

Learning beginning algebra in a computer-intensive environment

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Abstract We present a design research on learning beginning algebra in an environment where spreadsheets were available at all times but the decision about using them or not, and how, in any particular situation was left to the students. Students' activity is analyzed in Kieran's framework of generational, transformational and global/meta-level activity, and compared to the designers' intentions. We do this by focusing on the activity of one student in four sessions spread over several months and discussing the activity of 51 additional students in view of the analysis of the focus student. We show that the environment enables a number of different entries into algebra and as such supports students in becoming autonomous learners of algebra, and in making the shift from arithmetic to algebra via generational and global/meta-level activity before dealing with the more technical transformational activities.

Keywords Beginning algebra · Computer intensive environment · Algebraic generational activity · Algebraic transformational activity · Global/meta-level activity · Instrumental genesis and instrumental orchestration

1 Introduction

The framework of the present study interweaves three areas of research on learning mathematics: (1) students' difficulties while making their first steps in algebra (e.g. Kieran 1992; Yerushalmy 2005); (2) technological tools as support in students' transition from arithmetic to algebra (Sutherland and Balacheff 1999; Sutherland and Rojano 1993); and (3) research related to instrumental processes (Artigue 2002), the associated design of learning activities and their realization in the classroom.

We conducted an intervention study using a design research approach (e.g. Cobb et al. 2001) in a unique computer-intensive environment (CIE) with the following three characteristics: (1) full and unconstrained access to spreadsheets (Excel) for students in class and at home; (2) freedom to choose the strategy employed to deal with the proposed problem situations, including if, when and how to use the computerized tools; and (3) learning activities carefully designed as *generational* and *transformational* beginning algebra¹ activities (Kieran 2004). Except in some problems intended to familiarize students with the use of spreadsheets and their use as a tool, the teacher supported and legitimized any choices made by students regarding the ways, means and strategies used to solve the problems. Thus, the “intensiveness” of the environment relates to the availability of the computerized tools at all times—but not necessarily to the “intensiveness” of their use.

In this paper we report on the first half of a year-long classroom intervention, including the considerations

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¹ We use the term ‘beginning’ rather than ‘early’ algebra because we refer to a first course in which algebra is an explicit topic of instruction.

underlying its design and implementation. The research was aimed at characterizing the unique nature of students' transition from arithmetic to algebra as a process of accumulating understandings and skills in the designed learning environment. The designed activities in this first half of the year were mostly generational or global/meta-level. This order is opposite to the "classic approach" of teaching beginning algebra, in which transformational activities are introduced first.

2 Theoretical background

This section provides the theoretical, empirical and methodological framework for the study, and is therefore organized around the themes: learning beginning algebra, learning algebra with computerized tools, symbolic generalizations in a spreadsheet environment, instrumental genesis, and learning mathematics while computers are constantly available.

2.1 Learning beginning algebra

A classical approach to the didactics of algebra has been presented by Malle (1993). As other classical approaches, Malle starts from variables, symbols, expressions, formulas and equations, and stresses transformations of these. Consequently, a central chapter in Malle's book deals with a cognitive model enabling a systematic treatment of the schemas—correct and wrong ones—used by students when transforming algebraic expressions.

Kieran (2004) has characterized school algebra and algebraic activity as consisting of three main interrelated components: (1) *generational activities* that involve forming expressions and equations arising from quantitative problem situations, geometric patterns, and numerical sequences or relationships; (2) *transformational activities* that include mainly changing the form of expressions and equations while maintaining equivalence; and (3) *global/meta-level activities*—such as problem solving, predicting, modeling, generalizing and justifying—for which algebra is used as a tool.

In many countries students are introduced to algebra after 6 years of learning arithmetic, basic geometry, and possibly some data handling at elementary school. The shift from arithmetic to algebra requires abandoning many of the views and practices deeply rooted in arithmetic. For example, in arithmetic the equal sign is usually regarded as an invitation to calculate: " $3 + 2 =$ " means "add up the numbers and write the sum to the right of the equal sign", whereas in algebra an expression such as " $a + 3 =$ " remains as is and no calculation is possible. In algebra, the equal sign has other meanings that are new or almost

new to students, including equivalence as in $a + b = b + a$, or assignment, as in $f(x) = 3x$. Learning and getting used to these new meanings is problematic for many students (e.g. Knuth et al. 2006). What students experience when confronted with the many shifts in perspective occurring in the transition from arithmetic to algebra has been characterized as a "didactical cut" (e.g. Ainley 1996; Rojano 2002; Sutherland and Rojano 1993; Yerushalmy 2005) that needs to be investigated and understood by teachers, designers and researchers in order to support students in overcoming it.

2.2 Learning beginning algebra in a computerized environment

Many mathematics educators have proposed making use of the potential of technological tools to support students' transition from arithmetic to algebra. Several algebra projects based on partial use of different kinds of computerized tools have been developed, implemented and studied (e.g. Dettori et al. 2001; Haspekian 2005; Hershkowitz et al. 2002; Kieran 1992; Wilson et al. 2005; Yerushalmy, 2009). While the conclusions of these studies are not in full agreement, they led to new approaches to algebra, among them the functional approach, which recommends exploring changing phenomena that can be represented numerically, verbally, symbolically or graphically (e.g. Heid 1995; Hershkowitz et al. 2002; Radford 2000; Yerushalmy 2009) and hence provide them with opportunities for generalization and modeling in several representations (e.g. Bednarz et al. 1996; Yerushalmy 2005).

Kieran (2004) points out that because of the advancement of cognitive research and the emergence of technological tools, the dominant trend in teaching and learning algebra shifted from an emphasis on transformational work to a focus on generational and global/meta-level activities, which provide a starting point and motivation for transformational activities. We designed the sequences of activities for the present study in this spirit (see Sect. 4).

2.3 Symbolic generalizations in a spreadsheet environment

In pencil-and-paper learning environments, three stages of generalization processes by pre-algebra students have been identified (Arcavi 1994; Friedlander et al. 1989; Hershkowitz and Arcavi 1990). In the first stage, students tend to represent different quantities involved in a given situation by different letters, disregarding existing relationships between the quantities these letters represent. In the second stage, students start to express simple relationships, such as consecutive numbers, appropriately, but still use different letters in more complex relationships. Only in the third

stage are students able to express full relationships among related quantities symbolically (Tabach and Friedlander 2004). Dörfler (2008) considers all three stages of symbolic generalizations as reflecting algebraic thinking.

In a pencil-and-paper learning environment, algebraic manipulations on expressions which do not represent the full symbolic relationship do not usually lead to insightful results. With spreadsheets, however, students are able to operate, to experiment with, to reflect on and to learn the relationships between the quantities involved even in the earlier stages described above (Stacey and MacGregor 2001). Students may use the tool with a pure numerical approach (entering numbers one by one in the cells) without symbolizing any relationship. Symbolic generalization can then be expressed in one of several “algebraic” ways:

- “Multi-variable approach” “dragging” and considering a whole array of numbers (i.e. a column in the spreadsheet) as a “variable” dependent on another one (Dreyfus et al. 2001); for example, if the length of a rectangle is five times its width, students may enter the length and width of the rectangle in columns A and B, respectively, and in a third column calculate its area by $A*B$, concealing the dependence between the length and width;
- “Recursive generalization” the use of recursive expressions, which emphasize local relationships between consecutive elements such as contiguous cells of a same column (Tabach et al. 2006);
- “Explicit expression” expressing the general relationship and displaying the full relationships among the variables in an Excel symbolic language.

Previous studies indicate that initially many students prefer recursive generalizations (based only upon local connections) over explicit formulae, which require a generalization of a more global nature (Friedlander and Tabach 2001; Stacey and MacGregor 2001). Spreadsheets provide different students the possibility of working at different stages of symbolic generalization, and yet all approaching and successfully dealing with the same problem situation. Students may stay with numerical, “multi-variable” or recursive strategies for as long as need, while they slowly gain confidence in working with symbols. All three levels of symbolic generalization serve not only as descriptive of the structure identified by students (Dörfler 2008), but also as the key for creating sequences of numbers describing the phenomenon under study.

In spite of using different strategies, and obtaining their results in different presentations, students are able to generalize, mathematize, communicate and judge their results and their peers’ results, and discuss them with each other. In fact, students can explore functional relationships even

without being explicitly aware of them. Yackel and Cobb (1996) state: “...students can take over some of the traditional teacher’s responsibilities only to the extent that they have constructed personal ways of judging that enable them to know in action both when it is appropriate to make mathematical contribution and what constitutes an acceptable mathematical contribution” (p. 473). The socio-mathematical norms, developed in classrooms where students learn to make their own choices according to their mathematical needs, encourage students to be more autonomous in their mathematical actions. Whereas mathematization and generalization are at the core of the activity, spreadsheets enable students to temporarily remain within and to rely upon the numerical realm (handling large sets of numerical data by “dragging”) and to slowly get acquainted with the use, the purpose and the power of symbols. Thus the ideas of algebra become the core, and algebraic language is introduced according to students’ needs and ability. The Excel language is mediating this shift, allowing students to become accustomed to the new ways of thinking within meaningful contexts. In this sense the spreadsheet becomes a semiotic mediating tool in learning algebra.

Some researchers view spreadsheets as a tool for making sense of the dynamic aspect of functional relationships between the value of a variable and the value of the expression—the dependent variable (e.g. Drouhard and Teppo 2004). For example, if in a spreadsheet one writes the expression “ $=2*A2 + 1$ ” into cell B2, then the content of cell B2 is *dynamically* dependent on the content of cell A2. Moreover, many more cell-pairs like A2, B2 can be created, together representing the function $f(x) = 2x + 1$. Hershkowitz et al. (2002) pointed out that the use of spreadsheets to investigate processes of variation enables students to spontaneously use algebraic expressions. Spreadsheet users employ formulae (expressed in a spreadsheet syntax) as a natural means to construct numerical tables describing a given phenomenon. The present study benefits from this approach.

There is a difference between using letters in a functional situation and using letters in an ‘equation with unknown’ situation. In functional situations, generalization is usually an intrinsic part of the generational activity, because number patterns tend to be involved and the generated expression reflects the relationships in the number patterns. Situations with letters as unknowns frequently do not involve generalization. In this paper, we consider generalization from functional situations.

2.4 Instrumental genesis lens

The powerful theory of instrumental genesis can be used to characterize aspects of students’ learning in a computerized

environment. Instrumental genesis is a theoretical construct (Vérillon and Rabardel 1995) proposed on the basis of empirical findings to describe how computerized tools become instruments for the students and how diverse this process can be. For example, strategies to solve the same task, using the same tools, within the same classroom, can be diverse among individuals and groups in the classroom (Artigue 2002; Mariotti 2002). When students begin to use computerized tools, they construct an image of what the tool can and/or should do for them. This image is strongly related to their initial experiences, beliefs, the perceived nature and goals of the activities, dialogs with peers, and results of spontaneous explorations and serendipitous discoveries, especially when the initiative to use the tool (or not to use it) is left to students and their needs. Vérillon and Rabardel (1995) defined *instrumental genesis* as the process by which individuals are creating and changing their image of a tool during the performance of different tasks. Instrumental genesis is considered to be a bidirectional process, in which both tool and user are changing. Trouche (2004) called these two aspects of the process *instrumentalization* and *instrumentation*.

Whole-class discussions, orchestrated by the teacher (Trouche 2004), can serve as an appropriate forum to talk about and share students' personal instrumental geneses in order to further enhance them. Thus, instrumental genesis involves not only cognitive processes, which change the nature of the mathematics learned and re-position the difficulties thereof, they also involve socio-cultural processes concerning both individuals and whole classrooms, which change the dynamics of learning and teaching (Laborde 2003; Lagrange et al. 2003). In this sense, instrumental genesis in the classroom is a diverse socio-cognitive process, like other learning processes, and more than that: it emphasizes again the characteristic of autonomy it inserts into students' mathematical actions (Yackel and Cobb 1996). In this study, instrumental genesis will serve as the lens through which we will analyze findings and draw conclusions.

2.5 Availability and use of computers

Recently, there has been a growing interest in learning environments in which computers are available to students and teachers at *all* times. Studies on such environments (e.g. Gardner et al. 1993; Rockman et al. 1999) usually focus on outcomes rather than processes, showing advantages and gains, such as improvement of reading and writing skills, better organization of written work as a whole and argumentation in particular, improvement of self-esteem, involvement, etc. Although there are reports of partial uses of computers in mathematics classrooms or labs, there are almost no reports on the teaching and learning of

mathematics in an environment where computer tools are available at all times, and the students have autonomy to choose if, when and how to use them. An exception to this are investigations on the use of Computer Algebra Systems (CAS) with hand-held computers, as reported, for example, by Goos (2012). A study in a CIE that did not relate to CAS is the experimental setting reported by Shternberg and Yerushalmy (2003) that aims at developing software to support modeling activities by middle school students. Moreover, students who have had experience in a general CIE report that mathematics is the subject in which computer use is the lowest, and that problem solving usually takes place with pencil and paper only, even when computers are fully available (Rockman et al. 1999).

3 Methodology

3.1 Aims of the design research

This study aimed at investigating learning at the transition from arithmetic to algebra as it occurs in a purposefully designed course in a CIE and is expressed by generational, transformational and global/meta-level activities. Our main design goal was a “smooth transition” from arithmetic to algebra, as envisaged by the *CompuMath* project (Hershkowitz et al. 2002). Our working hypothesis was that the students will more flexibly control their own transition, and the transition will be more transparent for us as researchers because of the full and unconstrained access to computerized tools afforded by the CIE, where the decision if, when and how to use the tool is in the hands of the user.

The design intention was to develop independent algebraic problem-solvers, who can approach an algebra problem presented in several representations and make use of available tools in their effort to solve the problem. Two questions guide our analysis. First, in light of the unique learning environment, in what ways does the CIE shape cognitive and socio-cultural aspects of learning beginning algebra in the seventh grade? Second, to what extent does actual learning shift students from arithmetic to algebra at their autonomous pace?

3.2 Participants

The classroom study was conducted during two consecutive school years, each year in one seventh grade class (26 students in each cohort, working mainly in pairs). In both cohorts, the first author served as teacher and researcher (Tabach 2011a) and the second author was one of two observers. At the beginning of the year, students were given a test in order to assess their problem-solving skills and to investigate their initial use of algebra.

Amy, a student from the second cohort, exhibited throughout the course a positive attitude towards the use of computer software while also incorporating other resources during her algebraic work. She scored below the class average on the pre-test. By the end of the year, she had progressed and scored above average on the final exam. In this paper, we will focus on Amy as a paradigmatic student, because she was flexible and insightful in her thinking, and because of the changes she underwent during the course. Note that Amy changed her peer from one activity to the other; hence in order to follow changes along the year, our analysis of data focuses on Amy's work in the dyad and not on the dyad as a whole. As each student kept her own records, we were able to trace Amy's notebook. In addition, data about the entire class will be described and analyzed.

3.3 The course

The learning materials were developed as an adaptation to the specificities of this CIE from an existing beginning algebra course designed for partially computerized learning environments (Hershkowitz et al. 2002). The course consists of a series of problem situations designed on the basis of a functional approach to algebra. Along the entire course, problems are intended to serve as openings and motivation that lead students (a) to generalize from real-world phenomena by means of symbolic representations, and (b) to carry out generational and transformational activities that emerge from the problems.

As mentioned above, the course is designed to propose generational and global/meta-level activities first, while engaging in problem solving with the mediation of Excel, and only later to add meaningful transformational activities. In this study we describe the first half of the course and hence focus mostly on generational and global/meta-level activities.

The students were initiated into the world of symbolic generalizations during the very first algebra activity (lasting two lessons). During the first lesson, they worked in a pencil-and-paper environment on a problem involving changing phenomena (see Fig. 1). During the second lesson, the use of Excel was introduced (see Sect. 4.1). Students were taught how to enter an expression (Excel formula) into a cell and produce a column of values by dragging (Tabach, Arcavi and Hershkowitz 2008).

Four activities from different points in time along the first half of the school year were chosen in order to demonstrate Amy's transition from arithmetic to algebra.² The

reasons for choosing these specific activities were the following. Each activity demonstrates a change in the development of the transition towards algebra. The very first algebra activity was chosen because it presents the first opportunity for the students to use Excel (with instructions). The second activity was chosen because it was the first opportunity for the students to use Excel and to organize the data without instructions. In the third activity Amy employed a unique strategy, which suggested her need to generalize. And the fourth activity is different from the previous ones in that it does not involve a changing phenomenon, and yet encourages transformational activity.

3.4 The teacher and her role

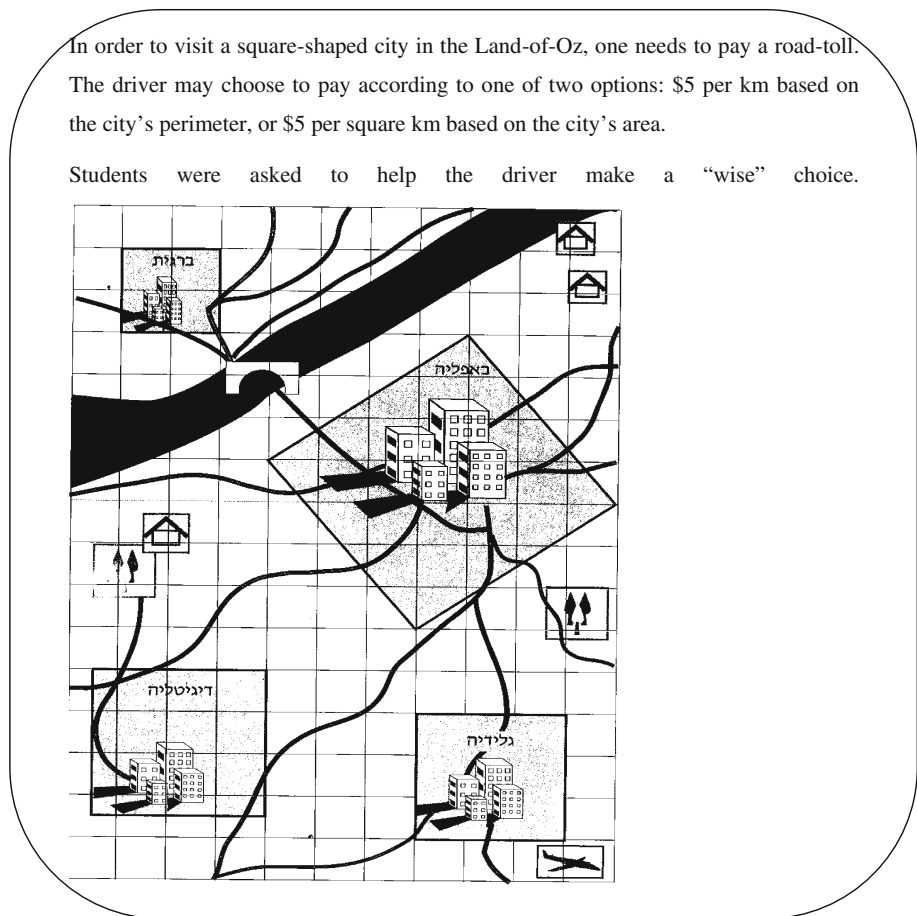
In the CIE learning environment in this course the teacher has quite a complex role. As in every designed learning environment the teacher's role is to bring the designed learning into realization. As mentioned above, in a technological environment students are going through an instrumental genesis process. Trouche "introduced the term *instrumental orchestration* to point out the necessity (for a given institution—a teacher in her/his class, for example) of *external steering* of students' instrumental genesis" (2004, p. 296, emphasis in the original). Instrumental orchestration also includes a socio-cultural aspect, as the technological medium serves as a boundary object between teacher and students, where "mutual negotiation and meaning-construction is the norm for both sides" (Hoyles et al. 2004, p. 321). We perceive the role of the teacher in light of the instrumental orchestration notion:

Instrumented orchestration is defined by four components: a set of *individuals*; a set of *objectives* (related to the achievement of a type of task or the arrangement of a work-environment); a didactic *configuration* (that is to say a general structure of the plan of action); a *set of exploitations* of this configuration. (Guin et al. 2005, p. 208)

In the present study, *the set of individuals* was composed of the students and the teacher, and was relatively stable across lessons. The *didactic configuration* of the lessons throughout the school year had a general uniform format with three lesson phases: opening whole-class discussion, student work in pairs, and summary whole-class discussion. At the beginning of each 90-min class period, the teacher presented a problem situation followed by a short class discussion to ensure that the situation and terms of the problem were understood. During the main part of the lesson, students worked on the problem in pairs, possibly using the computer, and the teacher acted as a facilitator. During this part the teacher, who was listening and

² A subset of the examples regarding Amy's work was presented by Tabach (2011b).

Fig. 1 The Toll Road situation and a partial map of the Land-of-Oz



observing the group work, accumulated information for handling a whole-class discussion as part of her instrumental orchestration. Finally, a whole-class discussion led by the teacher was conducted, based on the information she had gathered.

Regarding the *set of exploitations* of this configuration, in this case study the teacher prepared parts of her instrumental orchestration in advance, based on her familiarity with the curriculum materials. Other parts were created spontaneously during each lesson, based on her observations during the group-work phase of the lesson. In this way, a time dimension that is related to didactical performance (Drijvers et al. 2010) was realized.³

3.5 Data collection

Several tools were used to document classroom learning processes. Classroom observations by the second author and another researcher were conducted and a detailed

research diary was written immediately after each lesson. The work of five randomly selected pairs of students was audio-taped in all lessons; several lessons were also video-taped. All files that students saved on their computers were collected. The written work and the assessments of the students were also collected.

3.6 Data analysis

In the current study we use Kieran's (2004) model for a priori analysis of the four chosen activities, as well as for a posteriori analysis of students' work on these activities, in order to examine to what extent the designers' intentions concerning the hypothesized trajectories of learning (Simon 1995) were reflected in students' work.

Amy's work is described and analyzed in detail. For each example we also analyzed the work of the two cohorts as a whole, in a more quantitative manner. Hence, the presentation of each activity is structured into five parts: (a) description of the problem situation; (b) a priori analysis in terms of Kieran's model; (c) Amy's work; (d) other students' work and the teacher's role; and (e) conclusions.

³ See Tabach (2011c) for more information about the teacher's role in this project.

4 Findings

This section is organized around the four activities.

4.1 Activity 1: the *Toll Road* situation (September 8)

(a) *The situation* The situation was presented verbally and visually, by means of a small map (Fig. 1).

(b) *A priori analysis* The task is *generational* in nature, as there is a need to generalize the verbal–pictorial representation to a symbolic one in order to analyze the numerical representation of the two ways for paying the road toll, and comparing the payments. As mentioned above, this was the first use of Excel during the school year, and hence students got instructions regarding the table’s headers, and the expression to be entered for obtaining the toll in column B (only). If B represents one of the payments (or one of the payments is entered in column B of an Excel sheet), and C the other, the comparison may be carried out by using one of two symbolic expressions, $=B - C$ or $=C - B$, both of which are multi-variable expressions. In both cases, some of the resulting numbers will be negative.

(c) *Amy’s work* Amy used a four-column Excel table (Fig. 2) with the expressions $=A_4*5$ and $=A*A_5$ in columns B and C reflecting the two ways of paying the toll. To compare, Amy used *two* expressions in column D: in the first four rows she used the expression $=B - C$, while in the remainder she used $=C - B$. In this way, she avoided negative numbers, which had not yet been learned in class. Amy is acting at an intermediate level of generalization, which is enabled by Excel. Also, Amy generalized from the numerical output she observed in column D back into symbolic generalization, by changing the order of variables so as to subtract the smaller amount from the larger amount.

(d) *Other students’ work and the teacher’s role* All students used a four-column table and explicit expressions for the two methods for paying the toll (columns B and C). In column D, 56 % of the students used a single multi-variable expression (either $=B - C$ or $=C - B$), while

44 % used Amy’s strategy—using two multi-variable expressions in order to avoid negative numbers.

During the whole-class summary discussion that followed, the teacher invited students to report on their use of expressions for describing the saved amounts in column D. During this discussion, the meaning of signed (negative and positive) numbers was discussed, as well as the considerations that led students to use two expressions. It is an appropriate example for pointing out the way the teacher considers and performs her role; she encourages students to report and reflect on their mathematical actions, to listen to and analyze other students’ mathematical actions and, above all, she supports students in constructing new mathematical concepts upon need.

(e) *Conclusions* Amy’s behavior indicates her understanding of the situation and ease of access to this generational activity. She shows flexible use of expressions in a spreadsheet environment from the very beginning of the school year. Also, she observes the outcomes of her actions, and critically reflects upon them. For Amy, this task was not only *generational* in nature, but also a *global/meta-level activity*, as she devised a way to use symbolic expressions so as to avoid negative numbers. Amy’s behavior reflects, to some extent, other students’ work (44 %). During the subsequent whole-class discussion, as the students discussed the expressions for column D and reflected on their effects, all students were exposed to *global/meta-level activity*. Hence, the activity was not only *generational* as planned by the designers, but also included a *global/meta-level component*. We would like to emphasize that the process of instrumentation among the students had already begun—and it is quite a diverse process, as evidenced by the different headers (generalizations) proposed by students, and in particular by the different symbolic expressions students used in column D. In parallel the process of the teacher’s instrumental orchestration is taking place, as expressed in the whole-class discussion at the end of the meeting. The teacher also makes available the opportunity to discuss (implicitly) the new concept of negative number which emerges from the generational activity.

4.2 Activity 2: the *Growing Rectangles* situation⁴ (September 27)

(a) *The situation* A process of growth in three “rectangle families” over a period of 10 years was presented visually and verbally (Fig. 3).

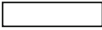
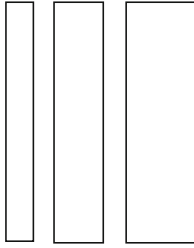
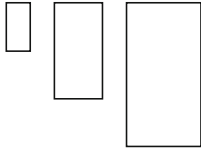
(b) *A priori analysis* The task entailed comparing the changing areas of the three “growing rectangles”. Initially,

	A	B	C	D
1	No.	Perimeter option	Area option	Saved amounts
2	1	$=A_2*4*5$	5	$=A_2*A_2*5$
3	2	$=A_3*4*5$	20	$=A_3*A_3*5$
4	3	$=A_4*4*5$	45	$=A_4*A_4*5$
5	4	$=A_5*4*5$	80	$=A_5*A_5*5$
6	5	$=A_6*4*5$	125	$=A_6*A_6*5$
7	6	$=A_7*4*5$	180	$=A_7*A_7*5$

Fig. 2 Amy’s symbolic generalizations in Excel for the *Toll Road* situation

⁴ An analysis of the work of the first cohort on this activity was presented by Tabach and Friedlander (2004).

Fig. 3 The *Growing Rectangles* activity

Rectangle C	Rectangle B	Rectangle A
   . . .	 . . .	 . . .
At the end of the first year, the rectangle's width is one unit, and it grows by an additional unit each year. The length of this rectangle is always twice the length of its width.	At the end of the first year, the rectangle's width is one unit, and it grows by an additional unit each year. The length of this rectangle is constantly 10 units.	At the end of the first year, the rectangle's width is one unit, and it grows by an additional unit each year. The length of this rectangle is always longer than its width by three units.
At what stages during the first ten years does the area of one rectangle overtake that of another one?		

students were asked to conjecture, without performing any calculations or formal mathematical operations, which rectangle area will overtake the others and when. Then students were asked to organize their data regarding the growing rectangles in a spreadsheet table, in order to verify or refute their conjecture. No instructions were given as to the organization of the data or the expressions to use. This task involves conjecturing and verifying, which are *global/meta-level* actions. Deciding on the organization of the data may also be considered *global/meta-level*, as it is not unique to algebra. The process of verifying the conjecture is *generational*, as it involves modeling the situation with symbolic expressions, entering them into Excel and dragging to compare the resulting numbers.

(c) *Amy's work* Amy's conjecture was that rectangle B will be the largest by the end of 10 years. She did not provide a priori support for her conjecture. She organized her data in an *extended table* (Tabach and Friedlander 2004) with ten columns: the year, six columns for the linear dimensions of the rectangles, and three columns for the corresponding area values. The columns were ordered by variable (Fig. 4). Amy used symbolic expressions to describe the change of each variable. Except for the length of rectangle C, which does not change (column G), the changes of the linear dimensions were all described by recursive expressions, which express a relationship between two consecutive numbers (cells) in a sequence (column). Amy entered each of the first elements in columns B to G as a number, thus, in most cases, missing the opportunity to express the relationships between the widths

(columns B, D and F) and the years (column A) as well as the relationships between the lengths (columns C and E) and the respective widths. The areas of the rectangles were described by multi-variable expressions, where the letters expressed the length and width cells used as the variables (e.g. $=B2 \cdot C2$). The use of multi-variable formulas conceals the full relationships between the changing of length and width and area of the rectangles. The advantage of using Excel for Amy is that even at this stage of algebraic thinking and skills, Amy, who was able to generate only multi-variable and recursive expressions, could still succeed in solving the problem with the use of the symbolic syntax.

(d) *Other students' work and the teacher's role* We do not have data concerning other students' *conjectures*. Regarding data organization, four categories of tables were observed in the students' work files: (1) *Separate tables*: students in this category constructed three tables—one table for each rectangle. Each table contained four columns to describe the year, the rectangle's linear dimensions, and the corresponding area value. Three groups used this way of organizing data. (2) *Extended table*: the way Amy organized her data. Eleven groups used this way of organizing data. (3) *Reduced table*: these students allotted only one column for the width measures, since they are identical for all three rectangles. Thus, the number of columns in these tables was reduced to 8. Six groups used this way of organizing data. (4) *Minimal table*: students noticed that the width measures are identical to the year number and omitted them altogether. Moreover, they omitted the length

Fig. 4 Amy's symbolic generalizations in Excel for the *Growing Rectangles* situation

	A	B	C	D	E	F	G	H	I	J
1	Year	Wid. A	Len. A	Wid. B	Len. B	Wid. C	Len. C	Area A	Area B	Area C
2	1	1	4	1	10	1	2	=B ₂ *C ₂	=D ₂ *E ₂	=F ₂ *G ₂
3	=A ₂ +1	=B ₂ +1	=C ₂ +1	=D ₂ +1	10	=F ₂ +1	=F ₃ *2	=B ₃ *C ₃	=D ₃ *E ₃	=F ₃ *G ₃

measures as well, and included their corresponding expressions directly into the area formulas of each rectangle. Thus, the number of columns in this case was further reduced to 4. Seven groups used this way of organizing data.

The students' use of symbolic expressions to generate an Excel table was reported in Tabach, Arcavi and Hershkowitz (2008). Most student pairs used two types of symbolic expressions, like Amy. Multi-variable expressions were used by 20 pairs, recursive expressions by 13 pairs, and explicit expressions by 15 pairs—all the pairs who chose a minimal table and some others. Thus students were engaged in *generational* activity, creating diverse symbolic models to represent the situation at hand. The symbolic models which were provided by the students express different degrees of algebraic relationship: the full relationship, as in the minimal table, and different partial relationships as in the other ways of organizing data.

The teacher initiated a reflective discussion in a whole-class forum, allowing students to compare and contrast the different ways of data organization, considering the various symbolic expressions, which were used by different students to obtain the same number columns. The aim of the whole-class discussion was for the students to get to know the diverse ways that were used, but not for the teacher to evaluate these ways. In other words, the teacher did not give preference to the minimal table or to the full relationships. By doing this she imparted to her students the awareness that all these diverse ways are appropriate and hence they were free and encouraged to use any or all of them.

(e) *Conclusions* While in the first example students' work was almost uniform in terms of data organization and symbolic expressions used, here we observed a variety of responses. This change can be explained by the idea of instrumental genesis and is supported by the teacher's openness to accept all of them. During the 3 weeks that passed between the implementation of the two activities in the classroom, students had a chance to develop their own instrumental scheme of the Excel tool. In addition to their own uses of the tool, they were exposed, during whole-class discussions, which were led by the teacher, to other students' ways of using Excel. Hence, even though the task was the same, students' ideas about data organization by symbolic expression varied considerably.

A second notable issue about this example is that here students were engaged in actions which reflect the task design: Amy was engaged in *global/meta-level* action as well as a *generational* action. This was also the case for the other students, as reflected by the various data organizations and different symbolic expressions that were produced.

4.3 Activity 3: *Number Sequence* activity (October 18)

(a) *The situation* In fact, in this case no situation was provided to the students. Rather, a table with a sequence of numbers (Fig. 5) was presented, and the students were asked to find an expression that can be written in cell B₆, using A₆ as a variable.

(b) *A priori analysis* This task is *generational* in nature, like the last two examples. The mathematical intention is for students to express the numbers in column B as a function of those in column A, assuming that the linear relationship indicated by the first five rows will continue. This assumption is left implicit, since students had become used to similar assumptions in earlier activities.

While the two previous examples evolved from problem situations, using visual and verbal representations, in this case the problem is presented only numerically. It is quite easy to realize that the number in B₆ is 15. But, students were already familiar with some symbolic mediation and could be expected to make use of it by suggesting a recursive generalization like $=B_5 - 3$ for cell B₆, or an explicit one like $=30 - 3*A_6$. Only the latter follows the instructions. We note that multi-variable generalization is not appropriate in this situation.

(c) *Amy's work* For Amy, finding an explicit expression as a generalization of a given numerical sequence was quite

Find an expression that can be written in cell B ₆ , using A ₆ as a variable.		
	A	B
1	0	30
2	1	27
3	2	24
4	3	21
5	4	18
6	5	?

Fig. 5 The *Number Sequence* task

difficult at this stage of the course. In her efforts to find the requested expression, Amy invented the following “story”: “At the beginning of the year there were \$30 in a savings box. Each week, the amount decreased by \$3.” Then she wrote the following expression in cell B₆: $=30 - 3*A_6$, which is generational at the highest level. It seems that inventing a context for the given sequence of numbers enabled Amy to create an appropriate symbolic expression. The idea of reverting numerical data to a “story problem” might have stemmed from the previous activities, in which the mathematical tasks were situated within a story context. Amy was confident in her solution, and did not check it with Excel.

(d) *Other students’ work and the teacher’s role* This task was rather challenging for other students as well; however, none of the other students took a path similar to Amy’s. All but four students found explicit expressions; these four found recursive ones. About half of the students wrote a correct expression without using Excel. The other half used Excel to verify their symbolic expression, by entering the two number columns as given, and then entering their symbolic expression in cell B₆, verifying that the resulting number was indeed 15. It seems that the students in the second half were less confident in their mathematical actions and needed symbolic–numeric confirmation. As a consequence of this activity the teacher made two pedagogical observations: (a) a few students still prefer recursive generalization to explicit generalization; (b) about half of the students do not need the Excel support for the generalization nor for the numerical validation.

(d) *Conclusions* Shifting between visual, verbal, numerical, symbolic and graphic representations of a mathematical situation was common classroom practice by this time in the school year. Nevertheless, Amy’s method of inventing a story for the task was unique. The fact that half of the students used Excel while the others used only pencil and paper is a good demonstration of the CIE’s feature that the decision whether to use the tool is in students’ hands, and shows that some students did not consider the use of Excel advantageous for this task. The task involves *generational* activity for all students, as they generate a symbolic model for the given numerical situation, as intended by the designers. Yet, for the second half of the students the status of their symbolic expression was a prediction, tested and verified by the use of Excel, and hence the task also had a *global/meta-level* component.

4.4 Activity 4: the *Secret Number* activity (March 16)

(a) *The situation* The teacher asked the students during a whole-class discussion to “Choose a number, add 3 to the chosen number, multiply the sum by 2, and subtract 6.” To the students’ surprise, the teacher was able to guess each

student’s original number from their final result. The task is finding out how the teacher discovers the original number from the final result, and to explain why it works. Later on, students were asked to create a similar number trick by themselves. In this activity, students did not use Excel.

(b) *A priori analysis* The structure of this activity differs pedagogically from the structure of the previous three examples in the role given to the opening whole-class discussion. The activity was presented orally by the teacher. Students were asked to explain the procedure, but no particular form of representation was suggested. This activity was among the first opportunities for the students to engage, in a meaningful context in a transformational activity, with the ideas behind simplifying, not necessarily using algebraic expressions. The concept of simplifying expressions had not yet been discussed in class. The activity follows recommendations by Kieran (2004) and Star (2007) about interweaving techniques with conceptual understanding.

For the second part of the activity—creating a new, similar number trick—students needed to be able to reflect globally on the process. Hence, students’ work involved *global/meta-level* activity.

(c) *Amy’s work* Asked to solve the “mystery”, Amy said during the whole-class discussion that the resulting number is twice the original number. The teacher agreed, and asked for an explanation. Amy began, “Let’s choose 2 as our number.” She then followed the steps with her chosen number and upon reaching the step “multiply the sum by 2”, explained that no matter which number is given, the result of this step is an even number. Although Amy was not able to fully explain why the resulting number is twice the original number, we see in her attempt to do so a form of “generalizing using a generic example” (Mason and Pimm 1984; Hershkowitz and Arcavi 1990). She used the number 2 as a place holder for “any number”.

As response to the request to compose a similar number trick, Amy proposed: “Choose a number, add 1,000,000 and multiply by 2, subtract 2,000,000 and then subtract the number you chose. You will get back your chosen number.” Amy’s number trick is modeled on the original one, in the sense that it follows the underlying structure of $(x + a)*n - an$, where x is the chosen number. However, she added the subtraction of $(n - 1)*x$ in order to end up with the original number as result and she used the very large number 1,000,000—an extreme case, apparently again as a generic example. The use of a generic example is considered to be a step towards symbolic generalization (Mason and Pimm 1984; Hershkowitz and Arcavi 1990).

(d) *Other students’ work and the teacher’s role* After the teacher presented the number trick at the beginning of the lesson, and correctly “guessed” the original number of several students, and Amy observed that the resulting

number is twice the original one, one of the students asked the teacher to write on the board the verbal steps of the riddle. This was an interesting request—if the steps are written, students do not have to keep them in mind, and can “observe” the different operations and the way they are combined to create a “simplified” expression. In other words, while at the beginning of the lesson students were engaged in following a sequence of oral instruction to calculate numerically, they now moved their attention to the verbal written sequence, in order to understand its structure.

Following Amy’s explanation of why the resulting number is even, other students raised the idea of representing the original number by “ x ”, and writing the calculation on the board as $(x + 3) * 2 - 6$. Although they were not trained yet to simplify this expression, they did observe that the product of 3 and 2 is 6, which is then subtracted and could also explain that the original number is also multiplied by 2, and hence the result is twice the original.

Focusing on the number tricks the students generated, 23 groups (most of them pairs) provided 38 different number tricks. Twenty followed the model of the teacher’s number trick, that is $(x + a) * n - an$. Three other proposals were very similar: two of them included a negative a , which for students who entered the world of negative numbers just 2 months ago is not trivial; the third included dividing by 2: $[(x + 6) * 4 - 24] / 2$. Possibly this was done in order to get a resulting number which is twice the original, as in the teacher’s riddle.

The remaining 15 number tricks followed three different models. (1) Three number tricks followed a pattern which may be seen as less sophisticated than the one presented by the teacher: $x + a - b$, including one case of $a = b$. (2) Five number tricks involved multiplication and in some cases also division, the simplest being $x * 2$, and the most complex $x * 2 * 1,000 / (100 * 5)$. (3) Six number tricks were of a different type: they aimed at a *target* number that was independent of the original number. In these cases, students were able to “guess” the resulting number rather than the original one. Three strategies were used to obtain such results. One is illustrated by the following: “Choose a number, multiply it by seven, and divide it by the original number. Find the letter which corresponds to the resulting number (according to the alphabetic order, $1 = a$, $2 = b$, ...), and find a mammal starting with this letter.” A second strategy was based on the idea that multiplying by zero results in a known answer—zero. A third strategy involved a characteristic of multiples of 9, namely that the sum of their digits is also 9.

In this activity the teacher has a main goal: she is the presenter of the situation as well as the orchestrator of the discussion which followed. In both parts of the situation

(solving the teacher’s trick and inventing of students’ tricks), she encouraged a diversity of strategies and by doing so legitimated the autonomy and individual thinking of her students.

(e) *Conclusions* The discussion at the beginning of the lesson shows that students had a good sense of the mechanism behind the teacher’s number trick, even if they were unable to articulate it explicitly by a symbolic model. The *transformational* nature of this task was quite notable; students followed and used the numerical path of the calculation, in order to explain what is going on, more than creating and using a symbolic model. Amy’s example which uses a number as large as a million is a good example of the fact that the general aspect of the riddle was clear, but the way to express it was still carried out by an example, and not by symbols.

Concerning students’ creation of their own number tricks, we can see that all of them were able to generate at least one. All 38 number tricks created by students were built purposefully: the students decided what they wanted to create and expressed it in verbal-symbolic form. The fact that 15 riddles did not follow the teacher’s model suggests that, for those students, the general idea of performing a sequence of calculations and their undoing, or disguising a simple calculation behind a sequence of actions, was clear. In this sense the task involved also a *transformational* activity, even if not symbolically, while students created their own number tricks. Another interpretation of the creation of these 15 riddles is that the students felt free and welcome to be creative, and even to take the riddle outside of the realm of mathematics, as was the case with using an alphabetic value of the letters. As noted earlier, such freedom and autonomy is one of the characteristics of the CIE environment, and was realized by means of the teacher’s supportive actions as detailed above.

5 Discussion

In this paper we have tried to convey the flavor of the learning trajectory intended by the designers for beginning algebra in a CIE learning environment, and its implementation over more than half a year. We chose to do so by presenting four representative example activities through which the readers are able to get a sense of this intervention in the classroom and its consequences: the goals of the CIE environment for learning mathematics in general and beginning algebra in particular; the kinds of activity that were purposefully designed and the ways in which students approached them; the kinds of thinking that students engaged in while working on the tasks; the spirit of the environment in which the intervention was carried out; and the teacher’s role in implementing this spirit in the

classroom. By describing and analyzing one particular student (mostly qualitatively) and the work of over 50 other students (mostly quantitatively) in the same situations, we have tried to provide a deep understanding and, in parallel, a more global view on the learning processes that occurred. This course was never implemented on a larger scale, hence the results presented here should be considered as an existence proof showing that under suitable conditions our CIE approach to beginning algebra can facilitate the transition from arithmetic to algebra.

We note that pre- and post-tests were conducted with the experimental classes as well as with students with similar background learning in the same school in parallel classes. The tests show no statistically significant differences. Although the post-test results of the experimental group are slightly higher than those of the control group, the differences are not statistically significant. Both groups performed well, and seem to have learned algebra during the course (see Tabach, Hershkowitz, Arcavi and Dreyfus 2008 for more information).

The following two related questions guided our analysis. First, in light of this unique learning environment, in what ways does the CIE shape cognitive and socio-cultural aspects of learning beginning algebra in seventh grade? And, second, to what extent did actual learning shift students from arithmetic to algebra at their autonomous pace? Students working in this environment clearly followed the message that they are responsible for choosing a working method, which is suitable for them in any specific problem situation. As early as September 8, we observed variation in students' work. This freedom to choose one's working method and the affordances of the various mathematical/symbolical generalizations provided by Excel expressed themselves when students developed symbolic models as keys to generalizations for a given situation, or created a number trick which did not follow the teacher's model. In short, we have evidence for the diversity of students' instrumental processes, afforded by the CIE, and for students becoming autonomous. In a sense our conclusion is similar to the one by Yackel and Cobb (1996): "...in the process of negotiating sociomathematical norms, students in these classrooms actively constructed personal beliefs and values that enabled them to be increasingly autonomous in mathematics" (p. 474).

Another aspect of this freedom was reflected by the choice of working with or without the aid of the computerized tool. In the third example, about half of the students chose to use it while the other half chose not to use it. In other words, students took seriously the message that it is in their purview to work according to the needs they feel. This also provides evidence that the students became autonomous.

The variety of students' solution methods can be attributed in part to the presence of the computerized tool. As noted by others, the process of instrumentation results in a large collection of solution methods (Artigue 2002; Mariotti 2002). This phenomenon is clear when discussing the use of Excel. Three levels of generalization may be used effectively in this environment, while in a pencil-and-paper environment only the third and highest level of symbolic generalization may be used effectively. So, in a sense, the availability of Excel triples students' possible working strategies.

The use of Excel was meaningful for these algebra beginners. They were able to investigate complex changing phenomena, as demonstrated by the first three activities, due to the possibility of using partial generalizations with the tool. However, even by March 16 (fourth example) students were not yet able to simplify a simple expression like $(x + 3) \cdot 2 - 6$. In other words, students were engaged in generational and global/meta-level activities successfully, long before mastering transformational activities. This is a notable change from the classical approach for learning beginning algebra, which begins from transformational skills and only later, if at all, proposes generational activities. The presence of the tool was crucial, as it allowed the students to use a collection of symbolic expressions efficiently, and explore the numerical representation of these phenomena. Students could adjust their symbolic thinking gradually, progressing towards using explicit expressions. As reported by Tabach, Arcavi and Hershkowitz (2008), by the end of the school year the students were able to function without the aid of the tool, creating and manipulating explicit symbolic expressions for various changing phenomena. In other words, the symbolic generalizations created by students have changed along the school year, towards the third and highest level.

A variety of solution methods is considered to be a characteristic of creative thinking. It also serves as an indication that students indeed enacted the flexibility, which this environment afforded, to the benefit of their learning. When a group uses a collection of methods, it means that the students as individuals show autonomous thinking concerning the given tasks, and choose their problem-solving strategy accordingly. That is, the individual student becomes responsible for his own learning—students become autonomous learners.

As mentioned above, our approach to beginning algebra in a spreadsheet environment began from *generational* activities and added *transformational* ones gradually; the Secret Number activity, for example, started as more *generational*, and gradually became more *transformational*. Similar findings were reported by Tabach and Friedlander (2008), who studied the same students' work on a sequence

of three different tasks designed as transformational activities. That study showed that the activities were first perceived by the students as *generational*, and gradually a transition towards *transformational* activity occurred. As in the present study, Tabach and Friedlander (2008) reported that students' work did not necessarily follow the designers' intentions. Yet, students made sense of the situation, followed their solution path, and did acquire the mathematical knowledge intended by the design. These findings are in contrast to the "classic" approach of teaching algebra, in which transformational activities and skills are learned first, and only after mastering them are students requested to engage also in generational and global/meta-level activities.

A priori analysis of activities according to Kieran's model has the potential to inform curriculum developers and teachers about the scope of tasks to be given to students. However, as this study shows, the way students interpreted and performed a task does not necessarily coincide with that of the designer or the teacher, as shown, for example, by Amy in the third activity. In almost any of the examples provided here we note this difference. Eisenmann and Even (2010) found that the same task may be enacted by the same teacher in a different classroom with different emphasis as *transformational* or *global/meta-level* activity. We perceive this ambiguity on the nature of the algebraic activity as a potential entry point to students with different knowledge and approach to the same activity.

Yerushalmy (2009) warned mathematics educators that designing a long-term curriculum based on the use of technology may overcome some known didactical cuts but at the same time may produce new ones: "I argue that in designing a new curriculum, based on the use of new (technological) tools, attention must be paid to identifying points of discontinuity that may be different in order or quality from those revealed by previous research" (p. 56). During the two successive years of implementing the current year-long course, no signs of such discontinuities were observed. The students carried out all activities, each pair choosing a level of generalization and a use of symbols as afforded by Excel that suited them. It is possible that a 1-year course is not long enough to identify such discontinuities, but it is also possible that the freedom to choose and the ensuing autonomy of students as learners reduces or even eliminates these didactical cuts. More specifically we emphasize that didactical cuts are mentioned in the literature as emerging from students' limited ability to generalize with algebraic symbols and as finding their expression mainly in *generational* and *meta-level* activities. Such didactical cuts did not appear in the students' mathematical actions in the activities demonstrated above. It remains an open question how this curriculum affects didactical cuts that may emerge in later *transformational* activities.

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