

Transitions among different symbolic generalizations by algebra beginners in a computer intensive environment

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Abstract The transition from arithmetic to algebra in general, and the use of symbolic generalizations in particular, are a major challenge for beginning algebra students. In this article, we describe and analyze students' learning in a "computer intensive environment" designed ad hoc and implemented in two seventh grade classrooms throughout two consecutive school years. In particular, this article focuses on the description and analysis of how students initial generalizations (which relied on computerized tools that enabled different students' to work with different strategies) shifted to recursive and explicit symbolic generalizations.

Keywords Beginning algebra · Symbolic representations · Computerized environment · Long term study

1 Introduction

This paper describes components of a long term project with the overall goal to observe, analyze and better understand the learning processes of a beginning algebra course in a specially designed *Computer Intensive Environment (CIE)*. By *CIE* we mean a learning environment in which (1) computerized tools are available to learners at all times (in class and at home), and (2) the learners are free to choose if, when and how to use these tools while working on problem situations. Thus, the intensiveness of the environment consists on the availability of the computerized tools at all times—but not necessarily on the intensiveness of their use. In this *CIE*, seventh grade students worked with an ad-hoc algebra curriculum based on a functional approach, which is rich in problem-situations, and was designed to provide opportunities to work with spreadsheets (with only very few instructions as to when and how to do so). The realistic problem-situations, the functional approach and the availability of spreadsheets provide opportunities to use algebraic symbols, in particular to express and handle generalizations. The environment and the tasks

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allow for different levels of symbolization, in such a way that different students can make sense of symbols and their potential at their own pace. This is possible because the problems require the students to model phenomena from real life situations (e.g. the growth patterns of different plans for saving pocket money) using numerical, symbolical and graphical representations while keeping in mind the meaning of the situation. A detailed description of this environment, including some problem situations and the flexible way of working (freedom to choose a representation, classroom discussions on advantages and disadvantages of solution strategies and the teacher as a consultant) has been reported elsewhere (Tabach, Hershkowitz, Arcavi & Dreyfus, 2008).

The present study is part of a larger research project whose main goal was to investigate student learning processes in a year long beginning algebra course in a *CIE*. The findings from the different components of this project are described elsewhere. One of these components focused on students' use of spreadsheets throughout the course, and their adoption of different learning trajectories while making transitions from large numerical sequences to their corresponding symbolic generalizations (Tabach, Hershkowitz, & Dreyfus, 2008).

In this article, we focus on the changes that occur throughout the year long course in the expressions that students use in order to make generalizations to solve the problem-situations on which they work. We begin with a brief background on the initial stages of algebra learning in general, and the use of computers in mathematics classrooms. We focus on the role of spreadsheet as mediator for understanding the meaning and the uses of symbolic generalizations. Then we present the research goals, the methodology, the data collection and the analysis. Finally, on the basis of the findings, we discuss the mediating role of spreadsheets in the transition from arithmetic to algebra.

2 Background

2.1 The transition from arithmetic to algebra

In many countries students begin to learn algebra after 6 years of learning arithmetic, geometry, and possibly some data handling, at elementary school. At that stage, instruction shifts from working with numbers, their properties and operations to the introduction of symbols for generalizations, properties and equation solving. Students are expected to learn the syntax of symbolic expressions and to apply it. Moreover, students are implicitly requested to abandon some of their views and practices. For example, whereas in arithmetic the equal sign is usually regarded as an invitation to calculate (e.g. $3+2=$ means “add up the numbers and write the sum on the right hand side of the equal sign”), in algebra, the equal sign may also have many other meanings: general equivalence of two expressions, a way of defining the expression to its left as the expression to its right (e.g. $f(x)=3x$) and more. The subtle shift in the meanings of the equality sign is problematic for many students (see, for example, Knuth, Stephens, McNeil, & Alibali, 2006). This and other shifts in perspective have led researchers to characterize what students experience as a “didactical cut” (Sutherland and Rojano 1993) that needs to be understood and attended to.

Different approaches to algebra learning were proposed in order to meaningfully bridge the gap between arithmetic and algebra. Such approaches include “generalization of numerical and geometric patterns and of the laws governing numerical relations, problem solving, equation solving aided by the use of concrete models, introduction of functional

situations, and the modeling of physical and mathematical phenomena” (Bednarz, Kieran, & Lee, 1996, p. 3).

2.2 Teaching beginning algebra in partially computerized environments

The advent of computers turned the attention of many researchers to the potential of technological tools to support students’ transition from arithmetic to algebra. It has been claimed that computerized environments amplify student mental capabilities and significantly change the nature of mathematical activity itself (e.g. Pea 1985). Also, researchers suggest that explorations with computerized tools encourage students to plan, to reflect, to produce explanations usually through classroom discussions (e.g. Heid 1995).

In the last two decades, several algebra projects based on the partial use of different kinds of computerized tools were developed, implemented and studied (e.g. Dettori, Garuti, & Lemut, 2001; Haspekian 2005; Hershkowitz et al. 2002; Kieran 1992; Wilson, Ainley, & Bills, 2005; Yerushalmy and Schwartz 1993). These studies are not always fully convergent in their conclusions.

Appropriate and successful uses of technological tools in beginning algebra have been described, as in the following examples: explorations of everyday life problem situations using several representations (e.g. Heid 1995), numerical experimentation which evolves into functional connections (e.g. Kieran 1992), and manipulations of symbolic and graphical representations of functions (e.g. Yerushalmy and Schwartz 1993).

Computerized tools may contribute to the learning of algebra, and have a potential to address syntactic aspects as well as enhanced understanding, mathematical modeling and the development of symbol sense (Arcavi 1994). Such aspects of algebra can be learned by making use of graphical, numerical and symbolic representations whose static nature with paper and pencil can become dynamic in computerized environments. Graphical, numerical and symbolic representations can be used in parallel or be chosen by the students according to their needs and/or personal preferences. These various representations might be used to contrast and operate upon mathematical objects, thus, as Kaput (1992) states, they turn *display* notation systems in a pencil and paper environment into *action* notation systems in computerized environments. The change of representations can be observed, initiated, and reflected upon, and hence it becomes the source of investigations and insights. Furthermore, the claim is that the functional approach (i.e., equations consist of point-wise comparisons within a dynamic phenomenon and syntactic algebraic skills are integrated into, and are at the service of, the mathematical activity related to these problem situations), enables the teacher to present students with different kinds of changing phenomena and opportunities for generalization and modeling within several representations (e.g. Bednarz et al. 1996; Yerushalmy 2005).

In contrast, other studies result in questioning the benefits of introducing technologies into the algebra classroom. For example, the potential of spreadsheets to express algebraic relationships in the form of equations is challenged (Dettori et al. 2001; Yerushalmy and Chazan 2008). Others are concerned with the kind of mathematics with which students engage (Hershkowitz and Kieran 2001), and/or with the shift of students’ difficulties from one area to another (Yerushalmy 2005).

The Algebra course we designed and implemented is organized around a sequence of problem situations dealing with phenomena which can be described by changing quantities (a function approach to algebra). Following Hershkowitz et al. (2002), we adopted three criteria for the selection of appropriate computerized tools for teaching algebra. These

criteria are based on the potential of the tools to support: (1) *generalization* (2) *mathematization* (in the sense of Treffers 1987; see also, van Reeuwijk 1995) and (3) *communication*. Spreadsheets (e.g. those widely available, like *Excel*) seem to fulfill these three criteria, and were widely studied as tools for learning beginning algebra (e.g. Ainley 1996; Filloy and Sutherland 1996; Friedlander and Tabach 2001a; Haspekian 2005; Hershkowitz et al. 2002; Sutherland and Rojano 1993; Wilson et al. 2005).

2.3 Spreadsheet as a mediator for understanding and constructing the use of symbolic generalizations

Spreadsheets allow students to observe, handle, and generate a large number of numerical instances, and thus to potentially bridge the sometimes rapid and disconcerting transition from numbers to symbols, from arithmetic to algebra (Wilson et al. 2005; Haspekian 2005). With the creation of numerical sequences out of existing ones by using either rules (expressions, or formulae) and “dragging”, or by representing numerical data in graphical form, spreadsheets support the functional approach to algebra, and the envisioning of patterns which leads to generalizations. Whereas mathematization in general and generalization in particular are at the core of the activity, spreadsheets enable students to remain within and to rely upon the numerical realm (handling large sets of numerical data by “dragging”) and to slowly get acquainted with the use, the purpose and the power of symbols. Thus the ideas of algebra become the core, and learning of the algebraic language to express and operate is introduced according to the student needs, allowing for time to become accustomed to a new field, within meaningful contexts.

Other researchers view spreadsheets as a tool for making sense of the dynamic aspect of functional relationships between the value of a variable and the value of the expression—the dependent variable (Drouhard and Teppo 2004). For example, if in a spreadsheet one writes the expression “ $=2 \times A2 + 1$ ”, then the content of that cell is *dynamically* dependent on the content of cell A2. Moreover, many more cells can be created as the list of values which are a function ($2x + 1$) of the corresponding values of the cells A_i (x).

Hershkowitz et al. (2002) indicated that the use of spreadsheets to investigate processes of variation enables students to spontaneously use algebraic expressions. Spreadsheet users employ formulas (expressed in spreadsheet syntax) as a natural means to construct numerical tables that describe a certain phenomenon.

The use of spreadsheets for learning the symbolic language of algebra is questioned by several researchers in mathematics education. Some point to the intricacies and potential complications that underlie symbolization with spreadsheets: one concern refers to the role of spreadsheets as a mediator for the meaning of variable and for understanding symbolic representation. The main claim is that while working with symbols, which represent locations in the spreadsheet table, these symbols are neither unknowns, nor variables.

They represent particular locations and in that sense seem too particular to be variables, although of course the values in cells to which they refer can change; the cells to which they refer either do or do not have values; when they do, it seems funny to call them unknowns (Yerushalmy and Chazan 2002, p. 735).

Similar complexities were pointed out by Dettori et al. (2001)—students do not grasp immediately what is a variable or an unknown, and “the sign of equality used in a spreadsheet is actually the assignment of a computed value to a cell” (p. 199).

But, precisely the ambiguous nature of a cell in a spreadsheet environment is viewed by Haspekian (2005) and others as a possible source for bridging between arithmetic and algebra:

A number in a cell can have several meanings; it can be a specific number or a cell representing a general number, or a cell representing an unknown number or a cell representing a relationship between numbers. The spreadsheet-algebraic approach is to view a cell as 'x', either as an unknown or a general number and to express relationships with respect to this 'x' (Sutherland and Balacheff 1999, p. 22).

A different and very important concern about the use of spreadsheets refers to the nature of the symbolic generalizations students prefer when using this tool. While working with spreadsheets, one can refer to *Excel* as a table, in which one may input numbers. After typing numbers in two consecutive cells, one may mark the two cells, and use the dragging option to obtain an array of numbers. The spreadsheet detects the linear relationship between the cells, and extends the array accordingly. A third person, who looks at such an *Excel* file, may not be able to determine whether the student typed each number, or used the dragging option. We consider this latter approach as purely numerical (i.e. non-algebraic).

In pencil and paper environments, three stages of generalization processes by pre-algebra students have been identified (Arcavi 1995; Friedlander, Hershkowitz, & Arcavi, 1989; Hershkowitz and Arcavi 1990). In the first stage, students, even when aware that they should use symbols, tend to represent quantities involved in a given situation, using different letters, disregarding existing relationships between the quantities these letters represent. For example, two consecutive numbers are often represented as a and b (rather than as a and $a + 1$), probably because consecutiveness is conceived within the alphabetical context, rather than as an arithmetical property to be expressed by letters and numbers. In the second stage, students start to express partial relationships using symbols. For example, two consecutive numbers are written as a , $a + 1$, but in other, more complex, numerical relationships, still a different letter is introduced for a new related quantity. Only at the third stage, are students able to express full relationships among related quantities symbolically (Friedlander and Hershkowitz 1997; Tabach and Friedlander 2004). Unfortunately, with paper and pencil, algebraic manipulations on expressions which do not state full symbolic relationships do not usually lead to insightful results, if any.

With spreadsheets, students are able to operate, to experiment with, to reflect and to learn the relationships between the quantities involved even in their earlier stages described above. This is possible because a large amount of numbers representing a certain variable can be produced and arrayed in such a way that the relationship is "visible" even without the explicit and general symbolic expressions. This can be produced by one of the following options:

- “Multi-variable approach”—“dragging” and considering a whole array of numbers (i.e. a column in the spreadsheet) as a “variable” dependent on another one (Dreyfus, Hershkowitz, & Schwarz 2001);
- “Recursive generalization”—the use of recursive expressions, which emphasize local relationships between consecutive elements such as contiguous cells of a same column (Tabach et al. 2008);
- “Explicit expression”—expressing the general relationship and displaying the full relationships among the variables.

As noted earlier, students may also use the tool with a pure numerical approach (that is entering numbers one by one in the different cells).

Previous studies indicate that initially many students prefer recursive generalizations (based only upon local connections) over explicit formulae, which require a generalization of a more global nature (Friedlander and Tabach 2001b; Stacey and MacGregor 2001). Spreadsheets provide these students with the possibility of working within all three stages of symbolic generalizations, and yet approaching and successfully solving the same problem situations. Hence students may remain in the numerical, “multi-variable”, or recursive strategies for as long as needed while they slowly gain confidence working with symbols. Students may use different strategies but they are still able to generalize, mathematize and communicate their results to each other. In fact, students can explore functional relations, even if unaware of it.

From this point on, we focus the discussion on the present study

3 Research questions of the present study

In this article we present answers to the following interrelated research questions from our study.

1. What kind of symbolic generalizations do students use at the beginning of a *CIE* algebra course using spreadsheets?
2. Do students’ symbolic generalizations change during the year long course? If yes, in what ways?
3. Can students wean themselves from spreadsheets by the end of the year? In other words, can students be fully functional in a paper and pencil environment with explicit symbolic expressions? This question is of special interest for several reasons. From a theoretical perspective, it is important to know the extent to which this innovative approach (using spreadsheets as an entry tool) results in a “different way” of doing algebra, or whether the tool can function just as a facilitator and fade when students fully grasp the nature of the symbolic enterprise, and if so, how does this process happen? From an educational perspective, many students will encounter the traditional school curriculum (centered on paper and pencil only) in subsequent courses and therefore it is imperative to know whether their knowledge and skills will be “impaired” in any sense in comparison with their peers.

4 Methodology

In this section, we describe the setting, research design and tools, and the data sources.

4.1 Setting

As mentioned above, the *CIE* we designed is characterized by: (1) full and unconstrained access to spreadsheets (*Excel*) during the learning of beginning algebra, (2) freedom of choice about if, when and how to use the computerized tools, and about the strategy to solve the given problem situations, (3) an ad-hoc learning textbook consisting of a sequence of problem situations and simple tasks. Only in about one-fifth of the problems, mainly at the beginning of the course, there is an explicit invitation to use spreadsheets as a working tool (including the technical instructions thereof) in order to get acquainted with it. In all

other cases, the teacher supported and legitimized any choices students made regarding the ways, means, and strategies used to solve the problems (for more details see Tabach et al. 2008). Our choice to use spreadsheets was based on the assumption (grounded in several previous pilot experiences) that the tool enables students to benefit from a smooth transition between arithmetic and algebra. Spreadsheets allow the handling of problem situations (around changing phenomena) at different levels of sophistication, from purely numerical to fully symbolic as we describe in this paper. Therefore students with different learning approaches, different paces and preferences can live together productively in the same class, solving the same problems with different approaches and still be able to exchange ideas meaningfully with each other.

4.2 Research design

This *CIE* was implemented twice during a year long course (seventh grade), taught by the first author, who served both as the teacher and as a researcher (Tabach 2006).

In this study, we focus on nine of the problems administered during each of the two rounds of the year-long course.

4.3 Participants and data sources

Data were gathered twice, throughout two consecutive school years, one cohort each year. A total of 52 students (in the two cohorts), working in 26 pairs (producing a maximum of 26 files for each problem) participated in the study. The school is a magnet school, with three classes for each grade level: the experimental class which learned the beginning algebra course with *CIE* and the other two classes which learned the beginning algebra course in a partially computerized environment (with *Excel*). Since there were no differences between the two consecutive experimental cohorts with respect to the use of symbolic generalization, we will refer to the student experimental population as one whole. For each problem, all the files from both cohorts were saved on the school server. The files were downloaded, and the expressions found in them were analyzed and grouped into categories (numerical, multi-variable generalizations, recursive generalizations and explicit generalizations). In addition, audio-records of the verbalizations of 10 pairs of students (randomly selected) while they worked on the second and third problems were transcribed.

4.4 The flowchart of the research design

Our data collection during the school year is described in the flowchart in Fig. 1:

The analysis of the use of symbolic generalizations by students in the *Growing Rectangles* problem (Fig. 2) at the beginning of the school year will be the basis for the answer to our first research question. In order to confirm that working strategies were indeed typical for the course beginning (rather than being specifically related to this particular problem), we also examined data on the use of generalizations in three other problems administered during the first 6 weeks of the course (Fig. 1, item b). Next we analyzed the generalizations found in another three problems administered during the last 6 weeks of the course (Fig. 1, item c). The comparison and the combined analysis of the generalization strategies of the seven problems (four at the beginning of the course and three towards its end) allowed us to describe changes in the use of generalizations throughout the year (our second research question).

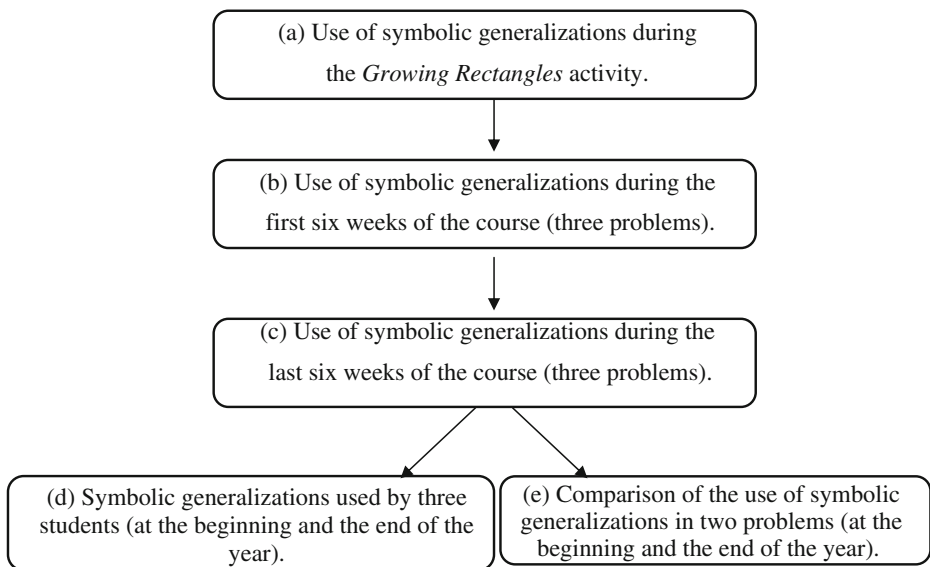


Fig. 1 Data collection flowchart

Next, in order to probe the hypothesis that individual students persevered with a specific type of strategy throughout the school year, we examined in more detail the generalizations used by three students in all seven problems (Fig. 1, item d).

In addition, the analysis focused on the working processes of one pair of students (on the basis of their audio recorded dialogues) on a couple of problems, administered at the beginning of the year (October) and almost 6 months later (March)¹—this analysis provided information about the shifts in the uses of generalizations (Fig. 1, item e). On the basis of this analysis, we discuss whether students wean themselves from the technological tool (our third research question).

We will describe in detail three of the seven problems: *Growing Rectangles* (Fig. 2), *Buying a Walkie-Talkie* (Fig. 7), and a *Present for the Parents* (Fig. 8). These are representative of the problem situations on which students worked throughout the school year.

5 Data analysis

5.1 Use of generalizations during the growing rectangles problem

The *Growing Rectangles* problem (Fig. 2) was administered to students three weeks into the course (Fig. 1, item a). First, students were asked to predict (hypothesize) which area will overtake the others and when, without performing any calculations or formal mathematical operations. Next, the students were asked to organize their data regarding the growing rectangles in a spreadsheet table, record the ways used to construct their table, and compare (first numerically and then graphically) their predictions with their findings. There were no specific instructions on how to use the computer. In this case, students had to decide how to

¹ The school year begins September 1st and ends June 20th. Due to the many holidays in September, the month of October is normally when the intensity of the school year starts.

Growing Rectangles

<u>Rectangle A</u>	<u>Rectangle B</u>	<u>Rectangle C</u>
...	...	...
At the end of the first year, the width of this rectangle is one unit, and it grows by an additional unit each year. The length of this rectangle is always longer than its width by three units.	At the end of the first year, the width of this rectangle is one unit, and it grows by an additional unit each year. The length of this rectangle is 10 units and remains constant.	At the end of the first year, the width of this rectangle is one unit, and it grows by an additional unit each year. The length of this rectangle is always twice its width.
At what stages of the first ten years does the area of one of the rectangles overtake the area of the others?		

Fig. 2 The growing rectangles activity

organize their data, how to create generalizations and how to express them in a symbolic format (for more details, see Tabach and Friedlander 2004).

The students used a variety of strategies to generate their data. An example of the various symbolic generalizations for the first rectangle (the width grows by one unit per year and the length is always three units more than the width) is shown in Table 1.

This table illustrates that:

- Explicit generalizations make use of one independent variable to express the full relationships between changing quantities (see Table 1, Explicit generalizations).
- Multivariate “generalizations” use more than one variable to express generality. Such use does not fully express the relationships between length and width established by the problem (see Table 1, Explicit and multivariate generalizations and Recursive and multivariate generalizations).
- Recursive generalizations express relationships between two consecutive elements in a sequence (see Table 1, Recursive and multivariate generalizations).

Table 2 shows the distribution of generalization strategies related to the growing rectangle problem, which were found in the working files of the 26 teams of students.

Table 1 Sample of symbolic generalizations in the *growing rectangles* problem

	A	B	C	D
Explicit generalizations				
1	Year	Width	Length	Area
2	1	=A2	=A2+3	=A2×(A2+3)
Explicit and multivariate generalizations				
1	Year	Width	Length	Area
2	1	=A2	=A2+3	=B2×C2
Recursive and multivariate generalizations				
1	Year	Width	Length	Area
2	1	1	4	=B2×C2
3	=A2+1	=B2+1	=C2+1	=B3×C3

Only two teams of students used a purely numerical approach which did not include any symbolic expressions in their work files. These students calculated the relevant numbers and typed them into the spreadsheet's cells. As noted above, we cannot tell whether students used the dragging option or not. The other 24 teams used 48 symbolic expressions altogether. Thus, the 26 teams produced a total of 50 strategies (48 different types of symbolic expressions and 2 purely numerical). On average, each team used two kinds of symbolic representations during this problem. It was surprising to find, at such an early stage of algebra learning, 15 files with explicit symbolic generalization (Table 2). It is worth noting that these explicit expressions that were produced within this environment could still be fragile and cannot be taken to indicate students' ability to perform algebraic manipulations.

5.2 Use of symbolic generalizations during the first six weeks of the course (three problems)

The first problem administered contained clear, detailed instructions for *spreadsheet* use (Fig. 1, item b). These clear instructions in addition to the problem formulation, resulted in a uniform and (almost) exclusive use of explicit symbolic representations (by about 96% of the students). However, such use should be taken neither as an indication of students' learning nor be attributed to previous knowledge. As mentioned above, we also analyzed results from three additional problems administered at the beginning of the year, in order to confirm that the variety of symbolic generalizations was not related to a particular problem. The three different graphs (Fig. 3) describe the use of the different generalization strategies in the three different problems (Fig. 3b corresponds to the *Growing Rectangles* problem).

In these charts, the *y*-axis displays the different types of generalization chosen (1—numerical, 2—multi-variable generalizations, 3—recursive generalizations, and 4—explicit generalizations), and the *x*-axis displays the students' numbers (assigned arbitrarily and preserved throughout the three charts). Each dot in the charts denotes the type of generalization used in the problem by a certain student. Each chart has two dates indicating the time the problem was administered to the two respective cohorts of students. A

Table 2 Symbolic representations for *growing rectangles* ($N=52$ students, in 26 files)

	Numerical use	Multiple variables	Recursive expressions	Explicit expressions	Total
Number of symbolic representations	2	20	13	15	50

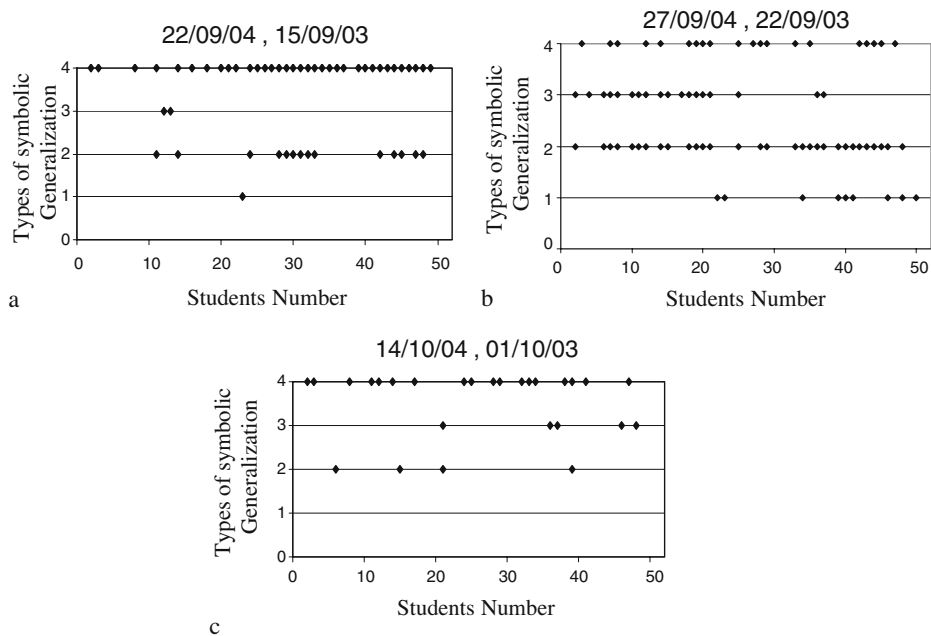


Fig. 3 Use of symbolic generalizations in three activities (a, b and c) at the beginning of the year ($N=52$). The y-axis: 1—Numeric, 2—Multiple, 3—Recursive, 4—Explicit

methodological aside: Students changed working partners several times during the year. We chose to present the data (in each of the three problems: a, b and c) per individual student (who were traced according to the pairs in which they participated).

As already mentioned, also these graphs show the variety of symbolic generalizations used for the same problem (even by the same students). As in the data presented in Table 2, most of the symbolic generalizations are not explicit. However, the spreadsheets allowed students who used different kinds of generalizations to be efficient in producing numerical representations of the phenomena at stake.

It is worth noting from these data that, in this *CIE*, students enacted their freedom of choice, and worked with a variety of generalization strategies, according to their needs, preferences, knowledge, and also the contextual idiosyncrasies of each problem (as reflected by the differences in the dot distribution in the graphs for the different problems).

5.3 Use of generalization strategies during the last six weeks of the course (three other problems)

We now turn to data concerning three problems, collected during six weeks towards the end of the school year, in which students still made extensive use of spreadsheets (Fig. 1, item c; Fig. 4).

We can see that, similarly to the data from the beginning of the year, there are still a large variety of generalization strategies, with students choosing, also here, to use more than one for the same problem. However, we can see that the use of plain numbers and multi-variable solutions almost disappeared, whereas almost all of the generalizations are based on either recursive or explicit expressions.

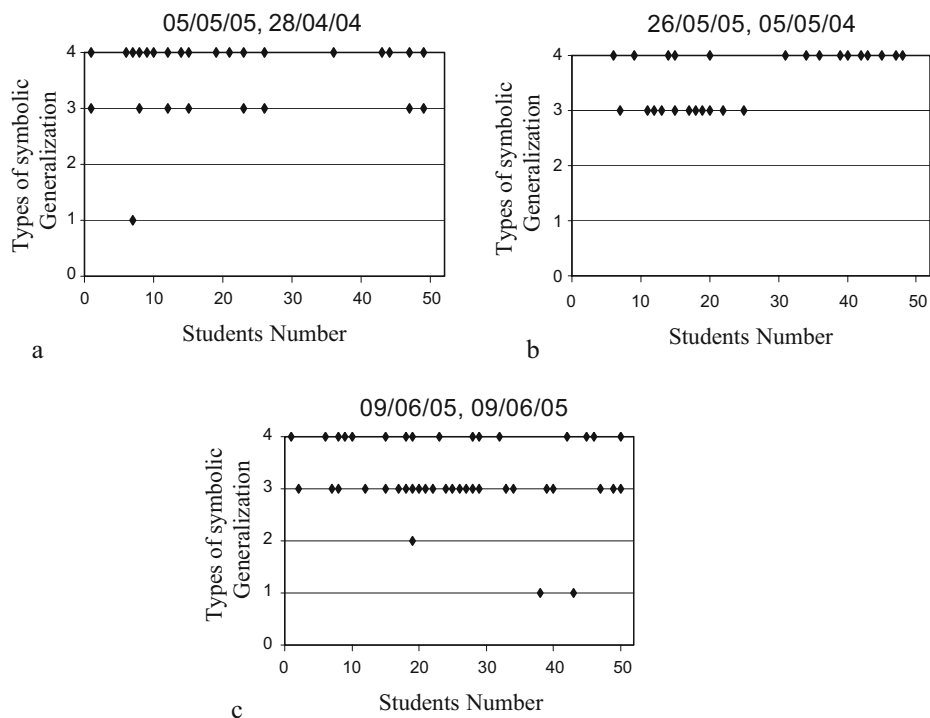


Fig. 4 Use of symbolic generalizations in three activities at the end of the year ($N=52$). The y-axis: 1—Numerical, 2—Multiple, 3—Recursive, 4—Explicit

In order to better identify this shift in generalization approaches, we calculated the relative frequency of symbolic generalizations used by students in three problems at the beginning of the school year (omitting the first problem, in which students were guided to use a certain generalization type), and the three problems at the end of the year. Figure 5 also shows that the use of numerical and multi-variable strategies had almost disappeared

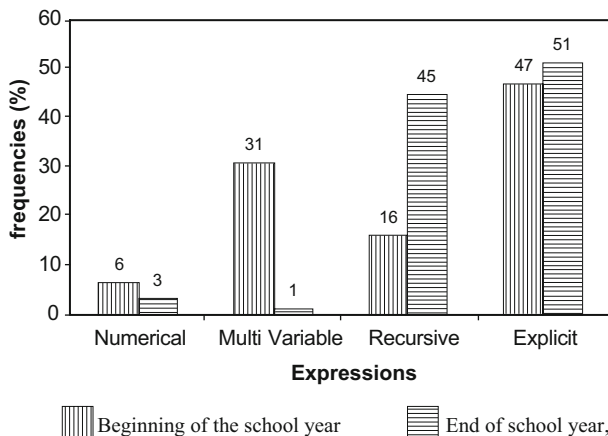


Fig. 5 Comparison of frequencies of symbolic generalizations used by students at the beginning and end of the school year ($N=52$)

towards the end of the school year, whereas the use of recursive generalizations increased most, while explicit generalizations, which were almost one-half at the beginning of the year, increased as well but not dramatically.

5.4 The work of individuals at the beginning and at the end of the year

In order to probe the hypothesis that individual students persevered with a specific type of strategy throughout the school year, we now “zoom-in” on the work of three students of different mathematical ability levels (as evaluated by the teacher and validated by their grades), on the same six problems (at the beginning and at the end of the course; Fig. 1d). The generalization strategies used by these three students (Amelia, mid-low ability level, Yuval, mid-upper ability level, and Raz, high ability level) during the six problems mentioned above, are presented in Fig. 6.

In this case, each chart displays data on one individual student, where the y -axis still displays the generalization type, but now the x -axis represents the six problems mentioned above. For each student, the left graph represents his/her generalization strategies at the beginning of the school year (Fig. 6a,c and e for the three problems respectively), and the right graph represents those generated at the end of the school year (Fig. 6b,d and f).

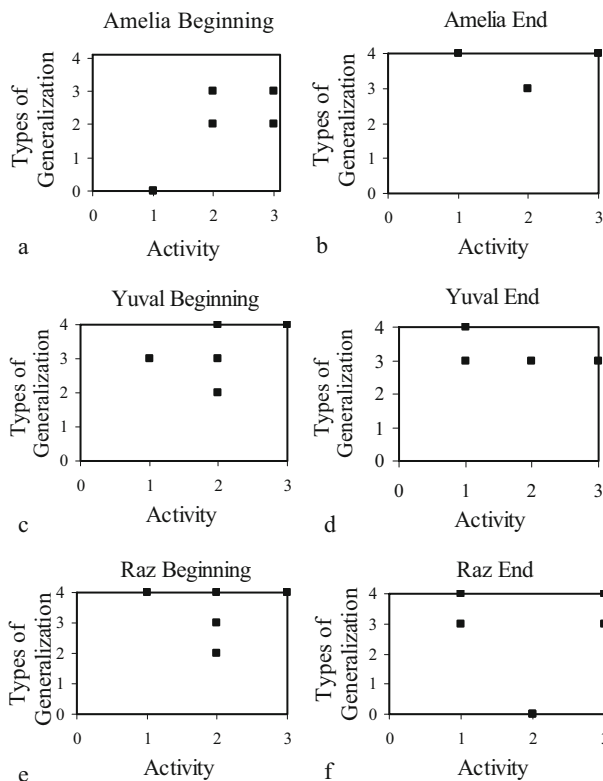


Fig. 6 Uses of symbolic generalizations by Amelia, Yuval, and Raz, at the beginning and at the end of the school year. The y -axis: 0—no use, 1—Numerical, 2—Multiple, 3—Recursive, 4—Explicit. *No use* the student did not use the computer

As the other students did, Amelia, Yuval and Raz used various strategies within the same problem. However, towards the end of the year, the absence of the numerical and multi-variable strategies is noticeable.

5.5 Two similar problems, administered in October and March

In this section, we present examples of the generalizations produced by students working on *Buying a Walkie-Talkie* (Fig. 7—at the beginning of the year) and *Present for the Parents* (Fig. 8—at the end; Fig. 1, item e). These problems are mathematically similar, to allow for comparisons.

Buying a Walkie-Talkie

The following expressions describe the amount of money (in NIS) in the savings box of different children (x denotes the number of the week).

Dina	$7x$	Yoni	300
Karin	$10x$	Rubin	$60 + 3x$
Moshon	$30 + 5x$	Eliran	$-20 + 4x$
Danny	$300 - 5x$	Moti	$-70 + 7x$

Example: At the beginning of the year Eliran had a debt, and each week he added 4 NIS to his box.

The children decided that, in order to be able to buy a Walkie-Talkie costing 400 NIS, they will join pairs of saving boxes.

1. *Find expressions, as short as possible, to describe the amounts of money in the joined saving boxes of the following pairs. (Try first to express verbally the joined savings for each week).*

Dina and Karin _____	Moshon and Danny _____
Dina and Moshon _____	Moshon and Yoni _____
Dina and Danny _____	Moshon and Rubin _____
Dina and Yoni _____	Moshon and Eliran _____
Dina and Rubin _____	Moshon and Moti _____
Dina and Eliran _____	Danny and Rubin _____
Dina and Moti _____	Danny and Eliran _____

2. *Which of these pairs (or perhaps another possible pair), will be the first to collect the 400 NIS needed to purchase the Walkie-Talkie?*

Fig. 7 The buying a walkie-talkie activity

Present for the Parents

The five siblings, Avi, Ben, Gal, Danielle and Hilla decided to buy together a present for their parents' 25th wedding anniversary, which will take place in half a year. Each of them reported on the present status of his/her bank account.

Avi had 300 NIS in his account, and each week he adds 40 NIS.

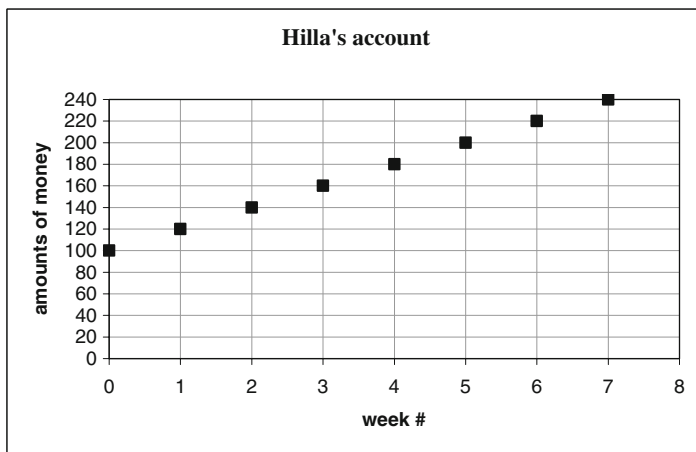
Ben had (-300) NIS in his account (a debt), and each week he adds 60 NIS.

Gal presented her account in the following table:

Week #	0	1	2	3	4	5	6
Amounts of money	500	480	460	440	420	400	380

Danielle presented her account with this expression: $-100 - 15x$, where x stands for the number of weeks.

Hilla presented her account with the following graph:



1. Find expressions to describe the amounts of money in the joined accounts of the following pairs. (Names of pairs were provided).
2. In what week will the joined account of each pair have 600 NIS?

Fig. 8 The present for the parents activity

The *Buying a Walkie-Talkie* problem does not include any guidance for using computerized tools. Note that when this problem was administered, students had learned neither how to add symbolic expressions nor how to solve linear equations.

The *Present for the Parents* problem was administered 6 months into the course and it does not suggest the use of computerized tools either.

While a majority of teams (about 73%) used spreadsheets for answering the problem at the beginning of the year, towards the end of the year only about one fourth of the teams (23%) used spreadsheets in a very similar mathematical problem. Thus about one-half of the students moved from computer use to a paper and pencil, even when the computer was fully available to them.

In order to better understand this shift, we looked at some of the working records on these two problems in chronological order.

When working on *Buying a Walkie-Talkie*, Nattaly and Eli were trying to find a pair of children who will reach 400 NIS first, when the following dialogue took place:

Nattaly: *Let's make a table with Excel.*

Eli: *They need 400 NIS, they will not have it.* [It is not clear to which pair she is referring].

Eli: *They will!!* [Still not clear to which pair she is referring].

Nattaly: *Yes, but we also need to know in what week.*

Nattaly states clearly *why* she would like to use *Excel*—the tool will help her in finding out *when* each pair of joined savings will reach 400 NIS. At this stage, she cannot solve an equation of the type $ax + b = 400$. Five months later, when working on *Present for the Parents*, the same team used an equation of the type $ax + b = 600$ in order to answer a similar question (the students were still not experts on techniques and they solved the equation verbally by considering possibilities). Creating a spreadsheet table was not their first choice of a working tool anymore. Rather, the students preferred to use a relatively recently learned tool—solving an equation with paper and pencil.

This shift in the tool use indicates that students did not stick to spreadsheets. Why? We propose that, whereas at the beginning, they found in the spreadsheets an easy way to answer “*when* the amount of 400 will be reached”, their incipient learning of algebraic tools offered to them later on another efficient way to solve an equation (which now they can state explicitly). We also propose that their appreciation for stating symbolically an equation (and for having rudimentary tools to solve it) developed precisely because it was preceded by a tool which helped them focus first by numerical means on the meaning of a situation which needs an equation in order to be solved (i.e. finding for what week a certain amount will be reached). Moreover, in the process students gained an exposure to an additional way of thinking: an introduction to recursion, an idea that will serve them well into their mathematical education.

6 Discussion

What kind of symbolic generalizations do students use at the beginning of a *CIE* algebra course using spreadsheets?

At the beginning of the year long course of beginning algebra students used a variety of generalization strategies, a few were numerical, while many were of the multi-variable type (Fig. 5). The use of explicit generalizations was relatively high, while the use of recursive generalizations was low. The variety expressed itself both among students but also within the same individual students, who often used more than one strategy. The variety can be attributed to the availability of the tool and to the freedom offered by this *CIE*.

Do students' generalization strategies change during the course? If yes, in what ways? Towards the end of the year, students used more sophisticated strategies than those used at the beginning. Almost no numerical nor multi-variable strategies persisted, while the use of recursive generalizations increased. Therefore, this study shows that students shift from arithmetic to algebra without experiencing the "cuts" so characteristic of similar courses. The students may have benefited from their exposure to alternative ways of solving problems, of communication with each other on their strategies, and of comparing and contrasting advantages and disadvantages of each.

Can students wean themselves from spreadsheets by the end of the year? In other words, can students be fully functional in a paper and pencil environment with explicit symbolic expressions? In our data and analysis described above, we showed that most students had shifted or started to shift to a pencil and paper environment, especially when they were confronted with situations requiring them to state and solve linear equations. It is interesting to note, that when solving linear equations students started to abandon spreadsheets; they returned to using them as a supportive and mediating tool in non-linear situations for which they reached again the boundaries of their symbolic knowledge.

Our *CIE* was based on (1) the availability of computerized tools to students at all times, (2) the freedom to choose if, when and how to use these tools, and (3) an ad-hoc algebra curriculum based on rich problem situations. It may be difficult to tease the role and influence of each of these characteristics (or perhaps their synergy) that resulted in such diverse and personalized strategies which seem to converge on the expected outcome of the successful learning of algebra.

More *CIE* research of this kind is a challenge worth pursuing in order better to understand the following: (a) how experiences reported here can be expanded for different schools and school systems, (b) how to harness the potential of technological tools further; (c) which kinds of pedagogies and classroom environments should accompany these tools. We can only speculate what kind of support a teacher needs in order to implement such an environment in her class.

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