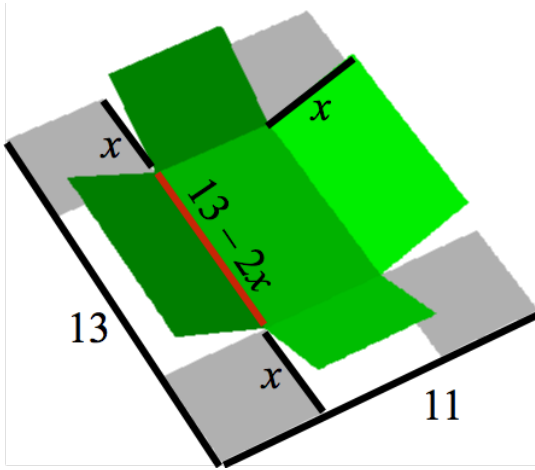


Figure 13.12. Seeing quantities' values in relation to each other as the value of  $x$  varies.

Finally, students' conceptualizations of the box as a quantitative structure supported their image of covariation of quantities' values. Students did not just imagine a family of boxes each being made from a different sheet of paper with different-size cutouts. Rather, they imagined one sheet of paper that had cutouts of a varying side length, which made a box whose shape varied continuously with the cutouts' side length, and therefore whose volume varied smoothly as the square cutout's side length varied smoothly.

In principle, envisioning a situation in terms of quantities and relationships among quantities provides students a foundation for reasoning about covariation as constrained variation. Students may reason statically about quantities to form a quantitative structure of a situation, and then envision one of the quantities' values varying. Other quantities' values will then vary according to the quantitative relationships they have envisioned. It also provides a basis for what Simon (1996) called transformational reasoning, or basing one's mathematical thinking on images of "how a system works."

It is worth emphasizing that students in Moore and Carlson's (2012) study who constructed a valid formula first used " $x$ " with the meaning of a parameter. The cutout initially had an unknown length, but it had a definite length that could be different for different boxes. Students who, afterward, thought about the value of  $x$  that would produce the largest volume changed their meaning of  $x$  from being a parameter to being a variable. Students imagined the side length growing or shrinking and imagined the box's other parts changing accordingly so as to maintain the quantitative relationships they had originally envisioned. This is what we meant earlier when we said that someone envisioning a functional relationship between quantities is able to do so by having conceived a situation as a quantitative structure. Relationships among quantities remain the same even though quantities within the structure have values that vary.

### Exponential Growth and Covariational Reasoning

Although we cannot focus individually on every concept that rests upon covariational reasoning, we single out exponential growth because of the difficulty it poses for a theory of continuous covariational reasoning as foundational to thinking about modeling natural

phenomena. As we mentioned earlier, Confrey's research group developed their construct of covariation by explicating ways to conceptualize exponential growth. They argued compellingly that, from a curriculum and instruction perspective, it is neither historically accurate nor conceptually productive to think of growth happening additively. Confrey and Smith, however, did not address the issue of how to conceptualize exponential growth happening smoothly. In their geometric model of exponential growth, one must always have successive values of quantities as they grow so that  $y_{i+1} = ry_i$ , where  $r$  is a percentage growth factor over same-size units of the independent variable.

With regard to conceptualizing what happens between values in geometric growth, Strom (2008) introduced the construct of a partial growth factor, which is that if a quantity grows geometrically by a factor of  $m$  over a time interval, then it grows by a factor of  $m^{1/n}$  over each of  $n$  equal subintervals of that interval. Ellis, Ösgur, Kulow, Williams, and Amidon (2012, 2015) used the idea of partial growth factor as a design principle in a study with three eighth graders. Ellis et al. focused on partial growth factors to address the question of how to help students think about values of an exponential function between values of its argument. The idea of a partial growth factor relies on understanding the reciprocal relationship between powers and roots. If an exponential function increases by a factor of  $m$  over an interval, then it will increase by a factor of  $m^{1/n}$  over each of  $n$  equal subintervals; if a function increases by a factor of  $p$  over  $n$  equal subintervals of an interval, then it will increase by a factor of  $p^n$  over the interval. The reciprocity exists once one operationalizes in thought the relationship that, for all natural numbers  $m$  and  $n$ ,  $(m^{1/n})^n = (m^n)^{1/n} = m$  for  $m > 0$ . However, Strom and Ellis et al. found that teachers (Strom) and students (Ellis) struggled to envision covariation of time and growth in a way such that time changed smoothly through intervals while growth happened geometrically at intervals' ends. This was true even after subjects understood the reciprocal relationship between powers and roots and after extracting time as an independent quantity.

The study by Ellis et al. (2015), which involved three eighth-grade students, centered on a dynamic Geogebra sketch that allowed students to place a Jactus (a mythical cactus) at different places on a timeline. The Jactus's height varied exponentially as students slid it from one place to another on the timeline. Students' tasks centered on predicting the Jactus's height in anticipation of moving to a different place on the timeline or moving it forward or backward a number of weeks on the timeline. We suspect that at least one student (Uditi) in Ellis et al.'s (2015) study was close to being able to think about the Jactus's height varying smoothly with conceptual time in anticipation of moving it. Uditi eventually sketched smooth qualitative graphs when Ellis asked her about the relationship without asking about specific values, but Uditi struggled with thinking about smooth change between calculated values. She had generalized her calculational activity to understand that she could calculate values of the Jactus's height at specific moments of time by using the formula  $h = ab^x$ , where  $h$  was the Jactus's height,  $a$  was the Jactus's initial height,  $b$  was the growth factor, and  $x$  was a number of weeks or parts of a week since measurements began. Our suspicion is that Uditi still tended to think of  $ab^x$  in terms of the calculational activities involved in actually determining a value. That is, she still had vestiges of an action conception of  $ab^x$  and still had to create a process conception of it (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Dubinsky & Harel, 1992). That is, Uditi thought of  $ab^x$  in terms of the chain of meanings and actions that she would actually draw upon to calculate specific values. However, she showed signs that she was close to thinking of the inscription " $ab^x$ " as representing the results of her actions. Our suspicion is that, with a process conception of  $ab^x$  (i.e., seeing it as representing values that could be calculated), Uditi would

have begun to think of  $ab^x$  as continuing to represent a height value even as she thought of time passing smoothly.

We dwelled on the case of Udit to highlight a general relationship between reasoning about a function expressed symbolically and reasoning covariationally. Our suspicion about Udit being near to a breakthrough is rooted in a hypothesis put forward by Thompson (1994c) and by Oehrtman, Carlson, and Thompson (2008). They hypothesized that for students to think covariationally about a function that is defined by a rule of correspondence, students must first build a process conception of the rule. We see this hypothesis as a potential foundational link between research on students' quantitative and covariational reasoning and past research on students' conceptions of functions that presumed a correspondence definition of function.

### **Covariation in Calculus and Trigonometry**

Covariational reasoning is important for students to conceptualize function relationships at all levels of schooling. However, we cannot make this case concretely in the space of this chapter. We therefore talk about covariational reasoning in calculus and trigonometry as a way to highlight ways of thinking that culminate the learning that must happen in elementary and secondary school. We are not claiming that all school mathematics must be aimed at preparing students for calculus. Rather, we claim only that ideas of calculus are based on much of the mathematical learning that we desire of students in school. More simply, it takes 12 years to learn calculus.

Larsen, Marrongelle, Bressoud, and Graham (2016, this volume) explain the history of research on calculus learning. They point out that early research on calculus learning uncovered myriad deficiencies in the ways that students understand calculus. One study in particular is important for our purposes here. Carlson's (1998) cross-sectional investigation of students' understandings of the function concept included tasks aimed at revealing how students coordinated changes in two quantities over intervals of a function's domain. Her sample contained students who had just completed honors calculus 2. Building on work of Monk (1992) and Kaput (1992), her study included tasks that required students to consider and represent how two quantities changed together in real-world contexts (e.g., describing how a ladder's height on a wall varies as its foot slides away from the wall) and to interpret rate of change graphs over an interval of a function's domain. The study revealed that the honors calculus 2 students in her study were unable to construct an accurate graph of the height of the water in a spherical bottle in terms of the volume of water that had been added to the bottle, and they were unable to describe how changes in volume affected changes in height (Carlson, 1998, pp. 123–126). They also had weak meanings of what a concave up or concave down graph conveyed about how two quantities in a situation are changing together on intervals of a function's domain (p. 117). Carlson characterized these weaknesses in high-performing calculus 2 students by reporting that her subjects were unable to interpret and graph dynamic situations (pp. 138–139).

Students' written responses to the bottle problem included a strictly concave up graph; a strictly concave down graph; an increasing straight-line graph; and a mostly correct graph that is concave down, then concave up, and linear at the top. Follow-up interviews with five students were even more revealing—one student who constructed a strictly concave up graph indicated that the water would be getting higher and higher on the bottle, and the student who produced an increasing straight line justified his response by saying that the height would always be increasing. The few students who produced a correct graph provided various justifications. One student considered how the height of the water in the bottle changed while imagining equal

changes in volume. Another student considered the amounts of water that would be needed to fill successive equal changes in height, and yet another student considered the relative speed at which the height and volume would be changing together for the bottom half of the spherical bottle as compared to the top half.

Carlson's (1998) findings supported that covariational reasoning was foundational for both defining function relationships algebraically and constructing graphs to represent dynamical events involving continuously changing phenomena, common in most applied problems in beginning calculus. Engelke's (2007) study corroborated and extended this finding to related rate problems in beginning calculus.

With regard to building calculus from quantitative and covariational reasoning, the body of ideas that comprise calculus, including differential equations, can be framed by two foundational problems that have covariational reasoning at their core:

1. You know how fast a quantity is changing at every moment; you want to know how much of it there is at every moment.
2. You know how much of a quantity there is at every moment; you want to know how fast it is changing at every moment. (Thompson, Byerley, & Hatfield, 2013; Thompson & Dreyfus, in press)

These foundational problems are pregnant with ideas of rate of change, accumulation, and functional relationship. The ways of thinking that bind them into a coherent scheme constitute the fundamental theorem of calculus (Bressoud, 2011; Carlson, Persson, & Smith, 2003; Schnepf & Nemirovsky, 2001; N. Smith, 2008; Thompson, 1994a, 1994b; Thompson et al., 2013; Thompson & Silverman, 2008). Smooth continuous covariational reasoning is foundational for all of them.

The concept of constant rate of change entails thinking about two quantities co-accumulating so that their increments are in constant proportion regardless of their size. Students need to build this way of thinking in middle school so that it can be crystalized in functional relationships between quantities in later grades. The idea of a function having a nonconstant rate of change is actually constituted by thinking of the function having constant rates of change over small (infinitesimal) intervals of its argument, but different constant rates of change over different infinitesimal intervals of its argument.

The meaning of a rate of change function  $r_f$  is that each value  $r_f(x_0)$  is the rate at which values of an accumulation function  $f$  change at a moment  $x_0$  in the domain of  $f$ . By "the moment  $x_0$ " we mean a small interval that contains  $x_0$  such that  $r_f(x_0)$  is essentially constant over that interval, which then implies that the accumulation is essentially linear over that interval.

Values  $f(x)$  of the accumulation function  $f$  can be recovered from the function  $r_f$  by (1) knowing  $f(a)$ , an initial value of the accumulation at some value  $a$  in the domain of  $f$ , and (2) by aggregating bits of accumulation that happen as the value of  $t$  varies smoothly from  $a$  to  $x$  through small intervals of size  $\Delta t$ . Bits of accumulation  $df$  themselves vary as  $dt$  varies smoothly and repeatedly through the half-open interval  $(0, \Delta t]$ . The differential  $df$  varies at the constant rate of  $r_f(t_i)$  with respect to  $dt$ , where  $t_i$  is the value of each interval's left bound. Put more succinctly,  $df = r_f(t_i)dt$ ,  $0 < dt \leq \Delta t$ . This, in a nutshell, is a theory of integration, and it is grounded in the scheme of ideas constituted by constant rate of change, accumulation, function, and smooth continuous covariation.

Reversing the process—deriving a rate of change function  $r_f$  when given every value of an accumulation function  $f$ —constitutes a theory of differentiation. Thompson et al. (2013) and Thompson and Dreyfus (in press) describe a calculus course based entirely on these ideas. The course is designed with quantitative reasoning, smooth continuous covariation, and the symbiotic relationship between accumulation and rate of change at its core. As discussed in Larsen et al. (2016, this volume), Thompson and Dreyfus (in press) provide empirical evidence that a focus on calculus grounded in covariational reasoning benefits students' learning in ways that traditional limit-based approaches that build derivatives and integrals unrelatedly do not.

To have access to these ways of thinking about integration, differentiation, and the organic relationship between them, students must be able to think that variables vary smoothly and continuously, that they can vary in small bits, and that variation within bits is also smooth and continuous. Students must also have concepts of function that enable them to think in these ways when functions are expressed symbolically, so they must have a process conception of functions' rules of assignments. Students are unlikely to succeed in calculus if they meet these ways of thinking for the first time in calculus. Meaningful learning in calculus relies on students being able to inject meanings they have built in school mathematics into representations of them in calculus while at the same time creating a scheme that unites them symbolically. They must therefore begin building ideas of smooth continuous variation and covariation, constant rate of change, and process conceptions of rules of assignment in school mathematics.

We hasten to add that these ways of thinking about rate of change, function, and covariation are not essential for students to succeed in calculus courses that emphasize memorization of procedures and rules. An entirely different skillset is required in these courses. But such courses are inherently meaningless to students with regard to building a conceptual foundation, because mathematicians' conceptual foundations (e.g., limit, function, instantaneous rate of change) are not conceptual foundations for students. Instead, any meanings that students derive are in terms of problems they can solve with their memorized routines. The meanings of calculus that are grounded in covariational reasoning also fit precisely with the ways of thinking that science educators complain is lacking in their students' mathematics (Czocher, Tague, & Baker, 2013; Martínez-Torregrosa, López-Gay, & Gras-Martí, 2006; Osgood, 1907; Von Korff & Sanjay Rebello, 2014).

Trigonometric functions also play a central role in calculus, especially in applications. Trigonometric functions, however, are grounded in understanding their arguments, and understanding their arguments as varying continuously. These ways of thinking are inaccessible to students (or teachers) when they understand trigonometry as being about triangles, and understand sine, cosine, and tangent in terms of SOH-CAH-TOA (Sine is opposite over hypotenuse, etc.), a mnemonic that is common in the United States. When students learn sine, cosine, and tangent as SOH-CAH-TOA, then sine, cosine, and tangent are not functions to students, or if students do think of sine, and so forth, as something like a function, they are functions that have triangles as their arguments (Bressoud, 2010; Thompson, 2008a). The key difference between triangle trigonometry and trigonometric functions in trigonometry is that trigonometric functions have angle measures as their arguments, and thus to understand trigonometric functions productively, students must have an appropriate conception of angle measure.

Until recently, with the research by Moore (Moore, 2010, 2012a, 2012b, 2014), angle measure as an argument for trigonometric functions simply was not an issue in research on students' understandings of trigonometry. What Moore made clear is that quantitative reasoning

(e.g., “What are you measuring when you measure an angle?” and “What does ‘ $x$ ’ stand for in  $\sin(x)$ ?”) and covariational reasoning (e.g., “What is covarying that produces a sine graph?”) are central to students’ productive understandings of trigonometric functions.

Moore first established that most undergraduate mathematics education majors at his institution could not answer questions like our examples—not because they did not know the answers, but because these questions have no answers within triangle trigonometry, and triangle trigonometry is what students understood. Moore then demonstrated that students understand trigonometric functions productively when they understand angle measure as a relative arc length (subtended arc on a circle, measured in a unit that is proportional to the circle’s circumference) and thought of trigonometric functions covariationally (as angle measure varies, so do values of sine, cosine, and tangent). By “understood trigonometric functions productively,” Moore meant that students could answer the questions about what “ $x$ ” represents in “ $\sin(x)$ ” and why graphs appear as they do. They also could explain ways that conversion between degree and radian measure was like converting between feet and meters, they could explain how functions’ graphs were the same whether displayed in rectangular or polar coordinate systems, and they could use trigonometric functions to create models of periodic phenomena. Finally, they understood that triangle trigonometry is a special case of trigonometric functions conceived quantitatively and covariationally. In other words, students understanding angle measure quantitatively, and understanding trigonometric functions covariationally, brought a deep coherence to their thinking about trigonometry that was not present when they understood trigonometry as about triangles.

We emphasize that even in the case of trigonometric functions, students’ ways of thinking that lead to productive understandings must begin in school mathematics. When the idea of measuring an angle is first introduced in lower grades, it should be introduced quantitatively, as an intellectual problem: “What might we measure about angles to see if this angle is more open than that angle? How might we measure it?” Triangle trigonometry could be introduced in middle grades via similarity, which would also help students build the idea that an angle’s measure is invariant across circles of different size that are used to measure its openness. A focus on angle measure as relative arc length of a subtended circle would also lay the ground for seeing that degrees (a degree being  $1/360$  of a circle’s circumference) and radius length are equivalent ways to measure relative arc length, just as feet and meters are equivalent ways to measure absolute length. Finally, a focus on angle measure as relative arc length would provide instructional approaches to helping students envision angle measures varying smoothly and continuously. In other words, a focus in elementary school on preparing students to build coherent understandings of trigonometric functions in their later life would have the benefit of making their mathematical understandings in elementary school more coherent.

### **Research on Students’ and Teachers’ Conceptions of Function, Seen Through a Covariational Lens**

Steffe and colleagues (Steffe et al., 1983; Steffe & Thompson, 2000) distinguished between what they called students’ mathematics and mathematics of students. Students’ mathematics is the mathematical reality that one presumes students have, but which is wholly inaccessible to outsiders (e.g., researchers). Mathematics of students is the collection of second-order models that researchers create that they hope give useful ways to think about what students’ mathematics might be. Our hope is that our characterizations of quantitative reasoning,

variational reasoning, and covariational reasoning contribute positively to the literature on the mathematics of students. We intend them as useful models of students' mathematics in regard to conceptualizing functions dynamically in ways that are propitious for later learning.

The great majority of research on students' and teachers' concepts of functions has been conducted from researchers' set theoretic, correspondence view of function. APOS research on functions, with a few exceptions (e.g., Keene, 2007; Stalvey & Vidakovic, 2015, discussed earlier) focused on students' development of ever more sophisticated understandings of functions' rules of assignments in relation to domain and range. Even research that acknowledged that variables vary often looked past students' conceptions of variation and presumed that variation existed in the situation that was presented to students (Leinhardt, Zaslavsky, & Stein, 1990). In other words, most research on functions has been largely agnostic with respect to students' understandings of functions as relationships between quantities or variables whose values vary.

One area that we anticipated would be rich with attention to covariation was research on students' understandings of functions and graphs as expressions of embodied cognition (e.g., Monk & Nemirovsky, 1994; Nemirovsky, 1994; Nemirovsky, Tierney, & Wright, 1998). This research indeed paid very close attention to researchers' interpretations of students' experiences in the context of analyzing motion. Monk and Nemirovsky also introduced an interesting construct they called "fusion," or the melding of representation and phenomenon into a single conceptual entity. Unfortunately, Monk and Nemirovsky's analyses were largely blind, from our perspective, to how students were conceptualizing the phenomena that they modeled with graphs. In terms of what students said, we would have to say that they gave no evidence of covariational reasoning beyond gross variation and gross coordination of values. Students might have been capable of more refined thinking, but the researchers' probing was directed more at students' gestures and descriptions of gross behavior than in the nuances of how they conceptualized quantities in situations. We therefore simply do not know how students in these studies conceptualized that quantities varied.

As we said, past research on students' and teachers' concepts of function was largely agnostic with regard to different ways that they might have employed quantitative or covariational reasoning in thinking about tasks. We suspect that this was because of the interplay of two factors: (1) Covariational reasoning was not a part of school mathematics, and therefore few students employed it spontaneously, and (2) most researchers were operating from the perspective of a set theoretic meaning of function, so they felt little incentive to attend to students' covariational reasoning. Put another way, researchers did not find students reasoning covariationally because they were not looking for it.

Although many researchers in the 1970s to 1990s stated the importance of students understanding that variables vary, with rare exceptions the researchers did not explain what they meant. Three studies illustrate our point. Küchemann (1978), in one of the first studies of students' understandings of variables, used "variable" exclusively as representing specific or "generalized" numbers and appeared to use "change" to mean substitution of one value for another. However, this is our inference from his characterizations of tasks and students' responses. Küchemann did not actually say what he meant when he said that a variable changes. Karplus (1979) bemoaned students' dichotomous understandings of continuous functions, but he spoke of continuous change as if his meaning would be evident to the reader, not recognizing that it could mean different things to different people and not recognizing that he had located continuous functions in situations where students might see them or not. Leinhardt et al. (1990), in an exhaustive review of research on functions to 1989, mentioned covariation as one way to

understand functional relationships (the other being correspondence) and addressed distinctions between situations that have continuous or discrete change. However, they spoke of variation largely in the sense that a symbol is a variable if it can take on different values (Leinhardt et al., 1990, p. 26).

Although we see great value in understanding the challenges that students face in conceptualizing a function's rule of assignment productively, we also think that research on this topic would be stronger if it were melded with attention to students' conceptions of quantities and variables, how they vary, and their conceptions of how they covary.

### **Curricular Treatments of Function, With Attention to Covariation**

We argued in the previous sections that ideas of quantity, variation, and covariation were at the historical and cognitive roots of many concepts from early algebra through differential equations. We have discussed and cited research that has revealed the necessity of developing students' quantitative, variational, and covariational reasoning. We have made a case that developing these ways of thinking is complex and happens over many years of schooling by engaging students in tasks and interactions aimed at nurturing their development. Our cursory review of 17 U.S. secondary precalculus level textbooks ranging from algebra 1 to precalculus, revealed, consistent with Cooney and Wilson's (1993) textbook review, that all the textbooks used a correspondence definition of function. The research we have cited in this chapter further supports the fact that U.S. curriculum and instruction are failing to develop students' quantitative and covariational reasoning abilities, contributing to many weaknesses in students' conceptions of fundamental mathematical ideas, such as variable, function, and rate of change, that are essential for understanding calculus and modeling dynamically changing phenomena in the sciences and engineering.

In contrast, the 2008 Japanese Mathematics Course of Study (JMCS) contains this statement in its grade 4 standards under the topic "Quantitative Relations":

Students will be able to represent and investigate the relationship between two quantities as they vary simultaneously. (Japan Ministry of Education, 2008, p. 11)

Moreover, the JMCS refers repeatedly, at every elementary grade level, to the need for students to represent and investigate relationships between quantities and variables whose values vary. We were curious to see how the Japanese textbooks incorporated the JMCS's emphasis on variation and covariation. A popular Japanese grade 4 mathematics textbook (Fujii & Iitaka, 2013a) contains 18 pages (of 140) that explicitly draw students' attention to quantities whose values vary or covary by asking them to envision situations dynamically and to answer questions about the variation. The fifth-grade text in this series (Fujii & Iitaka, 2013b) asks students to investigate relationships between varying quantities on 24 of 136 pages. For example, a section on volumes of solids presents the exercise in Figure 13.13. We are struck by the way that variation and covariation of quantities is prominent, but largely as the exercise's context instead of as the focus of the question. The exercise does not use the words "vary," "covary," "variable," or "function," but all these ideas are clearly present. This is an example of what we meant earlier when we said that teachers and instructional designers should focus on meanings they want students to have before introducing a term that they intend will convey that meaning.

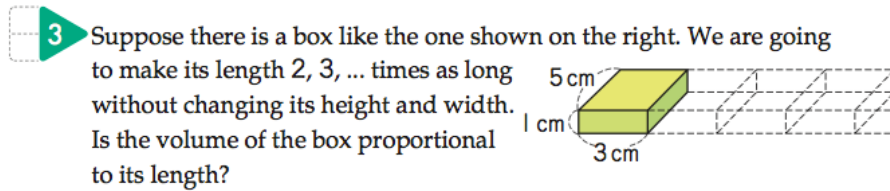


Figure 13.13. Japanese grade 5 textbook exercise. From “Mathematics International, Grade 5” (English translation, p. A24) by T. Fujii and S. Iitaka (Eds.), 2013, Tokyo, Japan: Tokyo Shoseki.

Another page in the fifth-grade text portrays someone wondering about a relationship between numbers of juice cartons and amounts of juice in language reminiscent of Newton: “As the number of cartons of juice *becomes* [emphasis added] 2, 3, 4, . . . times as many, I wonder how *the amount of juice changes?* [emphasis added]” (Fujii & Iitaka, 2013b, p. A29).

The 1992 Japanese grade 7 introduction to functions shows a photo of a weather rocket’s launch with a detailed description of how the rocket’s height above the ground, amount of fuel burned, and speed vary as the amount of time since the rocket was launched increases. This is followed by the following:

All around us, as in the advanced case of the rocket, we can see many instances of increasing and decreasing quantities. In this chapter we will be concerned with changing quantities, so let’s examine these situations. (Japan Ministry of Education, 1992, p. 96)

The text’s next section, “Introduction to Functions,” begins with the following passage, which could have been written by Euler:

When quantities vary in accordance with changes in other quantities, all these quantities are expressed as variables such as  $x$  and  $y$ . If we determine the value of  $x$ , the value of  $y$  is also determined. In situations like this, we say the  $y$  is a function of  $x$ . (Japan Ministry of Education, 1992, p. 99)

The Japanese primary mathematics texts have a clear, coherent focus on having students think about quantities whose values vary and about ways that quantities’ values vary together. By high school, Japanese texts’ authors presume that students think with images of variation and covariation and rely on this assumption as a matter of practice. In contrast to Japanese textbooks, many popular U.S. textbooks do not emphasize or support students in conceptualizing quantities and viewing function formulas and graphs as representing how two varying quantities change together. The idea of variable often is presented as representing a single unknown value. There are, however, some exceptions. In a review of secondary textbooks, Cai, Nie, and Moyer (2010) examined and contrasted two middle school mathematics programs, the Connected Mathematics Program (CMP) and Glencoe Mathematics. The authors report that CMP asks sixth-grade students to describe relationships using quantities that vary and that in seventh grade, variables are formally defined as quantities that change or vary. In contrast, Glencoe Mathematics formally defines a variable in the sixth grade as a symbol (or letter) used to represent a number and predominantly uses variables to represent unknowns in expressions and equations (Cai et al., 2010, p. 174).

Another encouraging example is the SimCalc Project housed at the Kaput Center for Research and Innovation in science, technology, engineering, and mathematics (STEM)

education. The SimCalc team produced an algebra 1 and algebra 2 curriculum that uses SimCalc MathWorlds to explore motion. The authors describe SimCalc as a piece of dynamic software “that allows the user to view and manipulate traditional functional representations such as graphs, tables and expressions (e.g.,  $y = mx + b$ ), and each of these are linked to a motion in an animated world” (Kaput Center, 2016).

These examples suggest that both curriculum developers and software designers are making attempts to support students in conceptualizing and representing covarying quantities. However, our own experiences suggest the complexity of developing these ways of thinking is often underestimated. As a result, we urge curriculum developers and software designers to collect formative data to assess the degree to which students are reasoning covariationally and developing the intended understandings. It is one thing to write curriculum that has a focus on variation and covariation and quite another to scaffold curriculum experiences that are effective in getting students to use these ways of thinking spontaneously when confronted with novel tasks that require them. In our experience, elaborated elsewhere in this chapter, multiple cycles of design, research, and revision are needed to achieve significant learning gains in students.

Another reason for optimism is the growing collaborations between mathematics education researchers and mathematicians on various projects aimed at improving student learning in mathematics. As one example related to improving and assessing curricular focus on variation and covariation, the Mathematical Association of America now offers, via Maplesoft, the Calculus Concept Readiness (CCR) exam (Carlson, Madison, & West, 2015) in its suite of mathematics placement tools. The exam’s taxonomy resulted from reviewing the literature on calculus learning and includes quantitative and covariational reasoning as two primary ways of thinking that are foundational for learning calculus. Of the exam’s 25 items, 10 require students to reason about quantities and use ideas of function, function composition, or function inverse to represent how two quantities change together (Carlson, Madison, & West, 2015, p. 215). The results from administering this exam to 631 students at four different universities, three public and one private, revealed fairly severe weaknesses in students’ ability to conceptualize and covary quantities. The mean score for these students was 10.37 (out of 25), with relatively low percentages of students choosing the correct answer on items that required students to develop formulas or graphs to represent or describe how two varying quantities change together. When

students were asked to describe the behavior of the function  $f$  defined by  $f(x) = \frac{1}{(x-2)^2}$ , only

37% of the students in the study selected the correct answer (out of five choices), that as the value of  $x$  gets larger the value of  $f$  decreases, and as the value of  $x$  approaches 2, the value of  $f$  increases. On a question that asked students to define a formula to represent the area of a circle  $A$  in terms of its circumference  $C$  only 16% of the 631 students selected the correct answer. When prompted to write a formula to define how an angle measure and vertical distance  $d$  change together (Figure 13.14), only 21% of the students selected the correct response.

Starting at  $P$  and ending at  $Q$ , an object travels counterclockwise  $k$  feet along a circle with radius 47 feet. If  $d$  represents the directed distance (in feet) from the horizontal diameter to  $Q$ , which of the following could express  $d$  as a function of  $k$ ?

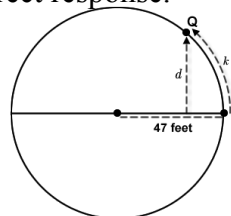


Figure 13.14. Sine function item from the Calculus Concept Readiness exam. From “A Study of Students’ Readiness to Learn Calculus” by M. P. Carlson, B. Madison, and R. D. West, 2015,

*International Journal of Research in Undergraduate Mathematics Education*, 1, p. 228. Used with permission.

These results corroborate the findings of many qualitative studies conducted in the context of beginning calculus that have revealed students' possessing static images of variables, formulas, and graphs, and limited ability to use function formulas and graphs to express invariant relationships between two covarying quantities in applied contexts.

We conclude this section by describing instructional materials developed by the second author (Marilyn P. Carlson) and her colleagues. These materials leverage research described in this manuscript related to developing students' abilities to conceptualize quantities, reason covariationally, and learn key ideas of precalculus that are foundational for calculus.

Carlson and colleagues developed curriculum and instructional resources (called Pathways) for a three-course sequence, algebra 1, algebra 2, and precalculus mathematics (Carlson, O'Bryan, Oehrtman, Moore, & Tallman, 2015). The precalculus materials are now in the fifth edition with revisions made yearly based on qualitative studies and formative data from quantitative assessments. The algebra 1 and algebra 2 materials have similarly undergone three major revisions based on formative data collected during their use.

Pathways Precalculus materials include student in-class investigations, an online textbook with embedded videos and dynamic animations, and online homework aligned with the investigations. The investigations are scaffolded to guide students in reasoning about how quantities vary together as an approach to constructing expressions, formulas, and graphs that are based in the meanings that students have constructed. There is a concurrent focus on structural aspects of algebra and equation solving. This occurs naturally after the invariant nature of two covarying quantities has been represented algebraically or graphically.

The course begins by engaging students in tasks that might be viewed as middle school mathematics in other countries. The course materials begin at this level because our early data led us to the realization that it was impossible to build rich meanings for exponential, polynomial, rational, trigonometric, and other function types when students viewed variables as unknown values, were unable to conceptualize quantitative relationships in a problem context, and had no intuition about the usefulness of function formulas and graphs for describing an invariant relationship between two quantities that change together. Our data early in the project revealed that fewer than 25% of students were able to define a formula to represent the distance between two girls (Lisa and Sarah) in terms of the rates at which they walked and the number of seconds since they began walking toward each another from a distance of 140 feet apart. Sarah was walking at a constant speed of 4.5 feet per second and Lisa was walking at a constant speed of 5.5 feet per second. Follow-up interviews with students revealed that they were not constructing an accurate image of how the quantities in the situation were related (e.g., the distance between Sarah and Lisa is originally 140 feet, it decreases at the rate of 10 ft/sec, and hence the distance between them at moments in time has decreased by 10 ft/sec times the number of seconds since they began walking). Even fewer students conceptualized  $4.5t$  and  $5.5t$  as varying distances. Other students wrote  $140 = 4.5t + 5.5t$  and attempted to solve for  $t$ . Many students wrote " $t = \text{time}$ " but did not view the letter  $t$  as representing a varying value nor as representing the number of seconds since Lisa and Sarah began walking.

The problem of being unable to conceive situations in terms of quantities is not limited to school students. We discussed the Sarah and Lisa problem with precalculus teachers in a recent summer workshop for teachers preparing to use Pathways materials. One teacher, Phyllis, wrote  $140 - 4.5t - 5.5t$  to represent the distance between Sarah and Lisa and explained that she now

needed to solve for  $t$ . When probed to explain what a solution would represent, she reread the question and then drew a picture of two stick figures with a straight line connecting them labeled “140 feet.” Phyllis initially defined the variables  $t$  and  $d$  as  $t$  = time and  $d$  = distance. The workshop leader worked with Phyllis extensively to help her conceptualize the quantities in the situation, how they are related, and how they vary together. Phyllis eventually wrote “ $t$  = the number of seconds since Sarah and Lisa spotted each other”, and “ $d$  = the number of feet between Sarah and Lisa since they spotted each other.” She then drew a line with an arrow pointing from each stick figure toward the center, and wrote  $4.5t$  above one line and  $5.5t$  above the other line. Phyllis explained that  $4.5t$  represented the number of feet Sarah had walked in  $t$  seconds. Her algebraic expressions now provided a mechanism for representing the quantitative relationships and how values of specific varying quantities change together. It is worth noting a crosscutting practice in the Pathways materials of asking students to construct new terms and expressions, such as  $4.5t$  and  $140 - 10t$ , for the purpose of representing specific varying quantities in the problem context. This explicit attention to thinking about terms, expressions, and formulas as ways to represent variation of quantities’ values in a problem context appears to bridge students’ conceptions of how pairs of quantities covary and the usefulness of algebraic symbols for representing this covariation.

This encounter with one teacher illustrates that it is not straightforward for teachers to conceptualize quantities in a situation when they have been teaching from traditional materials that fail to emphasize variational thinking but instead focus almost exclusively on methods for finding the unknown values of a variable. Phyllis said that she had never thought about approaching word problems in this way and that she had always thought of precalculus as a course where students refined their algebraic skills for calculus. The question of how much and what kind of support teachers will need to guide students in developing meaningful formulas and functions and understanding key ideas of precalculus that are foundational for calculus is an area where more research is needed and where the second author and her research colleagues (Tallman, 2015; Teuscher, Moore, & Carlson, 2015) are currently focusing one strand of their research.

Carlson and colleagues used the Precalculus Concept Assessment (PCA; Carlson, Oehrtman, & Engelke, 2010) over the past 7 years to track student performance in Pathways Precalculus classes. The PCA is a 25-item multiple-choice exam that assesses students’ function understandings and readiness for calculus. Classes of Pathways Precalculus students have mean post-PCA scores ranging from 13.8 to 19.5 (out of 25), as compared to mean post-PCA scores ranging from 7.9 to 10.2 in classes using traditional precalculus curricular materials. This data suggests that the curriculum’s focus on supporting students in seeing variables as representing values of a quantity that varies, and formulas or algebraically defined functions as how the values of two varying quantities covary, is improving students’ understanding of and ability to use functions productively.

In addition to using the PCA to track student learning, Carlson and colleagues regularly administer other select research items to Pathways students for which we have historical data. The two items (Sarah and Lisa item, and the sine function item) discussed above were administered to 1,132 Pathways Precalculus students at a large public university during the fall semester of 2015. Of these Pathways students, 87% selected the correct answer to the Sarah and Lisa question, and 62% selected the correct answer to the sine function item. As a comparison, 25% of 1021 students at four universities using traditional precalculus curricula answered the Sarah and Lisa item correctly and 21% answered the sine function item correctly.

The Pathways research group continues to study student learning in the context of Pathways curriculum and teachers' implementation of Pathways materials. They use the results of their studies to make adaptations that they conjecture will better support students in reasoning covariationally and acquiring critical understandings needed both to solve nonroutine problems and to learn calculus.

### **Concluding Remarks and Directions for Future Research**

We began this chapter by providing a brief historical account of the centrality of variational and covariational reasoning in the early evolution of the idea of function. Our historical account described how advances in real analysis led to a sharp decline in thinking of functions covariationally, which led to an emphasis in school textbooks that continues today on the set theoretic definition of function. We elaborated our meaning for variational and covariational reasoning, including subtle distinctions between chunky and smooth variational thinking and the necessity of coupling two quantities' values simultaneously (forming a multiplicative object) in one's mind, and provided a two-part framework that outlines levels of reasoning variationally and covariationally. Our description of covariation included other subtle and important ways of thinking, such as imagining that variables both pass through and assume all values in a continuum even when considering how a quantity varies over intervals of change. We then reported on studies that make a case for the necessity of developing students' abilities to imagine quantities' values varying smoothly and continuously for success in secondary and tertiary mathematics and commented briefly on the fact that most past research on functions was carried out from a correspondence view of function and therefore was agnostic about the role of covariational reasoning in students' understandings of function. Finally, we examined the role of covariational reasoning in current U.S. school mathematics textbooks. We found that covariational reasoning plays essentially no role in standard textbooks but appears strongly in three research-based text series, one at the middle school level and two at the high school level.

We also emphasized a number of themes repeatedly:

- Quantitative and covariational reasoning affords students powerful ways to conceptualize and represent dynamical phenomena.
- For students to learn advanced mathematical ideas that build on quantitative and covariational reasoning later in their schooling, they must build quantitative and covariational ways of thinking in school mathematics, and building these foundational ways of thinking in their school mathematics can have the salutary effect of making the mathematics they learn in school more coherent.
- Researchers must attend to the nature and substance of students' covariational thinking when they attribute covariational reasoning to students. How did students envision quantities or variables varying? What was the nature of the connection they made between quantities' or variables' values?
- Covariation happens in minds—students', teachers', and researchers'. When a researcher says that a situation involves variation or covariation, he is saying how he conceives the situation. The question remains as to how the student or teacher conceives the situation.

In prior sections we pointed to needing future research and promised to revisit our calls in this section. These are the areas in need of future research:

1. Students' development of smooth continuous variational and covariational reasoning

2. Students' conceptions of number lines
3. Students' uniting two quantities' values (creating multiplicative objects of them)
4. Connections between students' conceptualizing functions covariationally and conceptualizing them set theoretically (i.e., domain, range, rule of assignment)
5. Relationships between basic research on covariational reasoning and research on effects of different curricular approaches that support students in developing it
6. Relationships between reasoning covariationally and reasoning about functions via rules of assignment
7. Potential benefits of covariational reasoning for students' reasoning structurally about functions, and potential obstacles that an emphasis on covariation might create
8. Experiences of teachers who do not reason covariationally who are nevertheless called upon to support students learning it with instructional materials designed to support them
9. Relationships between basic research on students' learning to reason covariationally and the formation of policies and standards for mathematics teaching and curricula

We suspect that many of these research foci cannot be tackled independently of one another. We doubt that many students develop smooth continuous reasoning spontaneously, so researching its development must be done in the context of instruction that is intended to support it, using materials designed to support students and teachers in the teaching and learning process.

Second, the field is in dire need of longitudinal research on the learning and teaching of smooth continuous reasoning and on students' formation of multiplicative objects. We simply have no models of how such learning at early grades might be leveraged profitably at later levels when it is targeted systematically over students' schooling. We suspect that if students develop an early form of smooth continuous variational and covariational reasoning in lower grades, while also learning to reason quantitatively and to represent their reasoning symbolically, existing curricula in upper grades will not be relevant for what they are prepared to learn. We also suspect that teachers cannot teach students to conceptualize multiplicative objects through direct instruction. Instead, students will develop the ability by engaging in tasks and discussions that, from a designer's perspective, demand it. This is, however, an empirical question, awaiting careful and sustained study.

It is especially important to research the demands placed upon teachers, and their accommodations to them, when they become expected to support students' development of smooth continuous reasoning in the various areas that teachers already teach. Smooth continuous reasoning is important in precisely the ways that many teachers are unprepared to support—developing coherent understandings and meanings of mathematical ideas in contexts. This research direction is critically important, perhaps the most important of those we have mentioned.

At the same time, we should be prepared to find that moving curricula and instruction toward a greater emphasis on quantitative reasoning and smooth continuous reasoning will require a huge investment of financial and intellectual resources in teacher professional development and in transforming undergraduate teacher preparation. Why? Because our

experience in researching teacher professional development and our experience in designing undergraduate teacher preparation programs is that it is extremely difficult for adults to develop these ideas and ways of thinking when their entire mathematical experience has been about numbers and static variables. However, the investment must be made and sustained over a long period before we can start to recruit future teachers whose school mathematics entailed reasoning quantitatively, smoothly, and continuously.

We finish with remarks on standards. The Japanese Mathematics Course of Study (JMCS) is an example of standards that provide a concise, coherent statement of meanings and ways of thinking that students should develop. Of course, those meanings and ways of thinking are in the service of reasoning mathematically and learning symbolic methods, but the methods are proposed as being grounded in meanings and ways of thinking. Also, the JMCS has the advantage of existing within a culture of shared meanings, so they do not need to try to teach curriculum writers and teachers what they mean. Authors and teachers already know what they mean because the meanings are culturally shared.

The U.S. Common Core State Standards for Mathematics (CCSSM; National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA Center & CCSSO], 2010) does not have the same luxury as the JMCS. Variational and covariational reasoning are not culturally shared meanings in the United States, so it was incumbent upon the CCSSM writers to explain what they meant. Unfortunately, their explanations do not convey to readers a coherent picture of the robust practice of variational and covariational reasoning about quantities in the sense we have discussed in this chapter. For example, CCSSM refers repeatedly to the need to reason about quantities. Indeed, “reason abstractly and quantitatively” is one of CCSSM’s eight Standards for Mathematical Practice (p. 6). However, the CCSSM’s definition of a quantity is “a number with a unit” (p. 58). So, the practice of “reason abstractly and quantitatively” translates into “reason abstractly and with numbers that have units,” which is hardly a strong practice when the intent is that students create meaningful models of dynamic situations.

The CCSSM (NGA Center & CCSSO, 2010) refers repeatedly to functions. The word “function” is on 21 pages of the CCSSM. In every instance, “function” is used (1) to name a type of function (linear, quadratic, trigonometric, exponential, rational, polynomial); (2) to reiterate that a function takes an input and produces a unique output; or (3) to establish that a function is a correspondence between elements of two sets. Even where the word “function” is used in the context of relationships between quantities, it must be remembered that, in the CCSSM, quantities are “numbers with units.”

It is possible to identify passages in the CCSSM that attend to variation and covariation in some respect. Among other examples, in the standards for grade 6, one can find the following: “In a problem involving motion at constant speed, list and graph ordered pairs of distances and times, and write the equation  $d = 65t$  to represent the relationship between distance and time” (NGA Center & CCSSO, 2010, p. 44). And, in grade 8 the CCSSM includes a discussion of linear equations and rates of change:

Students use linear equations and systems of linear equations to represent, analyze, and solve a variety of problems. Students recognize equations for proportions ( $y/x = m$  or  $y = mx$ ) as special linear equations ( $y = mx + b$ ), understanding that the constant of proportionality ( $m$ ) is the slope, and the graphs are lines through the origin. They understand that the slope ( $m$ ) of a

line is a constant rate of change, so that if the input or  $x$ -coordinate changes by an amount  $A$ , the output or  $y$ -coordinate changes by the amount  $m \cdot A$ . (p. 52)

From the perspective of a mathematician or a mathematics education researcher, it is easy to see covariation in such statements. However, research tells us that students and teachers typically do not. For example, there is ample evidence that students do not understand that slope is a rate of change (e.g., Lobato & Thanheiser, 2002; Nagle, Moore-Russo, Viglietti, & Martin, 2013), and it will require much more than simply referring to it as such to convey to students or teachers what it means to understand rate of change. Moreover, it is a decidedly nontrivial leap to convey to students, teachers, or curriculum writers that rate of change entails variation and covariation. Mentioning “rate of change” will not convey variation or covariation to most readers of the CCSSM. Stump (1999) studied 39 U.S. in-service and preservice high school teachers’ concept of slope. She stated that most teachers in her study understood slope as a geometric ratio (“rise over run”), and made little or no connection to rate of change or function. We see no evidence that the situation is different today. Coe (2007) conducted an in-depth study of conceptual connections that three highly regarded high school mathematics teachers made among function, slope, rate of change, variation, and covariation. His semantic maps of their meanings for each showed that variation and covariation were largely absent in their thinking and that their meanings for function, slope, and rate of change were highly isolated. Indeed, one teacher exclaimed that she had never thought about why one divides to calculate slope until Coe asked her. Sofronos et al. (2011) surveyed 24 “national authorities in mathematics, particularly calculus” on what students should learn from first-year calculus. Not one authority mentioned variation or covariation, in any context. These ideas were not on their radar.

Given these hurdles, we leverage the opportunity this chapter offers to convey to our readers the difficulty that mathematics education faces in pushing for reforms in curricula and instruction. In particular, we emphasize that, when providing guidance for reform, we must go far beyond mentioning names of ideas or sharing examples of formulas as if they capture those ideas. To fail to do so would be a disservice to the students and teachers who must actually surmount the intellectual obstacles inherent in meaningful learning or teaching.

### Notes

1. We anticipate the objection that, on the surface, phrasing covariation as involving two variables or quantities whose values vary eliminates the possibility of considering, say,  $y = 5$  to be a function. This is the problem of the null case of a general definition, as in “Consider a square that has side lengths of 0 inches,” or “Consider an angle that has a measure of 0 degrees.” A discussion of students’ difficulties with null cases for general definitions, while important, is outside the scope of this paper.

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