

LEARNING AND TEACHING ALGEBRA AT THE MIDDLE SCHOOL THROUGH COLLEGE LEVELS

Building Meaning for Symbols and Their Manipulation

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From the days of al-Khwârizmî, and on through Vieta and Euler, algebra was all about procedures and notations. This view of algebra, as a tool for manipulating symbols and for solving problems, has been reflected in school algebra curricula as they developed and took shape throughout the 1800s and into the 1900s. Like the students of today, those of yesteryear no doubt struggled to adapt to the intricacies of this representational and procedural tool. Thus, research conducted during the first half of the 20th century on the learning of algebra focused, for example, on the relative difficulty of solving various kinds of linear equations (Hotz, 1918; Reeve, 1926), the role of practice (Thorndike et al., 1923), and the errors that first-year algebra students make in applying algorithms (Breslich, 1939). During the 1950s and early 1960s, research related to algebra learning was conducted mostly by psychologists with a behaviorist orientation who used the subject area as a vehicle for studying general questions related to skill development, memory, and the like. In contrast, the research that has been conducted at least since the late 1970s when algebra education researchers began to increase in number and coalesce as a community (Wagner & Kieran, 1989) has tended to focus on the kind of meaning students make of their algebra, as well as

on various ways in which to make algebra learning meaningful for students.

It is not surprising that more recent research in algebra learning and teaching has, in general, emphasized pathways to and the development of algebraic meaning. First, Piaget's ideas had begun to circulate among North American researchers during the 1960s and 1970s. The influence of his cognitive development psychology on mathematics education researchers was substantial. Next, studies over several decades had shown that an exclusively skills-based approach to the teaching of algebra did not lead to skilled performance among algebra students (e.g., Monroe, 1915, and the later work of Carry, Lewis, & Bernard, 1980). Nor, according to the ample number of studies of the late 1970s and 1980s, had such approaches led to students' being able to interpret adequately the various ways in which letters are used in algebra (Küchemann, 1981; Matz, 1982), or the structural features of algebraic expressions (Davis, 1975), or equivalence constraints on equations and equation solving (Greeno, 1982). Thirdly, the research on algebra learning since the 1980s has been influenced by socio-democratic forces, such as for example the "Algebra for All" movement in the United States (Chazan, 1996; Edwards, 1990). New ways of

thinking about the content of secondary school algebra, as well as the notion that introducing algebraic activity in primary school (Kaput, 1995) could help in removing some of the obstacles associated with secondary school algebra, began to emerge and be discussed. Lastly, the arrival of computing technology in schools in the 1980s led to reshaping conceptions of what could (and should) be taught in school algebra (Fey, 1989)—ideas that were to a large extent both initiated by algebra learning researchers as well as reflected in their subsequent work.

Whereas much of the 1970s and 1980s research focused on the reporting of primarily empirical findings with respect to students' understanding of and activity in algebra, recent theoretical advances in the field of algebra education research, both in North America and abroad (see, e.g., Lerman, Xu, & Tsatsaroni, 2002; Radford, 2000), have widened the community's perspective on the notion of algebraic meaning and have altered its ways of thinking about how students construct meaning for algebraic objects and processes. The theoretical framework of constructivism with its key tenet that knowledge is actively constructed by the cognizing subject, which had been inspired by Piaget's work and had blossomed in the 1980s, had led algebra researchers to move their attention from, for example, the errors made by students to the ways in which students craft their understandings of algebraic concepts and procedures. Although the focus was still cognitive, it tended to be broader than that suggested by the research analyses carried out just a decade or two earlier. Then, during the 1990s, the sociocultural perspective that had developed outside the mathematics education research community began to be felt within. The earlier constructivist/cognitivist orientation shifted for a large number of algebra researchers toward analyses of social factors affecting algebra learning (see, e.g., Cobb, 1994), with an accompanying interest in the mediating role of cultural tools (e.g., Meira, 1998). This shift also led to an increase in classroom-based studies with a focus on teacher-student and student-student discourse (e.g., Bartolini Bussi, 1995; Sfard, 2001). The increasing use of digital tools in learning, in combination with the sociocultural lens being applied to both the design of learning studies and the analysis of the resulting data, led to new questions about the nature of learning with these tools. Dynamic learning environments prompted the development of new theoretical perspectives in which bodily experience was considered to play a role in the creation of meaning for algebraic objects.

More recent research has thus had not only a stronger theoretical component to frame the empirical work, but also one derived from a more diverse set of theories

than in the past. Shifting views of the content of school algebra have also been exemplified in this research. In order to give adequate consideration to these shifts, the chapter is divided into two parts, the first part of which is devoted to their discussion. This part also includes the presentation of models of algebraic activity and examines various sources of meaning that shape algebra learning. The second part of the chapter deals with the body of research that has been carried out in the learning and the teaching of algebra, primarily since 1990. Although the research studies conducted prior to 1990 have already been discussed in Kieran (1992), this chapter will at times refer back to this earlier body of research to suggest the ways in which more recent work has been informed by the past.

This review draws upon the considerable number of algebra research resources that exist already in the domain, including research syntheses (Bednarz, Kieran, & Lee, 1996; Filloy & Sutherland, 1996; Kieran, 1990, 1992; Mason & Sutherland, 2002; Rojano, 2002; Sutherland, Rojano, Bell, & Lins, 2001; Wagner & Kieran, 1989), related monographs (Kilpatrick, Martin, & Schifter, 2003; National Research Council, 2001; Wagner, 1993), the Proceedings and final volume of the recent ICMI Algebra Study (Chick, Stacey, Vincent, & Vincent, 2001; Stacey, Chick, & Kendal, 2004), and the algebra research reports found in the Proceedings of both the International Group for the Psychology of Mathematics Education (PME) and the North-American chapter of PME (PME-NA).

In addition to the above sources, the journal literature that was extensively surveyed in preparing this chapter includes articles from the *Journal for Research in Mathematics Education*; *Educational Studies in Mathematics*; *Journal of Mathematics Teacher Education*; *Journal of Mathematical Behavior*; *Mathematical Thinking and Learning*; *International Journal of Computers for Mathematics Learning*; *For the Learning of Mathematics*; *Canadian Journal of Science, Mathematics, and Technology Education*; *Recherches en Didactique des Mathématiques*; *Petit x*; *Focus on Learning Problems in Mathematics*; *The Mathematics Educator*, and *Hiroshima Journal of Mathematics Education*. Additional resources, too numerous to mention here but noted in the Reference list, were also consulted.

PART 1. SCHOOL ALGEBRA: BACKGROUND ISSUES AND THEORETICAL CONSIDERATIONS

This part of the chapter deals with some of the issues surrounding current research in algebra learning and teaching, such as the content of school algebra, source-

es of meaning, and models of algebraic activity. It also includes a description of the model that will serve as a framework for the research studies on learning that are presented in the second part of the chapter.

The Current Content of School Algebra: Competing Views

Since the mid-1980s, the content of school algebra has been experiencing a tug of war between traditional and reformist views. The current situation is such that the content varies not only from country to country, but also within a given country. In the United States, for example, algebra classes might be following either reform-oriented or traditional programs, or something in between (see Lester & Ferrini-Mundy, 2004). Traditional (i.e., nonreform) algebra courses still tend to reflect the content and approaches of the texts written in the 1960s and 1970s by Dolciani and her collaborators (e.g., Dolciani, Berman, & Freilich, 1965). These courses, which are typically 1-year courses and have a strong symbolic orientation, include the simplification of expressions and the solving of equations, inequalities, and systems of equations by formal methods, as well as the factoring of polynomial and rational expressions. Interspersed among the various chapters of typical textbooks are word problems of various types that serve to apply the algebraic techniques just covered. Although functions, along with their graphical, tabular, and letter-symbolic representations, are also treated, they are generally accorded a more minor role—the main thrust of the course being polynomial and rational expressions and their manipulation. According to Saul (1998), this emphasis on polynomial and rational expressions is oriented toward recognizing form, which is considered one of the most important aspects of school algebra within this perspective:

This level [of computing with rational expressions] is reached when students learn patterns of factoring, for example. Seeing the factorization $A^2 - B^2 = (A + B)(A - B)$, a student who has mastered certain algebraic concepts can factor $x^4 - y^4$, $4x^2 - 1$, or $\cos^2 x - \sin^2 x$. Mathematically, the student is then letting the variables A and B stand for other rational expressions (or rational trigonometric expressions). (p. 138)

Saul is not alone in his view, one that is shared by several other mathematics educators and mathema-

ticians (e.g., Wheeler, 1996). Pimm (1995) similarly argued that, “algebra is about form and about transformation” (p. 88).

In contrast to the position enunciated above, nontraditional, reformist algebra programs tend to give a great deal of weight to functions, to various ways of representing functional situations, and to the solution of “real-world” problems by methods other than manual symbolic manipulation, such as technology-supported methods. These programs reflect the stance formulated by Fey and Good (1985) that practicing manipulative skills should be replaced with the study of families of elementary functions, a shift that “places the function concept at the heart of the curriculum” (p. 49).

The origins of this changed perspective regarding the core elements of school algebra can be traced back to the 1980s when some mathematicians and mathematics educators promoted the idea that computing technology could have significant effects on the content and emphases of school-level and university-level mathematics (Fey, 1984). Early visionaries soon realized that computing technology could be harnessed to more fully integrate the multiple representations of mathematical objects in mathematics teaching. Soon, a *functional perspective* on algebraic activity could be seen in some research projects, a few of them involving pilot programs in schools.¹

Heid (1988), one of the pioneers of a functional view of algebra (see also Fey et al., 1991; Schwartz & Yerushalmy, 1992), defined it as follows:

The functional approach to the emergence of algebraic thinking . . . suggests a study of algebra that centers on developing experiences with functions and families of functions through encounters with real world situations whose quantitative relationships can be described by those models. (Heid, 1996, p. 239)

The encouraging results emanating from the research program of Fey, Heid, and their collaborators had an impact not only on the developing projects of other researchers at the time but also on the recommendations to be found in the early *Standards* documents² and other related resources published by the National Council of Teachers of Mathematics (NCTM). The movement for function-based approaches to algebra continued throughout the 1990s, becoming more fully elaborated in integrated curricula (see Senk &

¹ This was not the first time that functions became a thread of secondary school mathematics programs. During the 1960s, for example, the modern mathematics movement had included similar proposals (see, e.g., the texts authored by Van Engen et al., 1964). However, at that time, the approach being considered was much more formal, emphasizing sets and distinctions between functions and relations.

² These documents included *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989), *Professional Standards for Teaching Mathematics* (NCTM, 1991), and *Assessment Standards for School Mathematics* (NCTM, 1995).

Thompson, 2003, for examples of National Science Foundation-funded, reform-based curricula that were produced). It culminated in the later publication of the *Principles and Standards for School Mathematics*, where the Algebra Standard for grades pre-K–12 emphasized “relationships among quantities, including functions, ways of representing mathematical relationships, and the analysis of change” (NCTM, 2000, p. 37).

If the number of research studies carried out within this perspective is any gauge, a functional perspective on the teaching of algebra became quite widespread in the United States. However, this perspective is not without its critics. For many researchers, algebra does not consist in the study of functions; algebraic expressions can be used without describing functions, and functions can be expressed without using algebra (Lee, 1997). Pimm (1995) has argued strongly against the position that algebra is about functions:

Ironically, technology is being used to insist on screen (graphical) interpretations of algebraic forms. There is a strong presumption that symbolic forms are to be interpreted graphically, rather than dealt with directly. . . . There is currently a rapid process of redefinition of algebra, triggered I feel more by the potentialities of these new [technological] systems and the drawbacks of an over-fragmented mathematics curriculum than by any novel epistemological insight. The view of algebraic objects that is being strongly promoted . . . is of algebraic expressions as functions. (p. 104)

One of the by-products of these competing views of algebra content is the development of hybrid versions of programs of study that attempt to include elements of both traditional (rational expressions and equations) and functional orientations to school algebra. However, such mixed approaches can create additional difficulties for algebra learners. As pointed out by Chazan and Yerushalmy (2003), students become confused regarding distinctions between equations and functions, not being able to sort out, for example, how equivalence of equations is different from equivalence of functions. According to Chazan and Yerushalmy, it is not the case that such questions are unanswerable, but rather that combining functional approaches with more standard treatments of school algebra leads to such dilemmas, and that only rarely are opportunities provided for students to inquire into these questions and to attempt to resolve them.

The various curricular approaches to algebra that are currently available in different parts of the United States, ranging from reform to traditional to in-between, are also to be found in the diversity of algebraic content worldwide (see, e.g., Kendal & Stacey, 2004). Sutherland (2002), who conducted a compara-

tive summary of algebra curricula in several countries, noted surprising differences in the ways in which algebra is specified in these various curricula:

For example, the algebra component of the Australian (Victoria) curriculum is organized around the ideas of “expressing generality,” “equations and inequalities,” and “function,” whereas in the Hungarian curriculum the algebra component refers only to work with systems of equations. . . . The Japanese curriculum is different from the algebra curricula of all the other countries in that there is an emphasis on symbolizing mathematical relationships from elementary school and there is a much more demanding expectation with respect to the use of algebraic symbols for the 15–16-year-old age group than in the other countries studied. . . . The idea of introducing algebra within the context of problem situations is evident within most of the countries studied, although these “problem situations” are sometimes more traditional word problems (e.g., in Italy, Hungary, France, Hong Kong) and are sometimes more “realistic modeling situations.” . . . In general where there is more emphasis on solving “realistic problems” there tends to be less emphasis on symbolic manipulation. (p. 29)

This latter observation by Sutherland regarding the greater the emphasis on realistic problems, the less the emphasis on symbolic manipulation, remains a fundamental issue in the arena of competing ideas on algebraic content. The orientation toward the solving of realistic problems, with the aid of technological tools, allows for an algebraic content that is less manipulation oriented. Such orientations also emphasize multi-representational activity with a shift away from the traditional skills of algebra. The logic underpinning this position is that technological tools can step in and assist with problem solving when students do not have the required manipulative skills. This has led to a certain amount of resistance in the United States to the NSF-funded reform-based curricula at the upper secondary level because the technical aspects of algebra would seem to be minimized. The fundamental issue is really one of values—that is, what it is that mathematicians, mathematics educators, and mathematics education researchers consider essential and important in algebra. The next section on sources of meaning takes a second look at some of the same issues, but this time from an epistemological perspective.

Sources of Meaning

This section examines first the general question of where algebraic meaning comes from and then goes on to describe each of four main sources of meaning: (a) meaning from the algebraic structure itself, involv-

ing the letter-symbolic form; (b) meaning from other mathematical representations, including multiple representations; (c) meaning from the problem context; and (d) meaning derived from that which is exterior to the mathematics/problem context (e.g., linguistic activity, gestures and body language, metaphors, lived experience, image building, etc.).

Algebraic Meaning: Where Does It Come From?

Several researchers have studied the specific question of meaning in school algebra (e.g., Arzarello, Bazzini, & Chiappini, 2001; Filloy, Rojano, & Rubio, 2001; Kaput, 1989; Kirshner, 2001; Lins, 2001; Sackur, Drouhard, Maurel, & Pécal, 1997). The different perspectives that have been described at various times in the literature have recently been reconceptualized by Radford (2004). For Radford, meaning in school algebra is produced in the “crossroads of diverse semiotic mathematical and non-mathematical systems” (p. 163) and is deemed to come from three primary sources: (a) the algebraic “structure” itself, (b) the problem context, and (c) the exterior of the problem context.

Although the third of Radford’s sources of meaning is quite new to algebra research, something akin to it has been discussed by Noss and Hoyles (1996) in their general treatment of “meaning in mathematics education” according to a triple perspective that includes *meanings from mathematical objects*, *meanings from problem solving*, and *meanings constructed by the individual learner*. The parallelism between the three sources of meaning in algebra from Radford and the corresponding view for mathematics at large from Noss and Hoyles has a certain appeal. However, a strictly threefold approach is not broad enough within the context of current views of algebra content in which graphical and other mathematical representations are considered very important. Thus, a second within-mathematical source of meaning in algebra could be added to Radford’s classification, namely *meaning from other mathematical representations, including multiple representations* (see Figure 16.1).

Meaning From the Algebraic Structure Itself, Involving the Letter-Symbolic Form

At the Research Agenda Project conference on algebra in 1987, Booth (1989) argued that

Our ability to manipulate algebraic symbols successfully requires that we first understand the structural properties of mathematical operations and relations which distinguish allowable transformations from those that are not. These structural properties constitute the *semantic* aspects of algebra. . . . The essential feature of algebraic representation and symbol manipulation, then, is that it should *proceed from* an understanding of the semantics or referential meanings that underlie it. (pp. 57–58)

In other words, “the sense of meaningfulness comes with the ability of ‘seeing’ abstract ideas hidden behind the symbols” (Sfard & Linchevski, 1994, p. 224), that is, the symbols become transparent.

This structural source of meaning not only links letter-symbolic representations to their numerical foundations but also provides connections among the symbolic forms of algebra, its equivalences, and its property-based manipulation activity (e.g., Cerulli & Mariotti, 2001). Although the algebra research literature often refers to *the structure of expressions*, the latter phrase both shuns definition (see, e.g., Arcavi, 1994; Linchevski & Livneh, 1999; MacGregor & Price, 1999) and proves difficult for students to grasp (e.g., Demby, 1997). Nevertheless, even though the nature of the meaning that students draw from algebraic structure can be elusive, this source of meaning is considered by many mathematics educators and researchers to be fundamental to algebra learning.

Meaning From Other Mathematical Representations, Including Multiple Representations

The diversity of perspectives underlying the content of school algebra suggests that the question of where algebraic meaning comes from is intimately related to the nature of the algebra courses experienced by students. Courses with a traditional focus on literal expressions, equations, equivalent forms, properties

1. Meaning from within mathematics:

1(a). Meaning from the algebraic structure itself, involving the letter-symbolic form.

1(b). Meaning from other mathematical representations, including multiple representations.

2. Meaning from the problem context.

Meaning derived from that which is exterior to the mathematics/problem context (e.g., linguistic activity, gestures and body language, metaphors, lived experience, image building).

Fig. 16.1 Sources of meaning in algebra (adapted from Radford, 2004).

and structure will offer quite different meaning-building experiences from those oriented toward functions and the interplay of tabular, graphical, and symbolic representations. The question then is, what role can be played by the tabular and graphical representations, in conjunction with the letter-symbolic, in algebraic meaning making.

Kaput (1989) has argued that the problem of student learning in algebra is compounded by (a) the inherent difficulties in dealing with the highly concise and implicit syntax of formal algebraic symbols and (b) the lack of linkages to other representations that might provide feedback on the appropriate actions taken. As a consequence, he has promoted the kind of mathematical-meaning building that has its source in *translations between mathematical representations systems*. Although tables, graphs, and equations can all be used to display binary relations, a critical difference, according to Kaput, between tables and graphs is that “a graph engages our gestalt-producing ability, which allows us to consolidate a binary (and especially a functional) quantitative relationship into a single graphical entity—a curve or line” (Kaput, 1989, p. 172). The opportunity to coordinate objects and actions within two different representations, such as the graphical and the letter-symbolic, is considered by many to be crucial in creating meaning in algebra (e.g., Fey, 1989; Romberg, Fennema, & Carpenter, 1993; Yerushalmy & Schwartz, 1993).

Meaning From the Problem Context

In contrast to the internal semantics of algebra as a site for meaning making (Booth, 1989), the external semantics of a problem permit the algebra learner to fuse symbols and notations with events and situations, thereby creating an external meaning for certain objects and processes of algebra. A strongly held belief in algebra education is the notion that problem-solving contexts are foundational to the emergence and evolution of algebraic reasoning (e.g., Bednarz & Janvier, 1996; Bell, 1996). This stance is based to a certain extent on historical grounds whereby problem solving made a major contribution to the development of algebra. As well, algebra grew in status to become the privileged tool for expressing general methods for solving whole classes of problems. Additional arguments for the role of problems in algebraic meaning building are related to notions of relevance and purpose (cf., Lee, 1997). Nevertheless, it has been argued that significant epistemological issues are associated with the whole idea of using word problems for creating algebraic meaning making (e.g., Balacheff, 2001; Pirie & Martin, 1997). Therefore, a broader perspective on problem solving has been advanced by Bell (1996):

For introducing and developing algebra, I understand problem solving to refer to the solving of problems by the forming and solving of equations; this is the narrow sense of the term. But the essential mathematical activity is that of exploring problems in an open way, extending and developing them in the search for more results and more general ones. Hence [all algebraic learning] . . . is based on problem explorations. This is the broad sense of the term. (p. 167)

Included within this source of algebraic meaning making are “real problems,” that is, those that involve *modeling* situations. Much of the current modeling activity that occurs in algebra classes uses physical artifacts or technological modeling tools as an integral part of the activity. In these cases, the contextual aspect of the situation is very much intertwined with the use of gestures and body language in the process of constructing a mathematical model. This activity thus combines two sources of meaning, one derived from the problem context and the other that is exterior to the context and that is presented immediately below.

Meaning Derived From That Which Is Exterior to the Mathematics/Problem Context

A recent approach by algebra researchers to thinking about meaning making concerns sources of meaning that are exterior to both the mathematics and the problem situation. (Note that much of this meaning building that is “exterior” to both the mathematics and the problem situation is related to that which is uniquely human, and thus is clearly “interior” in a very real sense.) This approach focuses on students’ processes of meaning production in terms of the way diverse resources such as gestures, bodily movements, words, metaphors, and artifacts become interwoven during mathematical activity (see, e.g., Arzarello & Robutti, 2001; Radford, Demers, Guzmán, & Cerulli, 2003). Past studies of students’ ways of thinking and of their errors in algebra had always suggested that students brought more to bear on their learning of algebra than was accounted for by the theories available at the time. Thus, the explicit focus on bodily activity, language, and past lived experience as a source of meaning would seem a natural evolution in research on the learning of algebra.

On the basis of his studies of students’ production of oral and written signs and the meanings they ascribe to them as they engage in the construction of expressions of mathematical generality, Radford (2000) has observed that

Students, at the very beginning, tend to have recourse to other experiential aspects more accessible to them than the structural one. . . . Novice students bring

meanings from other domains (not necessarily mathematical domains) into the realm of algebra. Hence it seems to us, one of the didactic questions with which to deal is . . . that of the understanding of how those non-algebraic meanings are progressively transformed by the students up to the point to attain the standards of the complex algebraic meanings of contemporary school mathematics. (p. 240)

Radford's closing remark, which implicitly raises the question as to the nature of the algebraic activity whereby students' meanings are gradually transformed and aligned with "the complex algebraic meanings of contemporary school mathematics," leads to the next section dealing with algebraic activity. As will be seen, the section integrates previously elaborated issues of content and sources of meaning in algebra.

Models for Conceptualizing Algebraic Activity

Bell (1996) has described algebra as a means to express generalizations, relations, and formulas; solve problems; denote unknowns; and solve equations. Usiskin (1988) has synthesized four conceptions of algebra: generalized arithmetic, the set of procedures used for solving certain problems, the study of relationships among quantities, and the study of structures. Kaput (1995) has identified five aspects of algebra: generalization and formalization; syntactically guided manipulations; the study of structure; the study of functions, relations, and joint variation; and a modeling language. Other researchers (see, e.g., Gascón, 1994–1995; Lins, 2001; Mason, Graham, & Johnston-Wilder, 2005; Rojano, 2004; Smith, 2003; Star, 2005; Sutherland, 2004) have offered variations on the above, each in accordance with their particular perspectives on algebra, algebraic thinking, or algebraic language.

Participants at an international colloquium on algebra in Montreal in the early 1990s focused on four approaches: generalization of numerical and geometric patterns and of the laws governing numerical relationships; problem solving; functional situations; and modeling of physical and mathematical phenomena (Bednarz et al., 1996). Concurrently, a study by Lee (1997) was in progress, one that adopted an even broader perspective on that which constitutes school algebra. In this study, which explored algebraic understanding, Lee presented the question, "What is algebra?" to a cohort of mathematicians, teachers, students, and mathematics education researchers. Seven themes emerged from her interviews on the question: a school subject, generalized arithmetic, a tool, a language, a culture, a way of thinking, and an activity.

If there was one theme that tended to permeate all others among Lee's interviewees, it was *Algebra is an activity*. Lee (1997) remarked that, "Algebra emerges as an activity, something you do, an area of action, in almost all of the interviews" (p. 187). For example, Pimm, one of Lee's interviewees, is quoted as saying that *action* is the central feature of school algebra: "Algebra . . . is so much more about doing, is actually about action on things, . . . with attention being more on the *transformations* [emphasis added] than on the objects themselves" (p. 187). On the other hand, Kaput, another interviewee, emphasized the alternate side of the coin when he pointed to the importance in algebraic activity of spending a great deal of time in the "building of algebraic objects" (p. 189). And Bell expressed concern about the question of purpose, about students not really "having the experience of what algebra is for" (p. 196).

Building on the idea of algebra as activity (and reflecting the key ideas expressed by Kaput, Pimm, and Bell), Kieran (1996) developed a model that synthesizes the activities of school algebra into three types: *Generational*, *Transformational*, and *Global/meta-level*. This model, hereafter referred to as the GTG model (see Figure 16.2), will serve in Part 2 as the structuring device for presenting findings from the research involving the algebra learner.

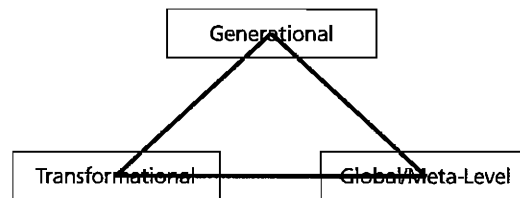


Figure 16.2 GTG model for conceptualizing algebraic activity.

The *Generational* activities of algebra involve the forming of the expressions and equations that are the objects of algebra. Typical examples include (a) equations containing an unknown that represent problem situations (see, e.g., Bell, 1995), (b) expressions of generality arising from geometric patterns or numerical sequences (see, e.g., Mason, 1996), and (c) expressions of the rules governing numerical relationships (see, e.g., Lee & Wheeler, 1987). Much of the meaning building for algebraic objects occurs within the generational activity of algebra. Therefore this activity includes, as well, work with variables, unknowns, and equality; even the notion of that which constitutes the solution to an equation can be initiated within generational activity. Clearly, the meaning that students develop for the notion of, for example, *variable* will

depend on whether the algebra content they have experienced has been derived from a polynomial-equation framework or a functions framework, that is, has tended to emphasize either the letter-symbolic or a combination of mathematical representations. These two frameworks each provide a unique transversal thread to the three types of algebraic activity.

Generational activity is the area where, according to Radford (2001), the role of algebra is that of a language to express meaning, and where the *habit of mind* of those who are in “algebra mode” can find expression (Cuoco, Goldenberg, & Mark, 1996). It may be tempting to equate generational activity with the conceptual aspects of algebra and, similarly, transformational activity with the skill-based aspects of algebra, but this parallelism is not intended. Indeed, as is being proposed in this section and argued throughout Part 2, conceptual work and meaning building occur as well within the transformational activity of algebra.

The second type of algebraic activity—the *Transformational* activities (referred to, by some, as the *rule-based* activities)—includes, for instance, collecting like terms, factoring, expanding, substituting one expression for another, adding and multiplying polynomial expressions, exponentiation with polynomials, solving equations and inequalities, simplifying expressions, substituting numerical values into expressions, working with equivalent expressions and equations, and so on. A great deal of this type of activity is concerned with changing the symbolic form of an expression or equation in order to maintain equivalence. In addition to developing meaning for equivalence, this activity also includes meaning building for the use of properties and axioms in the manipulative processes themselves. Even the technique of factoring has its conceptually meaningful aspects: for example, coming to see that, if the exponent in, say, $x^n - 1$ has several divisors, it can generally be factored in more than one way, and thus that this expression can be viewed in terms of multiple possible structures (e.g., seeing $x^6 - 1$ as a difference of squares and also as a difference of cubes). It is in this way that meaning can be said to be present for the transformational processes of algebra. This factoring example also illustrates that a given task can be both transformational and generational. It is generational to the extent that the learning of a new manipulative transformation, such as $x^6 - 1 = (x - 1)(x^5 + x^4 + x^3 + x^2 + x + 1)$, can induce an extended perspective on the algebraic object itself.

In this sense, transformational activity is not just skill-based work. It is much broader than this in that it can include conceptual/theoretical elements, especially during the period in which certain transformations are being learned. According to Lagrange

(2002), “Technique has a pragmatic role that permits the production of results; but it also plays an *epistemic* [emphasis added] role in that it constitutes understanding of objects and is the source of new questions” (p. 163, translated from the original).

Lastly, there are the *Global/meta-level*, mathematical activities. These are the activities for which algebra is used as a tool but that are not exclusive to algebra. Note that they are not referred to as *global/meta-level* because they involve taking some distance from the mathematical activity at hand and viewing it comprehensively. Rather, they suggest more general mathematical processes and activity. They also provide the context, sense of purpose, and motivation for engaging in the previously described generational and transformational activity. The global/meta-level activities include problem solving, modeling, working with generalizable patterns, justifying and proving, making predictions and conjectures, studying change in functional situations, looking for relationships or structure, and so on—activities that could indeed be engaged in without using any letter-symbolic algebra at all. These are the kinds of activities that provide the surround for engaging in mathematical activity in general, and algebraic activity in particular—and, thus, for constructing and working with algebraic objects and processes.

However, algebra does not equal problem solving, or modeling, or any of the other global/meta-level activities. Although some may find it helpful to think of these global/meta-level activities in terms of “approaches” to the teaching of algebra—such as a generalization approach to algebra or a problem-solving approach to algebra (see, e.g., Bednarz et al., 1996)—algebra includes as well generational and transformational activity, which may not be a necessary component of the designated approach to algebra. On the other hand, aspects of, for example, generalization and problem solving can be present in much of generational and transformational activity of algebra, especially during the early stages of learning. Thus, the global/meta-level activity of algebra is both broader than and at the same time not quite as broad as algebra.

PART 2. SCHOOL ALGEBRA: FINDINGS FROM RESEARCH STUDIES

This second part, which constitutes the main thrust of the chapter, describes the body of research work on the learning and the teaching of school algebra, emphasizing more recent studies but also integrating some

findings from earlier work. The research is presented according to three age ranges: the middle grade and lower secondary level student, the upper secondary and college level student, and the teacher of algebra. Research on the development of algebraic thinking at the primary school level is treated in another chapter (see Carraher & Schliemann, this volume).

Within each of the age-related categories dealing with research on the *learning* of algebra, the GTG model of algebraic activity provides the structural framework. Research where the primary focus is generational activity is treated first, followed by the research dealing with transformational activity, and then that of global/meta-level activity. Because the third type of activity is overarching and cross-cutting, it proved to be more practical, at times, to discuss it within the generational or transformational activity that it served. This was particularly the case for studies involving word problems with the middle grade and lower secondary level student. It is noted that the placement of research studies into the various categories of the GTG model was done according to rather general criteria: For example, if the main emphasis of the study appeared to be transformational activity, then the findings are discussed under that category, even if the study also included some generational activity; if several types of global/meta-level activity were involved, the study is reported in the section related to the major focus. Studies that did not fit this model, such as international assessments, are discussed separately.

Research on the algebra learner occupies the first two sections of this part of the chapter; research that has focused on the algebra teacher and the teaching of algebra is the subject of the third section. The chapter then closes with a fourth section that attempts to draw out that which this body of research on the learning and teaching of school algebra tells us.

The Middle Grade and Lower Secondary Level Student

The research that is reviewed in this section includes those studies carried out with students in the middle grade and lower secondary levels, that is, from Grades 6 to 9 (approximately 11 to 15 years of age). The majority of the research carried out with this age range of pupil has focused on generational activity, with considerably fewer studies dedicated to transformational activity. The studies focusing on global/meta-level activity have been concentrated in the areas of generalizing, proof and proving, and modeling (with problem solving being treated within the context of generational activity).

By way of introduction, this section starts with a brief commentary on results from some recent international assessments. This provides a first glimpse at what students of this age find easy or difficult in algebra. These introductory remarks are followed by the three main sections related to research on the middle grade and lower secondary level student, research categorized according to generational, transformational, and global/meta-level activity. A section on the impact of technology on algebraic activity is then presented, followed by brief concluding remarks.

Background: International Assessments

Two large-scale international studies, the Trends in International Mathematics and Science Study (TIMSS, formerly known as the Third International Mathematics and Science Study) and the Program for International Student Assessment (PISA), announced the results of their 2003 surveys in late 2004 (PISA, 2004; TIMSS, 2004). Forty-eight countries participated in the 2003 TIMSS mathematics achievement assessment for Grade 8 (Gonzales et al., 2004). The number of score points obtained on average in the algebra content area across the 48 countries was 25 out of a possible 53, whereas the U.S. eighth graders in algebra obtained on average 29 score points. Figures 16.3 and 16.4 provide two items from the algebra area of the survey.

The three figures below are divided into small congruent triangles.

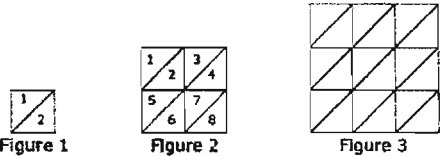


Figure 1 Figure 2 Figure 3

A. Complete the table below. First, fill in how many small triangles make up Figure 3. Then, find the number of small triangles that would be needed for the 4th figure if the sequence of figures is extended.

Figure	Number of Small Triangles
1	2
2	8
3	
4	

B. The sequence of figures is extended to the 7th figure. How many small triangles would be needed for Figure 7?

Answer: _____

C. The sequence of figures is extended to the 50th figure. Explain a way to find the number of small triangles in the 50th figure that does not involve drawing it and counting the number of triangles.

Figure 16.3 One of the released items from the 2003 TIMSS for eighth grade.

Sam wanted to find three consecutive even numbers that add up to 84.

He wrote the equation $k + (k + 2) + (k + 4) = 84$.

What does the letter k represent?

- (A) The least of the three even numbers
- (B) The middle even number
- (C) The greatest of the three even numbers
- (D) The average of the three even numbers

Figure 16.4 Another of the released items from the 2003 TIMSS for eighth grade.

The percentage of correct responses across all TIMSS countries for A, B, and C respectively of the item shown in Figure 16.3 was 36.6%, 22.1%, and 14%. Although the results on this item were universally low, the U.S. students scored higher (50.2%, 29.6%, and 19.4% respectively), perhaps indicating that U.S. students had had more opportunities to examine patterns in this manner than many students internationally. The percentage of correct responses on the item shown in Figure 16.4 (for which the correct answer is A), one with some letter-symbolic content, was 24.7% across all TIMSS countries and 23.1% among U.S. students.

In general, these low results are disturbing—less than 37% correct, worldwide, for each of the four questions shown in the two items of Figures 16.3 and 16.4. Despite the improvement of U.S. eighth graders since the 1995 administration of the test, the fact that only 8 of 48 countries outperformed U.S. eighth graders in algebra in the 2003 assessment suggests that, overall, students are doing quite poorly in algebra. The PISA results, similarly, confirm this situation. The research findings that are presented in this chapter offer a more detailed look at what students know, that is, the nature of the meanings that they attribute to algebraic objects and processes, as well as their approaches to tasks that involve the manipulation of symbols. Some of this research involves the study of alternate learning environments that can lead to more powerful algebraic understandings among students.

Generational Activity: The Middle Grade and Lower Secondary Level Student

The generational activity of algebra involves forming and interpreting the objects of algebra, usually within the context of some global/meta-level activity. This section is one of the larger ones of the chapter dealing as it does with an area where a significant number of research studies on the learning of algebra have been carried out. To render this area more accessible, it has been divided into subthemes according to the source of meaning being invoked in constructing

or interpreting the objects of algebra: (a) primarily a letter-symbolic focus, (b) a multiple-representation perspective, or (c) within the context of word problems. The relatively small number of studies that have focused on the fourth source of meaning, that which is “exterior” to both the mathematics and the problem context, usually included as well one of the other three sources of meaning and are thus treated therein.

Generational Activity with a Primary Focus on the Letter-Symbolic Form. The studies that have focused on the kinds of meanings that students of this age range attribute to the letter-symbolic forms of the objects of algebra have been concentrated in the following areas: variables, expressions, and equations; the minus sign and negative numbers; and the beginnings of structure sense.

Variables, Expressions, and Equations. Whereas past research on the ways in which students of this age range interpret algebraic symbols tended to focus on cognitive levels (e.g., Küchemann, 1981), prior arithmetic experience and methods of thinking (Booth, 1984), and difficulty with notation such as brackets (Kieran, 1979) and the equal sign (Behr, Erlwanger, & Nichols, 1976; Kieran, 1981), more recent work suggests additional factors impinging upon students’ interpretation of algebraic notation: (a) what one is able to perceive and prepared to notice (Sfard & Linchevski, 1994), (b) difficulties with operation signs (Cooper et al., 1997), (c) the nature of the questions asked and the medium in which they are asked (Warren, 1999), (d) the presence of multiple referents and shifts in the meaning of unknowns (Stacey & MacGregor, 1997), and (e) the nature of the instructional activity (Wilson, Ainley, & Bills, 2003).

More specifically, a large-scale study conducted by MacGregor and Stacey (1997) involving 2000 Australian students (aged 11–15) found that students’ interpretations of algebraic notation are based on intuitive assumptions and sensible, pragmatic reasoning about an unfamiliar notation system; analogies with symbol systems used in everyday life, in other parts of mathematics, or in other school subjects; interference from other new learning in mathematics; and poorly designed and misleading teaching materials. The nature of the interpretations brought to bear on algebraic notation by these students suggests the strong role played by the meaning derived from the exterior of the mathematics/problem context. These researchers have argued that recognition of these origins of misunderstanding is necessary for improving the teaching of algebra. In another analysis drawn from the same survey data, MacGregor and Price (1999)

assessed metalinguistic awareness³ and its relation to the use of algebraic notation. The researchers found that very few students with low metalinguistic scores achieved high algebra scores.

The transition from arithmetic to algebra remains an important theme of several studies involving variables, expressions, and equations. Building on earlier research on variables (e.g., Küchemann, 1981; Wagner, 1981), expressions (e.g., Bell, Malone, & Taylor, 1987), and equations (e.g., Kieran, 1988; Whitman, 1976), some of the more recent studies have examined students' abilities to express generalizations and to represent these generalizations in various forms in symbolic expressions (Arzarello, 1992; Lee, 1996; Mason, 1996). Fujii (2003) has introduced younger students to algebraic thinking through generalizable numerical expressions, using numbers as *quasi-variables*—for example, number sentences such as $78 - 49 + 49 = 78$, which are true whatever number is taken away and then added back. Finding that “generalizable numerical expressions can assist students in identifying and discussing algebraic generalizations long before they learn algebraic notation” (p. 59), Fujii claimed that these expressions “allow teachers to build a bridge from existing arithmetic problems to opportunities for thinking algebraically without having to rely on prior knowledge of literal symbolic forms” (p. 62). Other studies (e.g., Graham & Thomas, 2000) have illustrated how calculating technology can be effectively used to develop students' understanding of letters as specific unknowns, generalized number, and variable. In addition, some research studies have focused on the conceptual obstacles involved in moving from simple equations with an unknown on one side to more complex equations with unknowns on both sides and have described phenomena that they have called the *didactic cut* (Filloy & Rojano, 1989) and the *cognitive gap* (Herscovics & Linchevski, 1994).

The Minus Sign and Negative Numbers. In contrast to the findings already accumulated with respect to students' evolving knowledge of the different possible interpretations of the equal sign, we know comparatively little about students' interpretations of the minus sign in algebraic expressions and equations. The subtle distinctions in the ways that the minus sign is used in algebraic expressions and equations were the focus of a study by Vlassis (2004). Of the three possible interpretations of the minus sign, which she described as knowledge of *negativity*, she found that the binary sense predominated among the eighth graders of her

study. She noted a high degree of inflexibility in considering either a unary sense (whereby “subtract 3” means “negative 3”) or a symmetric sense (where, for example, in $-x = 1$, the $-x$ means “the opposite of x ”). Vlassis concluded that the various uses of the minus sign in algebra are counterintuitive to a majority of beginning-algebra pupils and constitute a significant conceptual obstacle in their coming to give meaning to algebraic symbols and processes.

Related research is that involving students' developing notions of negative numbers. Among the few studies in this area is the work of Gallardo (2002; see also Vergnaud, 1989), who applied historical/epistemological analysis to the study of students' extension of the natural-number domain to the integers within word-problem contexts. Gallardo has found that the first three of the four levels of acceptance of negative numbers within historical texts (i.e., subtrahend, relative number, isolated number, and formal negative number) were present among the 12- and 13-year-olds of her study. This suggested to her that students have an intuitive sense of negative numbers long before they are able to formalize this knowledge.

Beginnings of Structure Sense. Linchevski and Livneh (1999) extended some of the earlier research work related to students' structural sense in algebra (e.g., Kieran, 1989; Matz, 1982) in a study that tested whether students' difficulties with aspects of algebraic structure reflect difficulties they already have with numerical structure. They interviewed two classes of end-of-year sixth graders (average age: nearly 12 years) who had not yet learned negative numbers or any algebraic manipulation. They found that, on the whole, students' difficulties with interpreting equations containing several numerical terms and an unknown were a reflection of the same errors that they were making in purely numerical contexts. However, the researchers also noted that some students were inconsistent in that they gave an incorrect answer in one context but a correct one in another. Follow-up interviews with students after they had learned how to operate with negative numbers produced the same results. The researchers suggested that algebra instruction be designed to foster the development of *structure sense* by providing experience with equivalent structures of expressions and with their decomposition and recomposition. They emphasized that more systematic research in this area is needed. One recent thrust in this direction has been taken by Carpenter, Franke, and Levi (2003) whose research with younger students

³ *Metalinguistic awareness* refers, according to the authors, to the ability to reflect on and analyze spoken or written language, for example, being able to pay attention to sounds and spellings of linguistic signs instead of to their meanings. Both symbol awareness and syntax awareness are considered to be components of the metalinguistic awareness necessary for success in learning to use algebraic notation.

has aimed at having students explain and justify the properties they are using as they carry out their arithmetic calculations. Additional research in this area is discussed in the section on transformational activity.

Generational Activity with Multiple Representations.

Most of the approaches to algebra that make use of multiple representations define themselves as *functional approaches*. Tabach and Friedlander (cited in Kieran & Yerushalmy, 2004) suggested that three factors allow for a functional approach in the teaching and learning of algebra: the potential to produce ample numerical tables, the need to use general expressions to express the data in these tables, and the possibility of obtaining a wide variety of corresponding graphs. As an example of such an approach, they referred to the *Compu-Math curriculum* (Hershkowitz et al., 2002), which relies heavily on the use of spreadsheets, in particular. However, not all the research involving multiple representations includes technology environments. In addition, as will be seen, some studies have included multiple representations within word-problem or modeling situations. When the focus of the analysis has been the multi-representational aspect, then the study is included in this section; however, if the word-problem aspect has the higher profile, then the research is treated in the next section. Finally, if the modeling aspect is uppermost, then the study is reserved for the later discussion of global/meta-level activity. This section includes the following themes: tabular representations and the letter-symbolic, graphical representations and the letter-symbolic, and connections among representations.

Tabular Representations and the Letter-Symbolic. Recent studies involving the tabular displays of spreadsheets have investigated whether and how the use of spreadsheet methods lends meaning to algebra. Building on earlier research with spreadsheets as a means to introduce students to algebra-like notation (e.g., Sutherland & Rojano, 1993), Filloy, Rojano, and Rubio (2001) have found that the spreadsheet serves as a bridging tool to algebra because it helps students to create conceptual meaning for algebraic objects and operations, and to move from focusing on a specific example to describing general relationships. For Dettori, Garuti, and Lemut (2001), the functional orientation of spreadsheets allows for their effective use in the investigation of variation but can cause difficulties later when solving equations or inequalities. When these researchers investigated 13- and 14-year-olds working on algebraic problems with spreadsheets, Dettori and colleagues were led to conclude that “spreadsheets can start the journey of learning

algebra, but do not have the tools to complete it” (p. 206). As well, Hershkowitz et al. (2002) have described how students tend, in the spreadsheet environment, to generalize recursively rather than explicitly, which makes it more difficult for them to generate closed-form algebraic rules for patterns.

However, other research work that has included tables of values, but not within a spreadsheet environment, has noted that this representation can be used effectively within generational activity. For example, Cedillo and Kieran (2003) carried out a study with 13- and 14-year-olds in which the pattern of input and output values in the given table motivated the search for a computer “program” (algebraic expression) that would reproduce the pattern. Using TI-92 Plus calculators, the students were able to test the algebraic expressions they were generating to see if their expression would yield the same output values as were displayed in the given table. The researchers pointed out the beneficial role played by the calculator in the development of not only students’ ability to formulate algebraic expressions but also their emerging notions of variable.

Graphical Representations and the Letter-Symbolic.

In a study that focused on the slope-intercept form of a linear function, $y = mx + b$ (Lobato, Ellis, & Muñoz, 2003), the researchers found that students in a first-year reform algebra class developed the notion of slope as a difference (i.e., “it goes up by”) rather than as a ratio—as was intended. By means of the theoretical construct of focusing phenomena, Lobato and colleagues analyzed the regularities in the ways in which the teacher, students, artifacts, and curricular materials acted together to direct student attention toward certain mathematical properties over others. Their analysis revealed the presence of four focusing phenomena: the ambiguity of the “goes up by” language, the use of ordered tables of values, the language used when entering into the graphing calculator both the x -values of a function and the scale of the x -axis, and a lack of coordination of differences in x - and y -values when $x \neq 0$. The researchers suggested that a directed focus of attention on the coordination of covarying quantities would be more likely to lead students to generalize slope as a ratio.

Moschkovich (1998) has pointed to students’ tendencies to ascribe a role to the x -intercept when learning about linear functions of the form $y = mx + b$. A subgroup of 18 students from a ninth-grade pilot first-year algebra course that included the use of the computer graphing utility, *Superplot*, participated in videotaped discussion sessions with a peer of their choice, using the same graphing utility. Although 13 of these students used the x -intercept at least once at the outset

of the group study for either b or m in an equation, or to describe lines as moving along the x -axis as a result of changing b in an equation, their thinking nevertheless evolved over the course of the study. Moschkovich has described the resources used by students as they refined their conceptions of the presumed role of the x -intercept, such as narrowing the problem contexts in which they applied this conception and making connections between conceptions.

Using the graphing facility of the spreadsheet as a potential learning environment, Ainley (1996) explored some of the ways in which this tool could be used to introduce 12-year-olds to the power of generalizing through algebraic notation. Ainley found that spreadsheet activity allowed students to appreciate the purpose and power of the symbolic notation by providing them with a means to generate more data for the problem at hand and thus work towards a solution. In a study by Kieran and Sfard (1999) that used the software, *Algebra Connections*, students became so skilled at graphing linear functions by focusing on the slope and y -intercept that they could mentally approximate the solutions to equations such as $7x + 4 = 5x + 8$ by visualizing the graphs of the two functions and their intersection point. However, in another study, Kieran (2001) found that, for certain nonlinear situations involving rational functions, students experienced considerable difficulty in generating both a graphical and a symbolic representation for the problem situation—the rational function proving to be much more problematic than the linear function.

Researchers have also looked at the ways in which graphical representations are used to give meaning to inequalities. Bazzini, Boero, and Garuti (2001b), who studied the feasibility of a functional approach in the teaching of inequalities to eighth-grade students, have pointed to the positive role that graphical representations can play in helping students to conceptualize the symbolic forms of inequalities, as well as the pitfalls involved in attempting to apply to the solving of inequalities some of the transformational techniques used with equations.

Arzarello and Robutti (2001) have described a study introducing ninth graders to algebra and involving the interpretation of body-motion graphs within an environment that included symbolic-graphing calculators connected to motion sensors (the calculator-based ranger—CBR). Initially students were encouraged to try various running patterns in order to create different graphs. The continuous nature of the CBR graphing allowed students to test conjectures in a direct manner, controllable by their own physical movement. Arzarello and Robutti observed that

students' cognitive activity passes through a complex evolution, which starts in their bodily experience (namely, running in the corridor), goes on with the evocation of the just lived experience through gestures and words, continues connecting it with the data representation, and culminates with the use of algebraic language to write down the relationships between the quantities involved in the experiment. (p. 39)

Connections Among Representations. Although research continues to show that various technological environments can help in developing students' algebraic thinking by building links among symbolic, tabular, and graphical representations (e.g., Friedlander & Tabach, 2001; Koedinger, Anderson, Hadley, & Mark, 1997; see also Balacheff & Kaput, 1996; Ruthven, 1996), such improvements are not automatic. The quality of the tasks (e.g., Hoyles, 2002), the teaching (e.g., Kubinová & Novotná, 2002), and the general learning environment (e.g., Kramarski, 2000) all play significant roles. Furthermore, as pointed out in the review of graphing calculator research carried out between 1985 and 1995 by Penglase and Arnold (1996), graphing technology clearly helped students improve their understanding of function and graphing concepts; however, several of the studies included in their review suggest that students continued to experience difficulties with seeing the relationship between algebraic and graphical representations of functions.

A partial explanation of this phenomenon has been found to lie in the time needed by students to build an understanding for symbolic notation. For example, Nathan, Stephens, Masarik, Alibali, and Koedinger (2002) investigated seventh and eighth graders' ($N = 90$) abilities to solve problems using tabular, graphical, verbal, and symbolic representations (without the aid of technology) and to translate among these representations before and after instruction that involved bridging activities built on students' invented strategies and representations. Before instruction, students experienced more success solving problems using a given representation than translating among representations, with success strongly influenced by the particular representational formats involved. Results suggested that there are significant gaps in students' abilities to comprehend and to produce representations, and that students may attain more fluency with tables and pointwise graphs before symbolic equations and verbal expressions. (See also Koedinger & Nathan, 2004, whose study is described in a later section.)

Yerushalmy (2000) has studied the long-term impact of a problem-based, functional approach to the teaching of algebra, one that was supported by intensive use of graphing technology, and has reported findings similar to those of Nathan and his colleagues.

By means of classroom observations and clinical interviews with students over their 3 years of following the course, Yerushalmy found that the representation of problem situations evolved as follows: from numbers as the only means of modeling, to intensive work with graphs and tables, to the use of more symbolic representations. Students also moved from analyzing patterns of numbers by watching the behavior of increments to more explicit, closed-form formulations of functional relationships involving pairs of numbers. However, Yerushalmy wondered why it took so long for the students to appreciate algebraic symbols, and why they preferred to use the situation as the source for their answers. From the findings of another study she carried out (Yerushalmy & Shternberg, 2001), and long-term observations by Chazan (2000), Yerushalmy (2000) has argued that "a curricular sequence of algebra based on functions might have to address more subtleties and transitions than initially expected . . . and that the complexity of helping students to value algebraic symbols may require more than bridging between representations" (p. 145). In another related study (Gilead & Yerushalmy, 2001), ninth graders who had been taught by a functional approach to the interpretation of equations (comparison of functions) encountered an obstacle when dealing with a certain type of situation not easily modeled by an equation without introducing another variable or parameter. The researchers were led to suggest that graphical tools may be useful as an exploratory support only for canonical problems (i.e., where y -intercept and slope are both given).

In a study of 1 year's duration, Schwarz and Hershkowitz (1999) investigated ninth graders' learning of the concept of mathematical function in an environment that included problem situations, graphing calculators, and multirepresentational software tools, and where students were encouraged to make their own decisions about which representations to use, when and how to link the representations (especially graphs and tables), and in which medium to work. The researchers compared and contrasted the characteristics of students' concept images of functions that emerged in this environment with the concept images described in previous work involving different environments (e.g., Leinhardt, Zaslavsky, & Stein, 1990; Markovits, Eylon, & Bruckheimer, 1986; Schwarz & Dreyfus, 1995). The questionnaire that was presented to students included questions primarily on the interpretation and generation of graphical representations, and to a limited extent on the relation of certain graphs to algebraic representations. The researchers found that the students of the current study often used the

prototypical linear and quadratic functions but did not consider them as exclusive (for persistence of linear models in students' reasoning, see DeBock, Verschaffel, & Janssens, 2002). They suggested further that the numerous manipulative experiences with function examples, which were afforded by the multirepresentational software, influenced positively students' understanding of the functions' attributes. They also claimed that the process of beginning each class activity by small-group work on open-ended situations, followed by full-class discussions that were orchestrated by the teacher, was conducive to students' engaging in a discourse that led to higher levels of mathematical reasoning and conceptual change (see also Yackel & Cobb, 1996).

Generational Activity Within the Context of Word Problems. The solving of word problems has traditionally been considered to involve two phases: the setting-up of an equation to represent the relationships inherent in the word problem, and the actual solving of the equation. Although past studies focused almost exclusively on the equation as the form of problem representation (e.g., Chaiklin, 1989), more recent work has included a much broader range of representations. A further move has been occurring of late that involves an extension of the types of problems being worked on to include the modeling of more open-ended, "real-world" situations. This growing domain of modeling within school mathematics, to the degree that it intersects with the learning of school algebra, is reserved for the later discussion of global/meta-level activity. This particular section is therefore divided into two parts: The first examines the research related to word-problem solving involving multiple representations, and the second, involving the equation representation.

Generating Multiple Representations in the Solving of Word Problems. In a study involving 6 intact classes of seventh and eighth graders at three junior high schools, half of whom were in a treatment group that participated in a representation-based unit on functions and the other half in a comparison group using conventional textbook lessons on algebra that included solving equations, using equations to solve word problems, and a unit on graphing ordered pairs, Brenner et al. (1997) investigated whether instruction on representing word problems by means of multiple representations improved students' abilities to represent and solve problems. The researchers found that instruction on problem-representation skills can work, that is, students learned to create various multiple representations and to use these to find problem solutions. The reform-based instruction was also

found to have the same beneficial effects on language-minority students as on the overall school population of the study. Related research by Koedinger et al. (1997), involving the multirepresentational, problem-solving computer environment, *Cognitive Tutor Algebra I* (Carnegie Learning, 1998), and by Nathan et al. (2002), suggests however that students tend to favor tables of values as problem-solving tools before they gain facility with graphs and equations. Multiple representations of word problems have also been used to introduce students to the form of inequalities. In analyzing some of the videotape data from the TIMSS-R eighth-grade algebra lessons (Stigler et al., 2003), Kieran (2004) noted that Japanese students attempted to solve the inequality situation first by using a variety of representations and methods to solve for the corresponding equality, and then by making adjustments to their "solution" in order to provide an answer for the inequality. (Interestingly, none of the students used a graphical representation.) The teacher then used these various "equality" methods, in particular the table-of-values representation, to help students acquire meaning for the form of the algebraic inequality.

Generating Equations in the Solving of Word Problems. Generating equations to represent the relationships found in typical word problems is well known to be an area of difficulty for algebra students. Nevertheless, word problem situations not only continue to be used as a means for infusing algebraic objects with meaning but have also received increased emphasis in reform programs as vehicles for introducing students to algebra. Research in this area continues to provide evidence of students' preferences for arithmetic reasoning and their difficulties with the use of equations to solve word problems (e.g., Bednarz & Janvier, 1996; Cortés, 1998; Swafford & Langrall, 2000). For example, Stacey and MacGregor (1999) found that, at every stage of the process of solving problems by algebra, students were deflected from the algebraic path by reverting to thinking grounded in arithmetic problem-solving methods. Another study dealing with the setting up of equations within the activity of word-problem solving has reported that, for sixth and seventh graders, reasoning and symbolizing appear to develop as independent capabilities (van Amerom, 2003). Although some students could write equations to represent problems, they did not use these equations to find the solution, preferring instead to use more informal methods. Some students did, however, use the equations they had generated as a basis for their reasoning about the problems and for finding their solutions. Similarly, Malara (1999) has observed students using numerical substitution within problem representations in order to arrive at solutions (see also

Johanning, 2004). Kutscher and Linchevski (1997) have also noted the beneficial effect of numerical instantiations as mediators in solving word problems. Although contextualized problems are often considered an appropriate avenue for introducing students to algebra (NCTM, 2000), De Bock, Verschaffel, Janssens, and Claes (2000), who explored the influence of authentic and realistic contexts and of self-made drawings on the illusion of linearity in length and area problems, found no beneficial effect of the authenticity factor, nor of the drawing activity, on pupils' performance. Furthermore, the researchers suggested that realistic problems might, in fact, steer pupils away from the underlying mathematical structure of a problem.

However, according to Nathan and Koedinger (2000a), word problems presented in verbal form are easier for students to solve than comparable questions presented in other formats, such as equations or "word-equations." These researchers presented a set of 12 problems to a group of 76 high school students in the following formats: Two were in story-problem format (e.g., "When Ted got home from his waiter job, he multiplied his hourly wage by the 6 hours he worked that day. Then he added the \$66 he made in tips and found he earned \$81.90. How much per hour did Ted make?"), two were in equation format (e.g., "Solve for x : $x \cdot 6 + 66 = 81.90$ "), and two were in word-equation format (e.g., "Starting with some number, if I multiply it by 6 and then add 66, I get 81.90. What did I start with?"). For one problem in each format, the start value was unknown, and for the other, the result was unknown. The equation format was found to be significantly less likely to be correctly solved than either the story-problem or word-equation formats, whereas algebra story problems and algebra word-equation problems were found to be of equal difficulty. In a replication study involving 171 students (Nathan & Koedinger, 2000a), solution success rates for symbolic equations were 25% less than for story problems and nearly 20% less than for word equations. In both of these studies, the start-unknown format was more difficult than the result-unknown problems. Students' difficulties with problems presented in equation format challenge, according to these researchers, the oft-cited view that story problems are inherently harder than symbolic ones. The many possible entry points, as well as the availability of a variety of solving approaches for story problems, are deemed to be factors accounting for the higher success rates with the verbal format.

In a follow-up study, Koedinger and Nathan (2004) further explored their earlier finding that students are more successful solving simple algebra sto-

ry problems than solving mathematically equivalent equations. They found evidence that this result is not simply a consequence of situated world knowledge facilitating problem-solving performance, but rather a consequence of student difficulties with comprehending the formal symbolic representation of quantitative relations. In fact, students translated story problems into the standard equation format only 5% of the time. According to the researchers, students even after a full year of algebra were particularly challenged by the demands of comprehending the letter symbolic form of the equation: "The language of symbolic algebra presents new demands that are not common in English or in the simpler symbolic arithmetic language of students' past experience" (p. 149). The researchers suggested that more attention be focused on equation-solving instruction; however, they emphasized that this be done after students have had experience with translating back and forth between English and algebra within story problem situations.

Transformational Activity: The Middle Grade and Lower Secondary Level Student

As noted earlier, the transformational activity of algebra includes collecting like terms, factoring, expanding, substituting one expression for another, adding and multiplying polynomial expressions, exponentiation with polynomials, solving equations and inequalities, simplifying expressions, substituting numerical values into expressions, working with equivalent expressions and equations, and so on. Past research in this area tended to focus on the nature of the manipulative processes used in expression simplification (e.g., Davis, Jockusch, & McKnight, 1978; Sleeman, 1984) and in equation solving (e.g., Bell, O'Brien, & Shiu, 1980; Kieran, 1988; Whitman, 1976). The field has recently experienced a change that includes attention to the theoretical foundations of students' manipulative work. In other words, algebraic transformations are not viewed simply as procedural, but also as theoretical. Thus, notions of equivalence figure more prominently in some of this later work. Another movement has been the downward shift of symbol-manipulation technology from the college to the secondary school level, even to the lower secondary level. To reflect the current research emphases, this section is divided into the following three parts: theoretical elements of transformational activity, namely equivalence and theoretical control; technical elements of transformational activity, namely the manipulation of expressions and equations; and the use of concrete manipulatives in transformational activity. The reader is reminded that the age range of the student in this section continues to

be approximately 11 to 15 years of age, that is, the sixth to the ninth grader. Comparable research conducted with the older student is reserved for a later section.

Theoretical Elements of Transformational Activity: Equivalence and Theoretical Control. According to Pimm (1995),

One resource of algebra is a rich plurality of symbolic forms; one core notion, that of equivalence. Equivalence and transformation are linked notions, indicating sameness perceived in difference for some purposes, or indifference with respect to others. The existence of multiple expressions 'for the same thing' (highlighting 'naming') can suggest the very possibility of transforming expressions directly to get from one to another. (p. 89)

Despite the importance of this core idea, only a few of the earlier studies explicitly addressed students' notions of equivalence (e.g., Greeno, 1982; Kieran, 1984a; Linchevski & Vinner, 1990; Steinberg, Sleeman, & Ktorza, 1990).

An example of some of the later research in this area is the work of Cerulli and Mariotti (2001; see also Nicaud, Delozanne, & Grugeon, 2002). Cerulli and Mariotti have described a teaching experiment on equivalence that involved ninth-grade classes and the algebra microworld, *l'Algebrista*, created by Cerulli (2004). Their aim was to develop in students a theoretical perspective on algebraic manipulation, by basing activity on the concept of equivalence relation, and by distinguishing *proof* from *verification*. The equivalence of two expressions is *proved* if one expression is transformed into the other using the axioms. Students were taught that, with literal expressions, the use of axioms is the only way to prove the equivalence of two expressions, whilst numerical *verification* (i.e., substituting the letters with numbers and computing the expressions) is the main way to show that two expressions are not equivalent. Students worked at changing the form of given numerical and literal expressions by selecting axiom buttons of the computer tool or by creating their own "theorem/transformation buttons."

Cerulli and Mariotti's (2001) analysis highlighted two important issues. First, they noted the carryover of the button-icons to students' paper-and-pencil work. Similar iconic traces of the computer environment have been noted in the written algebra work of the participants of Sutherland's (1993) spreadsheet study. According to Cerulli and Mariotti, such signs guide the evolution of meaning in students' algebraic thinking. The second issue concerns the theoretical aspect of students' activity. Cerulli and Mariotti were able to

provide evidence to support their claim that they had moved students away from a purely procedural interpretation of algebraic manipulation and had inculcated a theoretical perspective by means of the emphasis on axioms and theorems. The importance of theoretical control in algebraic transformational activity has also been articulated by, among others, Balacheff (2001) and Dettori et al. (2001). However, as noted by Demby (1997), students have a great deal of difficulty in identifying the properties they use when they transform algebraic expressions.

Another element of theoretical control that is considered basic to transformational activity is the knowledge that relates the algebraic domain with the arithmetical. For example, numerical substitution activity within expressions and equations can help students make connections between the arithmetical and the algebraic world (see, e.g., Warren & Pierce, 2004). Graham and Thomas (2000) conducted a teaching experiment with nearly two hundred 13- and 14-year-olds in which students used the letter stores of the graphing calculator as a model of a variable and so evaluated different expressions for a variety of inputs. In the case of equivalent expressions, students came to see that different expressions were being used to represent the same process. This activity also had an impact on students' views of expressions and variables, which suggests that a task that is transformational in nature can simultaneously be related to generational activity if it leads to an evolution of students' conceptions of the objects of algebra.

The kinds of errors that students can make in algebraic transformational activity (e.g., Carry et al., 1980; Lemoyne, Conne, & Brun, 1993; Matz, 1982; Sleeman, 1984) have suggested to some researchers (e.g., Kirshner, 1989) that the issue is not an absence of theoretical control but rather a misperception of form. In a study that extended Kirshner's earlier work on the visual syntax of algebra, Kirshner and Awtry (2004) investigated the role of visual salience⁴ in the initial learning of algebra for students in four intact classes of seventh graders (about 12 years old). They found that students did indeed engage with the visual characteristics of the symbol system in their initial learning of algebraic rules: The percentage-correct scores for recognition tasks were significantly higher for visually salient rules than for non-visually-salient rules. Furthermore, visually salient rules turned out to be significantly more difficult to recognize than non-visually-salient rules when presented using tree notation. Similarly, Hewitt (2003), in a study of 40 teachers

and a class of 11- to 12-year-olds, found that the inherent mathematical structure, and the visual impact of the notation itself, had an effect on the way in which equations were manipulated.

Technical Elements of Transformational Activity: Expressions, Equations, and Equation Solving. Among the few recent studies dealing with the transformation of algebraic expressions involving this age group of students are a study by Bazzini, Boero, and Garuti (2001a) and another by Ayers (2000) on bracket expansion errors. A greater amount of research has dealt with equations. For instance, Herscovics and Linchevski (1994) conducted a study involving equations of the type $4 + n - 2 + 5 = 11 + 3 + 5$, with various numerical operations on one or both sides of the equal sign. They found that about 50% of the beginning algebra students whom they tested failed to obtain the correct solution, many of them having equated the left side of the equation with only the first member of the right side—a result consistent with Kieran's (1984b) findings with the same kind of equation. In addition, Herscovics and Linchevski observed that several of the unsuccessful solvers incorrectly grouped the left-hand expression to $4 + n - 7$, an error that they have referred to as the *detachment of a term from the indicated operation*. They noted as well that, with the equation $115 - n + 9 = 61$, some students transformed it to $106 - n = 61$, having committed the error of *jumping off with the posterior operation* (see Linchevski & Herscovics, 1996, for additional errors). Similar findings have been reported in research involving eighth-grade Belgian students (Vlassis, 2001).

In a study that focused on the checking of equation solutions, Perrenet and Wolters (1994) found that eighth graders made the same sorts of errors in their checking that they had made while solving the equation, thereby not detecting that their solutions were erroneous. Analysis of students' erroneous checking behavior disclosed their profound structural misunderstandings, uncertainty, and wishful thinking. In another study related to checking solutions, Pawley (1999) noted that students were more successful in both their equation solving and their checking if they were taught how to check sometime later than they were taught how to translate from sentence to equation.

Up until quite recently, researchers have known very little about the ways in which students of this age range approach the solving of systems of equations and the manner in which they think about its underlying concepts. The recent research of Filloy, Rojano,

⁴ According to the researchers, "visually salient rules have a visual coherence that makes the left- and right-hand sides of the equations appear naturally related to one another" (p. 229). Although $(x^2)^2 = x^4$ is considered a visually salient rule, $x^2 - y^2 = (x - y)(x + y)$ is not.

and Solares (2003, 2004) has focused on the spontaneous approaches of 13- and 14-year-olds, who had already been introduced to the solving of one-unknown linear equations, to problems that could be solved by systems of equations. One of the aims of this research was to document the transition from one-unknown representations and manipulations to the representation and manipulation of one unknown given in terms of the other unknown. It was found that students seemed more inclined to make sense of comparison approaches than substitution methods; however, the researchers noted that manipulation difficulties contributed to making the substitution method less accessible. They also observed that the extension of the notion of transitivity of equality from the numeric to the algebraic domain, as well as the idea of substituting one expression with another, was not at all obvious to students.

Further evidence of students' difficulties with the substitution method for solving systems of equations was found by Drijvers (2003). Within a Computer Algebra System (CAS) environment, 14- and 15-year-olds were asked to solve parametric equations, for example, "Solve $ax + b = 5$ for x ." Students experienced difficulty in accepting the expression $(5 - b)/a$ as a solution. According to Drijvers, this required that they conceptualize an expression as an object (Sfard, 1991). Other tasks such as, "Consider the equations $y = a - x$ and $x^2 + y^2 = 10$. Make one equation from these in which y does not appear; you do not need to solve this new equation" (p. 260), which required substituting a variable in one equation by an expression drawn from the other equation, were equally problematic. In some cases, the use of graphical representations helped students to make sense of the equations with which they were working. The obstacles that students encountered in working with these equations led Drijvers to suggest that more attention needs to be paid to the role of paper-and-pencil work throughout CAS activity and to the importance of focused classroom discussions. Friedlander and Stein (2001) found that a significant number of students who could solve equations with the various computer tools at their disposal actually preferred to use paper and pencil.

Within process-object theories (e.g., Sfard, 1991), the transition from process to object tends to be viewed as a leap, a sudden shift in perspective, whereby a process becomes reified as an object. Students who conceive of an expression as a process are considered to be unable to make much sense of, for example, composition of functions (e.g., Ayers, Davis, Dubinsky, & Lewin, 1988), until such time as they make that sudden shift from expression

as process to expression as object. However, recent research by Landa (2003) suggests a more gradual learning process, in that composition of function may be preceded conceptually by composition of argument. Landa, who worked with 14- and 15-year-old students in a spreadsheet environment, found that, before students could make sense of, for example, $g(f(x))$, where $f(x) = 2x + 3$ and $g(x) = x^2$, they first needed to be able to create meaning for replacing the argument in a single function by an expression, as in $g(2x + 3)$. The spreadsheet environment, by means of its numerical calculation of functions with composed arguments, was seen to play a vital role in mediating students' passage toward composition of functions.

The Use of Concrete Manipulatives in Transformational Activity. Research has produced mixed results with respect to the use of concrete models and manipulatives in learning about equations and equation solving. Filloy and Rojano (1989) found that the balance scale and geometric models were not effective for helping students construct meaning for the process of operating on both sides of the equation to solve for the unknown. Boulton-Lewis et al. (1997) reported that the eighth graders of their study did not use the concrete manipulatives (cups, counters, and sticks) that were made available for solving linear equations and suggested that the concrete representations increased processing load. In contrast, advocates of such models (e.g., Brown, Eade, & Wilson, 1999; Linchevski & Williams, 1996; Radford & Grenier, 1996) have argued that the balance scale facilitates the understanding of the operation of eliminating the same term from both sides of an equation. In particular, Vlassis (2002) found that, for the two classes of eighth graders whom she observed over a period of 16 lessons, the balance model was an effective tool for conveying the principles of transformation. According to Vlassis, some errors linked to its usage appeared during the first lesson; however, these tended to disappear over time. Nevertheless, Vlassis emphasized that if students have not already extended their numerical range to encompass the negative integers, cancellation errors will be inevitable.

Global/Meta-Level Activity: The Middle Grade and Lower Secondary Level Student

The global/meta-level activities of algebra include problem solving, modeling, working with generalizable patterns, justifying and proving, making predictions and conjectures, studying change in functional situations, looking for relationships or structure, and so on—activities that could indeed be engaged in

without using any letter-symbolic algebra at all. Although algebraic activity involving problem solving, more particularly word-problem solving, has been integrated into the previous sections on generational and transformational activity, three types of global/meta-level activity for which a significant body of algebra research exists with this age group of learner have been reserved for this section: generalizing, proof and proving, and modeling.

Global/Meta-Level Activity Involving Generalizing.

Generalization activity within algebra has its roots in the use of algebraic notation as a tool for expressing proofs (e.g., Bell, 1976; Fischbein & Kedem, 1982; Mason & Pimm, 1984). The position that generalization is also a route to algebra was developed by Mason, Graham, Pimm, and Gower (1985; see also Mason, Graham, & Johnston-Wilder, 2005). In some of the pioneering research on the use of algebraic notation as a tool for expressing the general form of geometric and numerical patterns, and for justifying equivalent forms of these patterning relations, Lee (1987; Lee & Wheeler, 1987) found that few students use algebra or appreciate its role in justifying a general statement about numbers. Similar findings have been reported by MacGregor and Stacey (1993), who observed that an additional difficulty was related to students' inability to articulate clearly the structure of a pattern or relationship using ordinary language.

Healy and Hoyles (1999) have pointed out that the visual approaches generated in tasks involving generalization of matchstick patterns can provide strong support for the algebraic representation of sequences and the development of a conceptual framework for functions but emphasized the need to work hard to connect the observed number patterns to symbolic form. Warren (2000), who tested 12- to 15-year-olds ($N = 379$) on tasks related to generalization of patterns, reported that students found this kind of activity to be rather difficult. As a result of follow-up interviews held with some of the students, she noted that the generalization process draws on a variety of skills not traditionally associated with the algebraic domain. In addition, Ainley, Wilson, and Bills (2003), who compared the process of *generalization of the context* with *generalization of the calculation*, found that generalizing the context did not seem to be sufficient to support pupils in moving to a symbolic version of the rule. However, Radford (2000), who has studied the transition from the particular to the general, has argued that such processes take time.

In a study that focused on the role of tabular representations in generalization activity, Sasman, Olivier, and Linchevski (1999) presented eighth grad-

ers (14-year-olds) with tasks in which they varied the representation along several dimensions, namely the type of function, the nature of the numbers, the format of tables, and the structure of pictures. Results showed that varying these dimensions had little effect on students' thinking. According to Mason (1996), a scrutiny of established school practice involving generalization in algebra reveals that often, starting from geometric figures or numeric sequences, the emphasis is on the construction of tables of values from which a closed-form formula is extracted and checked with one or two examples. This approach in effect short-circuits all the richness of the process of generalization. In fact, some researchers (e.g., Moss, 2005) have suggested that tabular representations may actually impede students' recognizing the general relationships underlying patterns and representing them algebraically. Mason suggests several possible investigative approaches that can lead to students' construction of algebraic formulas, including visualization and manipulation of the figure on which the generalizing process is based.

From a study of eighth-grade students and the generalization of patterns, Radford (2003) reported that whereas rhythm and movement play a central role in presymbolic generalizations, symbolic algebraic generalizations require a disembodiment of the mathematical from the spatial-temporal. This researcher noted that the passage from a nonsymbolic to a symbolic algebraic expression of generality involves two ruptures, one with the sensual geometry of the patterns and the other with their numerical features such as rank. Furthermore, these ruptures are, according to Radford (2000),

regulated by a socially established mathematical practice where the teacher plays a central role; this role is that of immersing and initiating the students into the peculiarities of the signs and meanings in which the practice of algebra is grounded. (p. 258)

Global/Meta-Level Activity Involving Proof and Proving.

A great deal of the research on proof and proving with students of this age includes problems drawn from elementary number theory. For example, Edwards (1998), in her study of 14- and 15-year-olds who had not yet been exposed to instruction in proof in their algebra class, found that none of them used standard algebraic notation to justify or explain their true-false decisions about simple statements involving the combining of odd and even numbers. Miyakawa (2002), who reported on a study involving 14-year-old ninth graders working on a proof problem dealing with the sum of two even numbers, found that the difficulty in constructing this kind of mathematical proof

relates not only to algebraic competence and proof conception, but also to general mathematical knowledge. Dreyfus, Hershkowitz, and Schwarz (2001), who observed seventh graders collaboratively constructing an algebraic proof, reported that constructing this proof, which the students were able to do, involved first generalizing the process and then justifying it by means of an algebraic representation.

In contrast to research involving students without prior experience in constructing algebraic proofs, a study by Healy and Hoyles (2000) examined the conceptions of proof held by students who had followed a curriculum that, several years earlier, had mandated activity in formulating and testing conjectures, as well as in explaining and justifying conclusions. The researchers surveyed nearly 2500 high-attaining 14- and 15-year-olds. Consonant with the earlier findings of Lee and Wheeler (1987), Healy and Hoyles found that, when students were asked to pick the argument that they thought would get the best mark, the majority of students chose the ones based on algebraic form. However, students were unlikely to base their own arguments on similar algebraic constructions, feeling that these arguments neither communicated nor illuminated the mathematics involved. Students preferred to give some numerical examples, followed by a narrative explanation—even if they were aware of the limitations of such arguments. According to the researchers, students' responses were influenced mainly by their mathematical competence, but also by curricular factors, their views of proof, and their gender.

Global/Meta-Level Activity Involving Modeling. The majority of the algebra studies involving problem situations have, in general, been carried out with typical word-problem contexts. Nevertheless, a growing number of studies are now including modeling activities that go beyond the traditional problems found in most textbooks. In his plenary address at the closing session of the 12th ICMI Study on The Future of the Teaching and Learning of Algebra, Burkhardt remarked that “modeling as a bridge to algebra” (see below) seemed to turn on its head the idea of algebra as a tool for modeling. However, a number of studies have used real or pseudo-real situations to stimulate algebraic symbolization, for example, the modeling of running with the aid of motion sensors to motivate the development of a symbolic representation of this activity (Arzarello & Robutti, 2001). Stacey and Chick (2004) have commented on the issue of modeling as a bridge to algebra as follows:

Why, at this point in time, is there an emphasis on “modeling as a bridge to algebra” rather than on in-

struction designed to help students use their algebra in modeling the world? It may simply reflect the dominance of interest in beginning rather than advanced algebra. It may reflect a feeling of stalemate in improving students' abilities to set up equations for solving word problems, whether presented traditionally or otherwise, and tackling traditional applications. Alternatively, it may reflect an intention to keep algebra and the real world closer together throughout instruction, so that special lessons to close the gap are irrelevant. In any case, it is important to recall the centrality of Kieran's global/meta-level activity. Proving, modeling, and applying are activities that give purpose to learning algebra. (p. 18)

Since the mid-1990s, there has been a noticeable shift toward interest in modeling activity. However, the term *modeling* is interpreted quite broadly in the research literature to include everything from typical word problems to complex physical situations. For Janvier (1996), modeling, in the true sense of the word, is a rare activity in school algebra (see also Doerr & Tripp, 1999). In line with Burkhardt (1981) and de Lange (1987), Janvier has emphasized that modeling involves two phases: formulation and validation. The formulation phase of modeling comprises first the examining of a phenomenon or situation in order to establish some key relationships between the variables involved. After initial assumptions have been investigated, a more or less complex series of mathematical transformations or operations ultimately lead to a model, often expressed as a symbolic expression. The second phase, that of validation, requires testing the validity of the model by going back to the reality that it is supposed to represent. Similarly, according to Zbiek (1998),

a mathematical model is a combination of one or more mathematical entities and the relationships among them that are chosen to represent aspects of a real-world situation (Niss, 1988); mathematical modeling requires the application of mathematics to unstructured problems in real-life situations (Galbraith & Clatworthy, 1990). (p. 184)

Research in this area includes the work of Lesh, Hoover, Hole, Kelly, and Post (2000) on model-eliciting activities, and Lesh and Doerr (2003) on a theory of models and modeling, as well as, for example, cross-national approaches to research on modeling (Molyneux-Hodgson, Rojano, Sutherland, & Ursini, 1999). (See also the chapter in this volume on problem solving and modeling.) In their research involving both U.S. and Australian middle school students, Doerr and English (2003) have described how students' modeling shifted their focus of thought from finding

a solution to a particular problem to creating a system of relationships that is generalizable and reusable.

Dynamic physical models are often an integral part of modeling studies. For example, research with students in the 11- to 15-years-of-age group has included devices with gears of different sizes (Bartolini Bussi, 1995; Meira, 1998), a screen version of a baby duck swimming to catch up with its mother (Kaput & Roschelle, 1997), a simulation of a moving elevator (Noble, Nemirovsky, Wright, & Tierney, 2004), and motion artifacts such as an electronic walking frog (Schorr, 2003). Winch movement was the focus of a case study by Izsák (2003), who analyzed how a pair of eighth graders developed knowledge for representing with algebra the phenomena produced by this device. Izsák reported that, as these students solved problems about the winch, they learned to distinguish equations that are true for any value of the independent variable from equations that constrain the independent variable to a unique value—distinctions that have been shown to be elusive for many algebra students. Izsák's analysis also demonstrated that students have and can use criteria for evaluating algebraic representations, and that they can develop modeling knowledge by coordinating such criteria with knowledge for generating and using algebraic representations of physical situations. Hines (2001), using a similar winch device in her research, has described the learning trajectory of an eighth grader as he created and interpreted tables, equations, and graphs intended to represent functions originating from explorations with this dynamic physical model.

Among the studies on students' modeling that have involved technology environments is the research of Yerushalmy (1997a). She studied seventh-grade Israeli algebra beginners working in a software environment, *The Algebra Sketchbook* (Yerushalmy & Shternberg, 1993), that allowed them to manipulate visual objects and to use two lexical sets (verbal and graphic) to model temporal phenomena related to functions of a single variable. Yerushalmy's descriptions of the linguistic aspects of students' modeling efforts provide substantive examples of student activity in an environment that does not require the use of algebraic symbols (see also the work of Nemirovsky, 1996, on mathematical narratives). In a related study involving situations in two variables, Yerushalmy (1997b) found that students' ability to produce visual representations and to think about and discuss concepts and processes using graphical terminology was an important factor in their modeling success. Visual considerations were a constant feature of students' analyses, even while they were using tables and talking about variables, and before drawing graphs.

The Impact of Technology on Algebraic Activity With the Middle Grade and Lower Secondary Level Student

Since the publication of the last *Handbook* (Grouws, 1992), research on the learning of algebra has experienced a significant increase in the number of technology-related studies. Although this surge must be balanced with the more modest growth in actual classroom usage of technology (e.g., Mullis et al., 2000, report that of the 23 educational systems participating in the 1999 TIMSS, only 21% of eighth graders reported some degree of computer use in their mathematics classes), the results regarding the effects of technology use on the initial learning of algebra have been quite remarkable. The various tools that have been employed include—as has been pointed out in earlier sections—spreadsheets, graphing calculators, computer algebra systems, calculator-based ranglers, and specially designed software environments. The aim of this section is to synthesize from the results discussed in the preceding sections the role that technology can play in the learning of algebra with students of the middle grade and lower secondary levels.

The greatest impact of the use of technology in algebra learning has been on students' understanding of graphical representations and the way in which these representations allow a significant number of students to visualize in a graphical way the symbolic form of functions. This has thereby also improved students' understanding of certain functions. However, many students continue to experience difficulties with seeing the relationship between algebraic and graphical representations of function (Penglas & Arnold, 1996). The long-term experience of students in problem-based, functional environments supported by intensive use of graphing technology suggests that their representation of problem situations passes through phases, starting with the use of numbers as the only means of modeling, to working with graphs and tables, and finally to using more symbolic representations (e.g., Yerushalmy, 2000). Several studies, involving various technological environments and an assortment of global/meta-level activities to provide context for algebraic work, have in fact pointed out that supporting students in moving to a symbolic version of a rule, a pattern, or a problem situation takes time. Nevertheless, in environments and with activities specially designed for symbolic work, students can and do develop conceptually with respect to thinking with and acting on symbols (e.g., Cerulli, 2004).

The use of technology, and mainly the use of graphing technology, obviously introduces into school algebra other representations besides symbols

and equations. One of the research-tested advantages of this introduction of a variety of representations in technology-supported environments is that such approaches are enabling traditionally unsuccessful students to gain access to the problem-solving aspects of algebra. In fact, "approaches to teaching and learning that emphasize problem solving and exploration, and within which students actively construct and negotiate meaning for the mathematics they encounter, find in this [graphing calculator] technology a natural and mathematically powerful partner" (Penglas & Arnold, 1996, p. 85). More specifically, Huntley, Rasmussen, Villarubi, Sangtong, and Fey (2000), whose study included ninth graders as well as older students, found that students who were not strong in symbol-manipulation skills, when they were able to use context clues and had graphing calculators available, could outperform symbolically capable students in tasks requiring formulation and interpretation of situations. This suggests that weaker students too are deriving a deeper conceptual knowledge of algebraic objects within technological environments. However, research needs to reveal more about how this happens.

There is a perception among some educators that, when powerful technological tools appear to "do all the work," there might be little left to teach in an algebra class. However, as the research in this and a later section shows, the opposite is the case. Nevertheless, the nature of the teaching changes, as does that of the tasks that are presented to students. To take advantage of the potential of technology would seem to require not only tasks that are designed to push students beyond the limits of their current algebraic thinking and encourage further development of that thinking, but also approaches to teaching that facilitate such growth in students. However, a great deal more research is needed in the two areas of task design and teaching within technological environments. Further discussion of the role of the teacher in technology-supported algebra classrooms occurs in a later section.

Concluding Remarks on the Algebra Research Involving the Middle Grade and Lower Secondary Level Student

The middle grade and lower secondary level student has received the bulk of the attention of algebra researchers. This would seem appropriate in that algebra is usually introduced with this age range of student. However, a comparison of the earlier studies of the 1980s involving the middle grade and lower secondary level student with the more current work that has been discussed in this particular section of the chapter discloses the presence of a shift in the

nature of the research. Earlier work tended to focus on student difficulties in making the transition from arithmetic to algebra, and on the nature of the algebraic concepts and procedures developed and used by students during their initial attempts at algebra. Later work, while building upon and extending some of this earlier research, has evolved in new directions. Three areas in particular were noted.

One concerns the sources of algebra meaning making. Meaning based on the letter-symbolic form and on the problem situation/context has been extended to include the meaning derived from the use of graphical and tabular representations, as well as the meaning that emerges within dynamic activity. More research, however, needs to be done in this latter area so as to provide further insights into how such meaning develops and becomes disembodied. Another more recent research focus that was signaled in this section, and which will be elaborated more fully in the next section dealing with the upper secondary and college level student, is the role of theorizing within manipulative activity. Although properties and axioms were always an implicit part of symbolic manipulation, they were rarely the theme of algebra research in the 1980s. In contrast, theorizing is now considered to belong not only to the domain of generational activity in algebra, but to transformational activity as well. Finally, the third new direction in algebra learning research is one that was just discussed in the prior section, that of the use of technology. In particular, spreadsheets and graphing technology have played significant roles in this technologically oriented movement in recent algebra research dealing with the middle grade and lower secondary level student.

This section closes with a few additional remarks regarding questions in need of further research. Earlier, it was noted that Radford (2000) had argued that

One of the didactic questions with which to deal is . . . that of the understanding of how . . . meanings are progressively transformed by the students up to the point to attain the standards of the complex algebraic meanings of contemporary school mathematics. (p. 240)

With the broad scope of generational and global/meta-level activity in current research, including functional approaches with their multirepresentational focus, Radford's question presents a challenge for researchers. School algebra has been taking on so many different forms that one might be led to ask: Are there still commonly accepted "standards of the complex algebraic meanings of contemporary school mathematics"? Although the view of that which con-

stitutes algebraic activity has been considerably enlarged, the answer to this question is, I believe, still *yes*. And, because algebraic symbolism remains foundational to school algebra, it is strongly recommended that further research related to the learning of the letter-symbolic form be carried out. Areas where the field could benefit from additional study include, for example, the question as to how students can be assisted in (a) becoming aware of structure in patterns and in using symbols to express these patterns, (b) seeing relations between graphical representations and the corresponding letter-symbolic forms, and (c) making connections between their verbal problem-solving activity and the generating of equations. With the increased use of technology in algebra learning, another area where further research is needed is in the interaction between paper-and-pencil and machine work with tasks that have been specially designed to take advantage of this interaction. Further discussion of this issue is offered in the next section of the chapter dealing with the upper secondary and college level student.

The Upper Secondary and College Level Student

This section of the chapter examines the algebra research that has been carried out with the upper secondary and college level student, from the 10th grade onwards (about 15 years of age and up). Investigations that have involved students in teacher training programs at the college level are, however, reserved for the later section on teachers. As with the previous section devoted to the middle grade and lower secondary level student, the first subsection below deals with generational activity, whereas the second treats transformational activity and the third, global/meta-level activity. A fourth section discusses research on the effects of technology use, followed by a fifth that offers concluding remarks. Note that the research studies carried out on algebra learning with the upper secondary and college level student are fewer in number than those conducted with the middle grade and lower secondary level student.

Generational Activity: The Upper Secondary and College Level Student

Creating the objects of algebra, and building meaning for them, is at the core of generational activity, not only for the middle grade and lower secondary level student but also for the upper secondary and college level student. This section examines the research bearing on generational activity accord-

ing to whether the object of study is treated from (a) a primarily letter-symbolic focus or (b) a multiple-representation perspective. The latter of these two sources of meaning is the theme of a larger number of studies than the former, reflecting both the increased attention to functional approaches across reform curricula as well as the traditional emphasis on functions and their various representations with the older student.

Generational Activity With a Primary Focus on the Letter-Symbolic Form. The studies that have focused on generational issues related to the letter-symbolic form among upper secondary and college level students have included research on form and structure, as well as on the kind of meaning building that occurs for the algebraic object, parameter. (A review of the work related to the learning of abstract algebra and of linear algebra is beyond the scope of this chapter.)

Form and structure. Algebra was at one time “all about form,” according to Goldenberg (2003, p. 11). Two examples of form-related questions from a test set for 3rd-year Italian secondary students (16-year-olds) are obtained from a study by Menghini (1994):

Find the value of the following expression:

$$z^4/c^2(b-a) + z^4/c^2(a-b) = \dots$$

For which values of x is the following inequality true?

$$x^2 + x < x^2 + x + 1.$$

Menghini reported that, for the first example, most students carried out very detailed calculations, and for the second, the best answers involved a comparison of the two parabolas represented by the two parts of the inequality. What seemed to be missing from the responses to these tasks, according to the researcher, was the ability to recognize more general laws and forms, or to be able to achieve an abstract understanding independent of the specific numerical context. Similarly, in their study of 11th graders, Hoch and Dreyfus (2004) found that very few students used structure sense and those who did were inconsistent. The more advanced students used it more often than the intermediate ones. The presence of brackets seemed to help students see structure, focusing their attention on like terms and breaking up the long string of symbols.

On parameters. While research with the older student has attended to the issue of shifts in the meanings of literal symbols (Bills, 2001)—including those related to the concept of variable (e.g., Ursini & Trigueros, 1997)—particular interest has been fo-

cused on parameters and their role in algebraic expressions and equations. According to Bloedy-Vinner (1994, 2001), understanding algebraic language related to parameters means (a) understanding from the context which letters are used as parameters, (b) understanding the role of parameters as opposed to the role of unknowns or variables, and (c) understanding the fact that the meaning of a letter as parameter might change throughout the process of solving a problem. In understanding this difference in roles, implicit quantifiers are involved. From a questionnaire designed to test students' understanding of implicit quantifier structures, administered to 82 Israeli students who had taken matriculation exams in mathematics, she found that most of the questions yielded a very low success rate (ranging from 3% to 69%). Furinghetti and Paola (1994), in their study of 199 Italian students, ages 16 and 17, found that only 20 of these students could adequately describe differences among parameters, unknowns, and variables.

Generational Activity with Multiple Representations.

The research interest in multiple representations within functional perspectives is expressed in themes related to the understanding of functions and the interplay between symbolic and graphical representations, with special attention paid to the role of visualization.

Functions and Their Meaning. A function can be regarded as a set of ordered pairs, a correspondence, a graph, a dependent variable, a formula, an action, a process, or an object (Selden & Selden, 1992). When introduced to the formal ordered-pair definition of *function*, students have been found to rely on process-oriented interpretations (Sfard, 1991, 1992) or on their own intuitive concept images (Vinner, 1992; Vinner & Dreyfus, 1989). A significant number of studies have focused on efforts to support students in moving from a process-oriented to an object-oriented view of functions by reifying their process views (see, e.g., Dubinsky & Harel, 1992a). Student misconceptions that have been documented in this area include, for example, identifying functions with just one representation (Vinner, 1992), neglecting domain and range considerations (Sfard, 1992), and relying on the limited information provided by computer screen windows (Goldenberg, 1988). Building on some of the earlier research, more recent studies have explored, for example, the nature of covariational reasoning within the context of exponential functions (e.g., Confrey & Smith, 1994, 1995), the elaboration of a property-oriented view of function that is based on visual aspects of functional growth (Slavit, 1997), the notion of chronicle as a source of error (Janvier, 1998), and

the role of concept maps in assessing conceptual understanding of functions (e.g., Williams, 1998).

Technological environments have been widely used in research on students' conceptual understanding of functions, for example, ISETL (Dubinsky & Harel, 1992b), *DynaGraphs* (Goldenberg, Lewis, & O'Keefe, 1992), and *Visualizing Algebra* (Schwartz & Yerushalmy, 1992). Graphing and graphical representations have been a major theme of the research (e.g., Leinhardt et al., 1990; Romberg et al., 1993), with a particular focus on the establishment of connections between the symbolic and graphical representation of function, often with the use of graphing calculators. This latter research is discussed next.

The Interplay Between Symbolic and Graphical Representations and the Role of Visualization. The role of visualization as a representational and interpretational tool has been an ongoing area of interest among algebra researchers (e.g., Arcavi, 2003; Dreyfus, 1991). Earlier work produced evidence that even advanced students demonstrate a reluctance to use visual representations (e.g., Eisenberg & Dreyfus, 1991), preferring letter-symbolic over graphical representations, even when the former are more complicated. More recently, drawing on findings by Schoenfeld, Smith, and Arcavi (1993) and Moschkovich, Schoenfeld, and Arcavi (1993), Knuth (2000) examined 9th- to 12th-grade students' understanding of the concept that the coordinates of any point on a line will satisfy the equation of the line, within the context of problems that require the use of this knowledge. Knuth found an overwhelming reliance on letter-symbolic representations, even on tasks for which a graphical representation seemed more appropriate. The findings indicated to Knuth that for familiar routine problems many students have mastered the connections between the letter-symbolic and graphical representations; however, such mastery appeared to be superficial at best. Similar results were found by Slavit (1998) in a study in which graphing calculators were available. According to Slavit, the students had difficulty in coming to view algebra from a multirepresentational perspective and thus tended to favor symbolic modes of work, using the graphing calculator only on graphing tasks. These findings contrast with those reported for younger students, where the use of symbolic modes of work lagged behind those of graphical representations as problem-solving tools. Visualization in advanced problem solving by both college students (the novices of the study) and mathematics professors (the experts) was the focus of a study reported by Stylianou and Silver (2004). The two groups judged visual representations likely to be useful with different sets of problems; furthermore, when it came to actual usage,

the experts constructed visual representations much more often than did the novices.

Dynamic, visual representations were a central feature of a study by Zbiek and Heid (2001), who investigated students' understanding of parameters. They have described an environment involving dynamic control of a slidergraph and provided an example involving a family of functions whose general member is represented by $f(x) = a/(1 + be^{cx}) + d$, where a , b , c , and d are real numbers. At the same time that the slidergraph-control moves along the screen, allowing a selected parameter to take on different values, the graphical representation responds to and reflects the various values that the functional expression takes on. Zbiek and Heid pointed out that the ease with which the slidergraph, unlike static graphing utilities, allows students to vary a single parameter quickly and consistently led to surprising observations and was, in fact, the key to launching student reasoning. This study is an example of one of a growing number related to the use of dynamic environments in algebra in which bodily motion plays a role in meaning building for the functional objects of algebra (see also Kieran & Yerushalmy, 2004).

Zaslavsky, Sela, and Leron (2002) found evidence of much confusion regarding the connection between algebraic and geometric aspects of slope, scale, and angle. Participants, who included 11th-grade students as well as teachers, mathematics educators, and mathematicians, responded to a simple but nonstandard task concerning the behavior of slope under a nonhomogeneous change of scale. Responses revealed two main approaches—analytic and visual—as well as combinations of the two. The researchers recommended that instruction on slope distinguish between the erroneous conception of *visual slope*—the slope of a line (for which the angle is a relevant feature)—and the *analytic slope*—the rate of change of a function. Similar approaches were noted in the Lobato et al. (2003) study involving younger students.

Other related studies have investigated students' interpretations of linear and quadratic graphs as displayed on the graphing screen. Although some studies (e.g., Cavanagh & Mitchelmore, 2000) have suggested that older students continue to experience some of the same difficulties that have been found with younger students, others (e.g., Pierce & Stacey, 2001a) have reported that the use of symbolic, graphical, and numerical representations helps to expand students' previous understandings by allowing them to link visual images to symbolic statements. In particular, Pierce and Stacey observed that college-level students operating in a CAS envi-

ronment experienced little difficulty in moving between graphical and symbolic representations and in fact seemed to prefer these two modes to the movement between tabular and symbolic representations. Furthermore, studies that have investigated the impact of long-term use of graphing calculators on students' conceptual understanding of function and its representations have reported positive results (e.g., Streun, Harskamp, & Suhre, 2000). The time spent in working with multiple representations has been found to be crucial to their learning.

Transformational Activity: The Upper Secondary and College Level Student

Research that has focused on the transformational activity of algebra with the upper secondary and college level student has investigated the following areas: notions of equivalence and meaning building for equivalence transformations, the solving of equations and inequalities, the factoring of expressions, and the integration of graphical and symbolic representations in transformational work. Before examining this research, a few theoretical remarks are offered to provide a context for some of the more recent work that has been carried out in this area.

Theoretical developments by French researchers have led to a rethinking of the nature of technical activity in mathematics. Combining elements from both anthropological theory (Chevallard, 1999) and cognitive ergonomics (Vérillon & Rabardel, 1995), Artigue (2002b) and Lagrange (2000) have elaborated the notion that, as students develop techniques in response to certain tasks, they engage in a process of theory building. In their theoretical approach, which they have applied to research involving Computer Algebra Systems (CAS), activity with technological tools is considered to promote both conceptual and technical growth in mathematics, as long as the technical aspects are not ignored. An increasing number of research studies on the transformational activity of algebra reflect this newly developed perspective.

Related to this perspective, especially with respect to the emergence of theoretical thinking within the context of CAS use, is the work done by Pierce and Stacey (2001b) on the development of algebraic insight. Algebraic insight, which is considered to consist of both algebraic expectation and the ability to link representations, involves a focus on basic properties, structure, key features, and links.

Transformational Activity Related to Notions of Equivalence. Among the research dealing with students' notions of equivalence and equality has been a study carried out by Sackur et al. (1997). These re-

searchers have described how 10th graders' thinking evolved with respect to these notions within a series of "make false"⁵ interviews. The researchers, who used Frege's sense-denotation distinction as their analytical framework, pointed out that,

When we say that a student must know what the expressions denote, we mean that the student must know the following: (a) that an expression like $y(2x + y)$ has a numerical value, (b) that this value depends on the values of x and y , and (c) that this value is not changed by the algebraic transformation rules to which an expression is subjected, for example, that which transforms $y(2x + y)$ into $2xy + y^2$. (p. 47)⁶

Their analysis led them to conclude that, unless students come to realize that algebra is an arena of sense-making and that they can arrive at rules that will permit them to obtain the same results as their teacher or classmates, they will never be able to control their algebraic work.

Research evidence exists that, without interventions of the type conducted by Sackur et al., students do not spontaneously develop an algebraic notion of equivalence, and that the obstacles associated with this notion cut across the age range. For example, Kieran (1984a) found that 8th graders showed little awareness that the equation-solving transformations they were able to execute proficiently preserved solutions. In a study using a larger sample of 8th- and 9th-grade students, and employing tasks similar to those used by Kieran (1984a), Steinberg et al. (1990) noted that although most students knew how to use transformations to solve simple linear equations, many did not spontaneously relate this knowledge to the production of equivalent expressions. Ball, Pierce, and Stacey (2003) developed an instrument to assess students' abilities to quickly recognize equivalent algebraic forms—the "Algebraic Expectation Quiz." On the basis of the performance of 50 students on this test, administered before and after students' progression from 11th to 12th grade and during which time they participated in algebra instruction involving the use of a computer algebra system, the researchers reported that "recognizing equivalence, even in simple cases, is a significant obstacle for students" (p. 15). Pomerantsev and

Korosteleva (2003) presented compelling evidence that difficulties related to understanding algebraic equivalence can extend well beyond the postsecondary level. Their study involved the use of a diagnostic test administered to a large ($N = 416$) sample of students enrolled in different stages of their K–8 teacher preparation program at a major American university. Test items were designed to assess students' abilities to discern and use structural aspects of algebraic expressions; results of this research revealed serious difficulties in doing so, cutting across students in all groups contained in the sample.

The growing increase in the use of CAS in secondary school algebra classes has been accompanied by research interest in the role that this technology might play in the development of students' conceptions of equivalence. Particular attention has been paid to the nature of the tasks that might provoke theoretical growth within the context of transformational activity. For example, Saldanha and Kieran (2005; also Kieran & Saldanha, 2005) reported on a study with 10th graders that involved a combination of CAS- and paper-and-pencil-based tasks on equivalence of expressions. Results suggested that the development of students' theoretical notions of equivalence was guided by the techniques that the tasks invited, and that the most productive learning occurred after the CAS techniques produced an output that conflicted with the students' expectations.

Transformational Activity Related to Equations and Inequalities. While the solving of linear equations has received a great deal of research attention, that of quadratic equations has not. One of the few studies was carried out by Vaiyavutjamai, Ellerton, and Clements (2005). Nearly 500 students from Year 9 classes in Thailand, Year 10 classes in Brunei Darussalam, and 2nd-year university students in the United States attempted to solve the same quadratic equations, all of the form $x^2 = K (K > 0)$ and $(x - a)(x - b) = 0$ (where a and b are any real numbers). All students had already learned to solve such equations before participating in the study. The responses to the second type of equation, in particular, suggested the presence of serious gaps in the theoretical thinking underpinning students' work when solving such equations. For example, in solving $(x - 3)(x - 5) = 0$, several students

⁵ A "make false" interview is one that is aimed at having students confront the limits of their own thinking and in which a student is asked to produce an equality that is always false. For example, the researchers presented pupils with tasks such as "Write a false equality for $7x/(7 + x) =$ " and then posed questions related to the responses provided by the pupils.

⁶ Similarly, according to Arzarello, Bazzini, and Chiappini (1994), "the 'denotation' of a symbolic expression in algebra refers to the number set that is represented by the expression; it is determined by the symbolic expression and by the universe in which the expression is considered (for example the equation $x^2 + 1 = 0$ denotes the empty set when it is considered in $\mathbb{R}$ and the set $\{+i, -i\}$ when considered in $\mathbb{C}$)" (p. 42).

who correctly solved the equation checked their solutions by substituting $x = 3$ into $(x - 3)$ and $x = 5$ into $(x - 5)$ and concluded that because $0 \times 0 = 0$ their solutions were correct. Related difficulties were encountered by the U.S. students who were asked to respond True or False to the following statement: *This equation $(x - 3)(x - 5) = 0$ is equivalent to $x^2 - 8x + 15 = 0$, which is a quadratic equation with two solutions. Thus, with $(x - 3)(x - 5) = 0$, the x in the first brackets always equals 3, and the x in the second brackets always equals 5.* Fifty-five percent of respondents answered that this was indeed True. To another question, many were unsure whether the x in the " x^2 term" represented the same variable as the x in the " x term" of the equation $x^2 - 8x + 15 = 0$. The finding that most of the students in this study were confused about how the solutions of a quadratic equation related to the equation itself led these researchers to suggest that, if quadratic equations are to remain an important component of middle- and upper-secondary mathematics curricula, then research is needed to guide teachers about how students think about quadratic equations.

Tsamir and Bazzini's (2002) study of 16- and 17-year-old Israeli and Italian students' solutions to algebraic inequalities found that students of both countries intuitively used the balance model to do the same thing to both sides of an inequality and continued to draw on equation-related analogies when dividing both sides by a not necessarily positive value. This reliance on the balance model can be related back to results described in an earlier section of the chapter where the introduction of this model with younger students was found by several researchers to be beneficial, even if some suggested caution. In related research with Israeli students, Tsamir, Almog, and Tirosh (1998) reported that the following methods were used by high school math majors for solving equations and inequalities: algebraic manipulations, drawing a graph, and using the number line. When graphical representations (without technology) were used to solve rational and quadratic inequalities, the solutions were usually correct; however, students often failed to reject the excluded values.

Goodson-Espy (1998) presented 13 college students with a set of eight word problems that involved linear inequalities. Even those participants who were known to be able to solve basic algebraic equations and inequalities when presented in the symbolic format solved the problems using arithmetic forms of reasoning. Similar observations were noted earlier in the research involving younger students who were found to make use of arithmetic forms of reasoning in solving word problems. In all but one case, the participants in the Goodson-Espy study who used algebra to

solve the tasks used equations rather than inequalities. This too echoes the findings reported with younger students who, when introduced to problems involving inequalities, chose to think about and solve for the corresponding equality situation and then make adjustments at the end to produce an answer for the inequality.

Transformational Activity Related to Factoring Expressions. With the emergence of CAS in secondary school mathematics classes, the transformational activity of factoring has been receiving research attention (e.g., Artigue, 2002a; Artigue, Defouad, Dupier, Juge, & Lagrange, 1998). For example, Lagrange (2003) has described the activity of classes of 16- to 18-year-olds (reported initially in Mounier & Aldon, 1996) attempting to find and prove general factorization of $x^n - 1$. The students who were in CAS-equipped classes tried to factor $x^n - 1$ for various values of n on their CAS; the machine produced as output complete factorizations of these expressions. This form of output tended to thwart the mathematical aim of the activity as designed by the teachers, which was to have the students arrive at an awareness of general regularities in the factored forms. A related study by Kieran and Drijvers (2006; see also Kieran & Saldanha, 2007) took advantage of the fact that the CAS produced complete factorizations of $x^n - 1$ in order to provoke a confrontation with the existing, but limited, theoretical thinking of the 10th-grade participants with respect to factoring. This confrontation was found to be very productive for most students, but especially so for those who, upon realizing that they could not generate the same factors as had the CAS, insisted on finding out how to do so either from the teacher or from their group-mates during the working part of the activity, or later during the plenary classroom discussion. These CAS-based encounters resulted in the evolution of not only students' paper-and-pencil factoring techniques but also their theoretical views of the structure of these expressions (e.g., seeing that $x^6 - 1$ could be viewed as a difference of squares, or as a difference of cubes, or as an example of the general rule that they had earlier generated, $x^6 - 1 = (x - 1)(x^5 + x^4 + \dots + x + 1)$).

Another study on factoring is that reported by Nguyen-Xuan, Nicaud, Bastide, and Sander (2002). Two classes of 10th-grade novices (but not beginners) learned factoring techniques in a computer-based tutoring environment for symbol manipulation that provided informative error messages. One class received instruction by means of observing 14 pairs of worked-out examples, each observation being followed by activity on a related example. The second class did not observe any worked-out examples; their activity

consisted only of factoring the 14 pairs that the other class had observed being factored. Both classes had from 4 to 6 hour-long sessions with the given activities. The researchers found that the second class learned more. This finding supported their hypothesis that, although observation may be an effective learning approach for rank beginners, it is less effective than learning by doing for students who are trying to become more expert. The researchers also noted that the time taken to move from novice to expert status was significant—a finding already reported in several other studies.

Transformational Activity Involving the Integration of Graphical and Symbolic Work. Ruthven (1990) conducted a study involving advanced-level mathematics students who had access to graphing calculators while working on double-angle trigonometric identity tasks. The most productive work started, according to the researcher, when students began to compare graphs of the different possibilities and used their manipulative skills to establish which of their different algebraic formulations were equivalent. This finding suggests an interesting approach for using the graphical to motivate the manipulation of the symbolic. In another study, Baker, Hemenway, and Trigueros (2001) noted that college students' work with transformations of functions was assisted by graphical representations when these activities were accompanied by written descriptions of what they were seeing on the screen of the graphing calculator.

The integration of graphical and symbolic representations has also been noted in the transformational work reported in studies on (a) finding the roots of quadratic functions (Mourão, 2002), (b) translating quadratic functions (Zazkis, Liljedahl, & Gadowsky, 2003), (c) working with cubic functions (Gómez & Carulla, 2001), (d) solving linear and quadratic equations and inequalities (Asp, Dowsey, & Stacey, 1993; Tsamir et al., 1998), (e) solving problems involving translations of quadratic graphs and parametric curves within a graphing-calculator environment (Drijvers & Doorman, 1996), and (f) the Newton-Raphson method for approximating the zeros of functions in a CAS environment (Hong & Thomas, 2002). Although the research involving CAS technology has focused on either the exclusive use of the letter-symbolic representation or the interplay between graphical and symbolic representations, some CAS research reports have included only graphing tasks, with no reference to the letter-symbolic (Zbiek, 2003; see also Thomas, Monaghan, & Pierce, 2004).

Traditionally, the study of transformations of functions has belonged to the domain of symbolic representation. However, in some technological environments (e.g., Schwartz & Yerushalmy, 1992, 1996), one graph can be transformed into another through direct actions on the graph. Borba (1993), who worked with 16-year-olds in the *Function Probe* environment developed by Confrey (1991), reported that the software environment influenced the mathematics that was studied and produced. He also argued that the nature of the tasks determined whether the students thought about transformations of functions in a static or dynamic way. Kieran and Yerushalmy (2004) have described other environments permitting dynamic transformation of graphical representations, accompanied by simultaneous change in the corresponding symbolic representation; however, they also noted that research on the impact of directly controlling a change in one representation to effect changes in another remains relatively undeveloped.

Global/Meta-Level Activity: The Upper Secondary and College Level Student

As with the middle grade and lower secondary level student, global/meta-level activity with the older student has included work in problem solving and in proving. However, in several of these studies, the two themes have been integrated and are therefore treated in the same section below. Some studies have also been carried out with the older, more mathematically advanced student in the global/meta-level activity of modeling; this research is briefly presented in the second section that follows.

Global/Meta-Level Activity Involving Problem Solving. Much of the research involving problem situations with the older student, as was the case with the younger student, has included technological tools. An example of such research is a study by Hershkowitz and Kieran (2001), who reported on classes of 16-year-olds exploring a situation involving three families of rectangles that increased in size according to a linear, a quadratic, or an exponential relation. In attempting to respond to the question as to which family of rectangles would eventually surpass the others in terms of area, students were found to go from entering lists to graphical representations by means of the regression tool of the graphing calculator without ever seeing or having to examine the algebraic representation of the situation. All of their problem-solving work involved attention to the data in the lists and the resulting graphs, which led the researchers to question whether the use of such tools within problem-solving situations

can, in fact, lead students away from the use of letter-symbolic representations.

Much of the research on the global/meta-level activity of problem solving with the older student has tended to integrate number-theoretic work and proving. For example, Arzarello et al. (1994) observed a link between expressing the relations of number-theoretic problems in suitable algebraic code and success in determining the solutions to such problems. According to the researchers, appropriate naming is linked to anticipatory aspects of solving and allows students to orient the solution process toward the aims of the problem. They also pointed out that some students who could express the elements of a problem by using natural language were unable to express them using the algebraic language. In contrast, Douek (1999) found that nonstandard representations of number-theoretic problems did not prevent students from producing valid proofs and problem solutions.

However, this may in fact depend on the methods used by students to generate their proofs. Alcock and Weber (2005) reported that students used both referential (involving some numerical instantiation) and syntactic approaches (involving symbolic manipulation) and that students who used referential approaches may have had a more meaningful understanding of their proofs but may not have been able to complete them. In another study with 10th graders who had been given the task of proving their conjecture that $x + 1$ is always a factor of $x^n - 1$ for even values of n , Kieran and Drijvers (2006) found that most students used in their proofs a combination of syntactic and referential approaches, and that the most successful of the proofs were generic in nature.

Global/Meta-Level Activity Involving Modeling. Dynamic physical models have been an integral part of modeling studies with the upper secondary and college level student. Schnepf and Nemirovsky (2001) reported on a study that involved graphing software and hardware that linked a computer to miniature cars on parallel linear tracks and to a miniature stationary bike. The software allowed the system to work in two ways: either by manipulating the cars (or bike) and then modeling this movement by means of graphical representation, or by constructing a graph on the computer that then moved the mechanical device according to the graphical specifications. The latter, which is the opposite of a modeling perspective, was found effective when combined with the former in the development of conceptions of movement, and its representation, among the 12th-grade calculus students who participated in the study. Related work

has been carried out by Rasmussen and Nemirovsky (2003) in an environment involving the physical tool, Water Wheel, with more advanced students.

The Impact of Technology on Algebraic Activity With the Upper Secondary and College Level Student

Just as with the younger algebra student, the presence of technology in the studies involving the upper secondary and college level student has been noteworthy. The main types of technology that have been employed in these studies with older students are the graphing calculator (without symbol-manipulation capabilities) and computer algebra systems, which also include graphing capabilities. Some of the impact of these tools has already been discussed in the individual research studies that have been presented. However, other technology-related studies that cut across the various types of algebraic activity have been reserved for this section. Some of these are large-scale studies, involving meta-analyses or comparison groups; others have focused on the ways in which roles in the algebra classroom change when technology is introduced.

Large-Scale Studies on the Effects of Technology Use.

The studies that are presented in this section are categorized as to whether they deal with calculators in general, graphing calculators, or computer algebra systems. The largest set of studies is that having to do with the impact of graphing calculator usage.

Calculators. The first study to be included herein is a meta-analysis that investigated the effects of calculators (basic, scientific, and graphing) on students' achievement and attitude levels by examining 54 studies, 26 of which dealt with the high school level (Ellington, 2003). It was found that, when calculators were included in instruction but not testing, the ability to select the appropriate problem-solving strategies improved for the participating students. Under these conditions, there were no changes in students' computational and conceptual skills. When calculators were part of *both* testing and instruction, the strategic skills, computational and conceptual skills, and problem-solving skills improved for participating students. Under these conditions, there were no changes in students' ability to select the appropriate problem-solving strategies. Students who used calculators while learning mathematics reported more positive attitudes toward mathematics than their non-calculator-using counterparts on surveys taken at the end of the calculator treatment. Ellington also reported that students received the most benefit when calculators had a pedagogical role in the classroom and were not available just for drill and practice or checking work. However,

she also noted that in only 11% of the studies analyzed had special curriculum materials been designed for calculator use and that this is thus an area in which more research needs to be conducted.

Graphing Calculators. The next two studies involved primarily the use of the graphing calculator within the computer-intensive algebra (CIA) curriculum (Fey et al., 1991). The first is a study by O'Callaghan (1998) that examined the effects of a one-semester computer-intensive algebra and traditional algebra curricula on college students' understanding of the function concept. He found that the CIA students achieved a better overall understanding of functions and were better at the components of modeling, interpreting, and translating. Further, the CIA students showed significant improvements in their attitudes toward mathematics, were less anxious about mathematics, and rated their classes as more interesting.

Hollar and Norwood (1999) tested four classes of intermediate algebra students (two experimental classes and two control classes), with two instructors each teaching one experimental and one control class in their study of the effects of a graphing-approach curriculum that extended O'Callaghan's (1998) CIA study. The control classes covered the same topics as the experimental classes but did not have access to graphing calculators. In the experimental classes, the calculators were used to explore, estimate, and discover graphically, and to approach problems from a multirepresentational perspective. It was found that students in the graphing-approach classes demonstrated significantly better understanding of functions on all four subcomponents of O'Callaghan's function test than did students in the traditional-approach classes. Additionally, no significant differences were found between the graphing-approach and traditional classes either on a final examination of traditional algebra skills or on an assessment of mathematics attitude.

Further evidence of the benefits of graphing-calculator use was obtained from a study that examined the effects of this technology upon students' understanding in elementary college algebra (Shoaf-Grubbs, 1993, as cited in Penglas & Arnold, 1996). The study involved 37 females from an all-women's liberal arts college, divided into an experimental class and a control class, with the only difference between the two groups being the use of the graphing calculator. Statistical analysis of pretest and posttest results showed that the majority of the students in the experimental group achieved significantly better than the control group in the test results, especially in their understanding of graphing concepts. However, Hall (1993), who compared four treatment classes with four control classes on the use of graphing calculators

in a 3-week study of trigonometric functions, found that the graphing calculators had no significant impact on students' achievement.

Another large-scale study involving graphing-calculator use was conducted by Streun et al. (2000) in the Netherlands. These researchers used a three-condition pretest-posttest design to analyze the impact of prolonged use of the graphing calculator throughout the entire school year for all topics of the curriculum (i.e., functions and graphs, change, and exponential and periodic functions). Three experimental classes used the graphing calculator throughout the year; a second set of five experimental classes used the graphing calculator with only one topic for 6 weeks; and four classes, which served as the control group, covered the same subject matter throughout the year but without the graphing calculator. The students who used the calculator throughout the year had enriched solution repertoires and a better understanding of functions. The students who used the graphing calculator for only a short period of time did no better on the posttest than the students in the control group. They merely replaced their symbolic approaches and guess-and-test procedures with graphing methods. Unlike the students who spent more time using the graphing calculator, they had not significantly altered their conceptual understanding of functions. The time factor, which was pivotal in this study, has also been found to be of crucial importance in other technology-related research findings (e.g., Chacon & Soto-Johnson, 1998; Lagrange, Artigue, Laborde, & Trouche, 2001).

Further research involving graphing calculator use is the comparative study of the effects of a 3-year integrated mathematics curriculum for Grades 9–11, the *Core-Plus Mathematics Project*, with more conventional curricula on growth of student understanding, skill, and problem-solving ability in algebra (Huntley et al., 2000). Results indicated that the Core-Plus curriculum was more effective than conventional curricula in developing student ability to solve algebraic problems when those problems were presented in realistic contexts and when students were allowed to use graphing calculators. Conventional curricula were more effective than the Core-Plus curriculum in developing student skills in manipulation of symbolic expressions in algebra when those expressions were presented free of application context and when students were not allowed to use graphing calculators.

Computer Algebra Systems (CAS). The number of large-scale studies involving the use of CAS is much fewer than for the graphing calculator. In one study, Shaw, Jean, and Peck (1997) evaluated the effectiveness of using CAS in an intermediate algebra course by comparing the grade distributions in a follow-up

course of three groups: students who went through the CAS technology-based developmental course ($N=34$), students who went through a traditional developmental course ($N=109$), and students who were not required to take a developmental course ($N=1335$). Students in the developmental program that had the technology-based course did at least as well as the students who did not need to take any developmental course. This was considered a real improvement inasmuch as the students who took the traditional intermediate algebra course did not do as well in subsequent courses as students who took the technology-based course.

Technology and the Changing Roles of Students and Teachers.

A few studies have investigated the work habits of students using technology. For example, Guin and Trouche (1999) categorized students' use of graphic and symbolic calculators into five profiles of behavior: random, mechanical, rational, resourceful, and theoretical. Doerr and Zangor (2000) classified five modes of graphing calculator use: as a computational, transformational, visualizing, verification, and data collection and analysis tool. These studies suggest the potential of technology for broadening the range of mathematical activity engaged in by students. Other studies have noted the changing roles of both students and teachers when technology is integrated into the mathematics classroom. For instance, Farrell (1996) observed that both teachers and students moved toward the roles of consultant and fellow investigator. Goos, Galbraith, Renshaw, and Geiger (2003) analyzed the impact of an assortment of technological tools on classroom interactions in terms of offering opportunities for students to engage constructively and critically with mathematical ideas, that is, as master, servant, partner, and extension of self. The 3-year Goos et al. study showed how technology can facilitate collaborative inquiry during both small-group interactions and whole-class discussions where students use the computer or calculator and screen projection to share and test their mathematical understanding. This finding is in contrast to that reported by Doerr and Zangor (2000), who observed that the graphing calculator as a private device led to a reduction in small-group interactions. However, Goos et al. (2003) have argued that, when calculators and computers are used as partners or extensions of selves and are permitted to become a part of face-to-face discussions, they facilitate communication and sharing of knowledge.

Similarly, Manouchehri (2004) found that the overall complexity of the mathematical discourse increased from before the technology was introduced

to afterward in the senior-level mathematics course that she observed. In particular, she noted that three factors appeared to control the construction of fruitful algebraic classroom discourse: the teacher, the tasks, and the technology. She pointed out that a common tendency in mathematics instruction is to focus on the power of the technology as a presentation device rather than as a discourse participant. In her study, even though the technology served as a powerful information provider, it only became a learning medium when supported by appropriate teacher intervention and tasks. This latter observation has been made time and again by several researchers.

Concluding Remarks on the Algebra Research Involving the Upper Secondary and College Level Student

Although the amount of research conducted with the upper secondary and college level student has been less abundant than that carried out with the middle grade and lower secondary level student, it has produced results that touch upon issues that are both similar to and different from those that emerged from the body of research involving the younger student.

The sources of meaning that have been the object of study in the research with the older student are principally those derived from within mathematics—focusing either on the letter-symbolic or on multiple representations. As was seen, the research involving the letter-symbolic form included studies on structure and on equivalence. With symbol-manipulating technologies being introduced in algebra classes, students' conceptions of equivalence of expressions were a theme of interest. The research involving multirepresentational sources of meaning in algebra, a comparatively larger body of work in relation to that with a letter-symbolic focus, included a broad range of studies. Several of the studies that examined students' notions and usage of symbolic and graphical representations noted students' preference for the symbolic over the graphical. This was in contrast to results from studies involving younger students, which noted their difficulties with using symbolic approaches. Studies dealing with the role of the graphing calculator reported, in general, that sustained use of this technology can provide effective support for multirepresentational thinking among algebra learners. However, the importance of according sufficient time for the conceptual advantages of the technological environment to take hold was noted.

As time has emerged as a key factor in technology-related studies, two other factors have been found to be equally important. One concerns the nature of the

tasks in which students engage; the other is the role played by the teacher in orchestrating the development of algebraic thought by means of appropriate classroom discussion. As is borne out by the results of several studies, the technology by itself is clearly insufficient.

The use of CAS technology in the learning of algebra was found to be more widespread in the research with this age group than with the younger student. Although some of the research with this technology makes use of multirepresentational approaches to algebra learning, a significant amount of this work focuses on the letter-symbolic mode. In line with advances by French researchers in the development of theoretical frameworks for studying CAS use, many of the studies reflect these frameworks by their attention to the emergence of theoretical thought within technical activity. Results from this newly developing sphere of research activity provide evidence that meaning building does indeed occur not only within the generational but also within the transformational activity of algebra. However, a great deal more research is needed in this area.

A related issue that arises with the use of technological environments in general, and with CAS in particular, is the question of the role of paper-and-pencil work in the development of understanding of particular concepts. For example, Asp et al. (1993) claimed that physically constructing tables of values for functions was essential to the development of students' understanding of the relationship between graphs and equations. Warren and Pierce (2004), on the basis of their review of Australasian research in algebra learning and teaching during the years 2000 to 2003, have further suggested that for successful CAS use students need to be able to do the following by hand: solve simple equations, use basic algebraic manipulations to rewrite equations in a form they can use, and be aware of standard techniques used to find solutions. However, they admit that further research is needed with respect to the question of the "by hand" skills that students require when working with CAS, as well as the issue of the place of by-hand work in the learning process (see also Forster, Flynn, Frid, & Sparrow, 2004).

The question regarding the by-hand skills needed by students embarking on CAS activity could usefully be researched by means of studies that carefully document the prior experience and incoming knowledge of students (i.e., their current by-hand skills) and then relate these to the theoretical and technical knowledge that develops (as well as to the obstacles encountered) as a result of the technological experience. Such well-documented research might indeed show that the prior by-hand skills needed by students

for CAS activity vary according to several yet-to-be-determined dimensions.

The second issue regarding the place of by-hand work in the learning process has been emphasized in several studies. For instance, recent research (e.g., Drijvers, 2003; Kieran & Drijvers, 2006) has suggested that it is in the interaction of paper-and-pencil work and CAS activity, particularly in tasks involving confrontation between students' existing theoretical thinking and CAS output, that students develop both technically and theoretically. However, research has just begun to explore this area in, for example, studies related to equivalence of expressions and factoring. Further research focusing on the interaction of paper-and-pencil work and CAS activity, on a variety of topics, is sorely needed.

Before closing this section, a few remarks on needed research that does not necessarily involve technology are offered. With the current curricular interest in problem situations as a means of motivating and inducting a large majority of students into the study of algebra, along with the use of multiple representations for exploring the domain of problem solutions, several issues arise. For example, the obstacles faced by younger secondary students in using letter-symbolic representations as problem-solving tools versus the oft-found reliance of older secondary students on this same mode of representation leads to several questions: What is the nature of the transition between nonusage and usage of algebraic symbols in problem solving? In particular, what are the factors at play when students shift from using non-letter-symbolic (numerical or graphical) to letter-symbolic representations to solve problems? How do students learn to make connections between graphical images and corresponding symbolic representations? More specifically, what is the nature of the connections that students make between graphical and symbolic representations? The mathematical content involved in the study of these questions also needs to move beyond linear situations to encompass, for example, quadratic and trigonometric situations, as well as systems of equations. Research is needed with the upper secondary and college level student on a much wider variety of algebraic content than has up to now been studied.

Algebra Teaching and the Algebra Teacher

In the algebra chapter of the 1992 *Handbook*, Kieran (1992) remarked

Even though the research community knows very little about how algebra teachers teach algebra and what

their conceptions are of their own students' learning, this is not to suggest that there has not been considerable research on new approaches to the teaching of algebra—there has. However, the analysis of results has usually been directed toward detailing learning phenomena. (Kieran, 1992, p. 395)

There are signs that the situation described in the early 1990s has changed in some ways. There has been a noticeable increase in the number of studies related to the teaching or teacher of algebra over the past 15 years or so. Some studies have dealt with the practicing teacher within the algebra classroom; others have been conducted within a context of professional development or inservice training; and yet others have involved the preservice teacher. However, researchers still know relatively little about algebra teaching. As well, there remains a noticeable disconnect between the research on the learning of school algebra and the research on the teaching of school algebra. This latter issue is elaborated throughout the sections that follow.

Research Involving the Practicing Algebra Teacher

One of the first differences one notices in moving from the research focused on the algebra learner to the research focused on the algebra teacher is that the theories used to frame the research tend to be quite different. Whereas studies of algebra learning have included theoretical frameworks based on, for example, constructivism, socioculturalism, semiotic mediation, discourse, embodied cognition, didactical situations, and instrumentation (see also Lerman et al., 2002), the most widely used perspective to frame studies involving the algebra teacher is based on Shulman's (1986) construct of teacher knowledge, which includes content knowledge, general pedagogical knowledge, curriculum knowledge, and pedagogical content knowledge. Shulman identified *pedagogical content knowledge* as the specialized knowledge that teachers need and described it as

the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for inspection . . . the category [of knowledge] most likely to distinguish the understanding of the content specialist from that of the pedagogue. (Shulman, 1987, p. 8)

Ball and Bass (2002) have pointed out that this notion of pedagogical content knowledge suggests that

even expert personal knowledge of mathematics often may be inadequate for teaching; knowing mathematics for teaching requires a transcendence of the tacit understanding that characterizes much personal knowledge . . . it also requires a unique understanding that intertwines aspects of teaching and learning with content. (p. 4)

This perspective has led Ball and Bass to focus on what it is that teachers of mathematics *do*, rather than on what teachers need to *know*, and to begin to develop a practice-based theory of mathematical knowledge for teaching. This distinction between what it is that algebra teachers know or need to know versus what they do may be helpful in examining the research that has focused on the teacher and the teaching of algebra.

Since the early 1990s, mathematics education researchers, wishing to characterize the components of Shulman's pedagogical content knowledge for various domains of mathematics, have worked at fleshing out this construct. For the domain of school algebra, the specification of pedagogical content knowledge has involved efforts to explore algebra teachers' knowledge and beliefs, as well as analyses of the practice of algebra teaching. Teachers' preexisting views, expectations, and beliefs are generally considered to act as a filter for both their pedagogical content knowledge and teaching practice, but little research exists on the influence of prior conceptions in the learning processes of algebra teachers. Thus, the research that has involved the practicing teacher of algebra has been concentrated in two areas that are treated sequentially below: the first focusing on knowledge and second on teaching, with the research on beliefs interwoven into both sections.

Algebra Teachers' Knowledge and Beliefs. Regarding the domain of school algebra, and functions in particular, Even (1990) has suggested that pedagogical content knowledge needs to include at least the following: different representations, alternative ways of approaching the concept, strength of conceptual knowledge, a basic repertoire of examples, knowledge and understanding of the concept, and knowledge of mathematics. However, as research has recently disclosed, and which will soon be seen, knowledge of students' ways of thinking about particular aspects of school algebra and functions is an equally important component of pedagogical content knowledge. Even (1993) has applied her framework to research involving prospective secondary level teachers and has found their understandings of the concept of function to be weak and fragile.

More recently, a related framework—the Multidimensional Grid for Professional Competence in elementary Algebra (MGPCA)—has been developed by French researchers (Artigue, Assude, Grugeon, & Lenfant, 2001, as cited in Doerr, 2004) with dimensions similar to Shulman's, but they have been elaborated specifically for the teaching of algebra: epistemological, cognitive, and didactic. The *epistemological* dimension includes knowing the following: the content of algebra, the structure of algebra, the role and place of algebra within mathematics, the nature of valuable algebra tasks for learners, and the connections between algebra and other areas of mathematics and to physical phenomena. The *cognitive* dimension includes knowing the following: the development of students' algebraic thinking, students' interpretations of algebraic concepts and notation, students' misconceptions and difficulties in algebra, different approaches taken by learners, ways to motivate learners, and theories of learning. The *didactic* dimension includes knowing the following: the curriculum, the resources, different instructional representations, different practices and approaches taken by other teachers, the connections across the grade levels, and the nature and development of effective classroom discourse. The MGPCA grid has been used as a tool for collecting data and analyzing preservice teachers' initial relationship with algebra, and its evolution, throughout their 1st year of combined teacher training and classroom teaching experience in the field. In general, Artigue et al. have noted that, with respect to the cognitive dimension, teachers begin to become sensitive to students' learning difficulties during their 1st year of fieldwork, but that the length of the training period is too short for the trainees to develop the means for analyzing these difficulties or for generating remedial strategies.

One of the current themes of the existing research inquiring into algebra teachers' pedagogical content knowledge concerns teachers' knowledge of students' algebraic thinking. Recent research suggests discrepancies between teachers' predictions of students' difficulties and students' actual difficulties. Sixty-seven high school mathematics teachers, as well as 35 mathematics education researchers, participated in a study conducted by Nathan and Koedinger (2000a) on teachers' and researchers' beliefs about the development of algebraic reasoning. This group was asked to rank order 12 mathematics problems from easiest to most difficult. Four were in story-problem format, four were in equation format, and four were in word-equation format. A comparison of their predictions with the actual difficulty as experienced by high school students, which was discussed in an

earlier section on algebra learning research, shows some important differences. The most salient discrepancy was that teachers and researchers predicted that story-problems and word-equation problems would be more difficult than symbol-equation problems, whereas students found symbolically presented problems most difficult. These differences between students' performance and teachers' predictions could have a significant effect on how teachers perceive students' reasoning and learning in the classroom and in how they teach to these perceptions. In a follow-up study involving 105 K–12 teachers and using similar tasks, Nathan and Koedinger (2000b) found that the middle school teachers came closest to predicting actual student performance. Furthermore, Nathan and Petrosino (2003) have provided evidence to support the argument that it was the well-developed subject-matter knowledge of the high school teachers (i.e., the “expert blind spot” syndrome), which underpinned their view of the equation-before-word-problem sequence, that led them to inaccurately predict student problem-solving difficulty.

Another study (Hadjidemetriou & Williams, 2002) addressing the issue of teachers' knowledge of what students find easy/difficult in algebra was one in which teachers' judgments of the difficulty of items dealing with graphical conceptions, and their awareness of errors and misconceptions, were analyzed and contrasted with learners' difficulty hierarchy. Teachers' judgment of what is difficult was found to be structured by the curriculum and by their mathematical knowledge. Also noted was a gap between pupils' difficulties and teachers' perception of these difficulties: Teachers underestimated technical difficulties of graphing and overestimated the difficulty of interpretive work. Similarly, in a study of 8 upper secondary school teachers in Sweden, Bergqvist (2005) found that the teachers tended to underestimate their students' reasoning levels on conjecture tasks and that the teachers erroneously believed that only a small group of students in each class could use higher level reasoning in mathematics.

Research has also begun to address the question of teachers' knowledge of mathematical links between lessons. Even, Tirosh, and Robinson (1993) compared the connectedness of lessons for two novice secondary teachers and an expert teacher on the topic of equivalence of algebraic expressions. They found that only the expert teacher used connections in mathematical content to guide the unfolding of her lessons. In a later study on teachers' knowledge of students' misconceptions, and the teaching approaches that would address these misconceptions, Tirosh, Even, and Robinson (1998) investigated four 7th-grade teachers'

awareness of students' tendency to "finish" algebraic expressions (e.g., to write the expression $2x + 3$ as $5x$ or 5) and their ability to respond on-the-spot to pupils who showed this tendency in class. Although two of the teachers were aware of this tendency, the means they used to address it in class were only partially successful. The authors argued that,

while one can use the mathematics education literature in order to raise teachers' sensitivity to students' ways of thinking, . . . this literature does not offer enough information and discussion regarding the impact of various approaches and teaching methods related to this conception; . . . it is important for teachers to be acquainted with various teaching methods and to be aware of their pros and cons in different contexts with different teaching aims, and with different students. (pp. 62–63)

Although the interaction between algebra teachers' knowledge and their beliefs is a complex and understudied area, it is widely held that both knowledge and beliefs shape teachers' teaching practice and that teachers' professed beliefs do not always match their instructional practices (Cooney, 1985). It has also been shown that limitations in both content knowledge and pedagogical content knowledge may influence teachers' beliefs and their instructional decisions and actions (Nathan & Koedinger, 2000b). Furthermore, findings by Nathan and Petrosino (2003) have suggested that strong content knowledge in algebra teachers produces beliefs that, in the absence of well-developed pedagogical content knowledge, lead to pedagogical decisions based on the structure of the mathematical domain rather than on the actual ways in which students think. An additional factor that has been found to influence algebra teachers' beliefs is the organization of mathematical material in textbooks (Nathan & Koedinger, 2000b). Cultural factors and related curricular demands also play a role, according to a study by Cai (2004), who found that U.S. and Chinese teachers viewed students' responses involving concrete strategies and visual representations differently. Although both U.S. and Chinese teachers valued responses involving more generalized strategies and symbolic representations equally highly, Chinese teachers expected sixth graders to use the generalized strategies to solve problems whereas U.S. teachers did not.

The existing research on algebra teachers' knowledge and beliefs has, up to now, barely begun to explore the various dimensions of knowledge that teachers need for teaching algebra. If there is one dimension that seems quite pressing, it would be the cognitive dimension of the MGPCA model, that is, the dimension

that includes knowing the following: the development of students' algebraic thinking, students' interpretations of algebraic concepts and notation, students' misconceptions and difficulties in algebra, different approaches taken by learners, ways to motivate learners, and theories of learning. These areas of pedagogical content knowledge are considered crucial to the expertise of the practicing teacher of algebra (Nagasaki & Becker, 1993) yet remain largely unresearched. Furthermore, these areas are exactly those where research on the algebra learner has produced abundant findings. Research on the algebra teacher has yet to tap into this resource and make connections between these two bodies of research.

The Practice of Algebra Teaching and Influences on That Practice. Although the above studies on teachers' pedagogical content knowledge have tended to use Shulman's framework, or derivatives of it, the emerging body of research on the practice of algebra teaching reflects a broader set of theoretical perspectives. For example, Boaler (2003) has drawn from Pickering (1995) to analyze the *dance of agency* occurring in reform algebra classes. Within this framework, she described the ways in which a teacher encouraged the interplay between human and disciplinary agency, that is, taught in such a way that students were able to interweave their own thoughts and ideas with standard methods and procedures of the discipline of mathematics. Atweh, Bleicher, and Cooper (1998) have employed a sociosemiotic perspective developed by Halliday and Hasan (1989) to investigate the social context of two 9th-grade mathematics classrooms that differed in the socioeconomic backgrounds and genders of their students. The researchers found that the students in the observed classes were perceived differently by their teachers and that each teacher conducted his classes in a manner that was consistent with his perceptions of the students' needs and abilities. Coulange (2001), a French researcher, has combined the anthropological approach of Chevallard (1999) and the theory of didactical situations of Brousseau (1997) in her study of the ways in which a 9th-grade teacher supported the approaches of students so as to ensure bringing the lesson to closure and achieving her teaching objectives.

Other frameworks that have been adopted in studies of algebra teaching include, for example, Davis's (1996) theory on evaluative, interpretive, and transformative listening, which was used by Coles (2001) to analyze the change that occurred in a given teacher's class as the listening of both students and teachers became transformative. Additional approaches to analyzing teacher change in the classroom have ranged

from using (a) perspective taking in middle-school mathematical modeling (English & Doerr, 2003), (b) students' ways of thinking as a motivation for recasting the tasks of teaching about functions (Doerr, 2003), and (c) the occasioning of the growth of teachers' mathematical understandings by focusing on students' mathematics within the classroom collective (Simmt, Davis, Gordon, & Towers, 2003). Another example of a framework for analyzing algebra teaching is presented in a study by Solomon and Nemirovsky (1999). They built upon "Chazan and Ball's (1995) idea that 'any discussion holds the potential for discrepant viewpoints' and that it is the teacher's role to 'manage' these views" (Solomon & Nemirovsky, 1999, p. 217) in their development of a tool for describing the complexity involved in managing the wide range of ideas and choices that emerge from the interactions among students and the teacher.

In some studies on the practice of algebra teaching, the focus has been on the beliefs of the algebra teacher and the role that these beliefs play in teaching. For example, Doerr and Zangor (2000) reported the results of a classroom-based study on the interaction between the role and beliefs of the teacher and the patterns of students' use of graphing calculators in support of their learning of precalculus mathematics. This interaction led to the creation and development of a set of ways the tool was used in the classroom and of related mathematical norms. Doerr and Zangor found that the teacher's confidence, flexibility of use, and awareness of the limitations of the technology led to the establishment of a norm that required results to be justified on mathematical grounds and to the devaluing of regression equations or appeals to the calculator as an authority in a mathematical argument. Simmt (1997), in a study of six 11th- and 12th-grade teachers' use of graphing calculators in the classroom, and the ways in which this use reflected their views of mathematics, found that the graphing calculators were primarily used as a device to provide graphical images from which the students were expected to observe and make generalizations about transformations of the quadratic function. The teachers' philosophies were manifested not so much in their choice of activities but in how they followed up those activities with questions and summary notes.

Another factor influencing teaching practice is the teacher's vision of the role of teacher talk in the algebra classroom. From her analysis of traditional and reform-oriented classroom teaching, Boaler (2003) reported that "the teachers in the traditional classes gave students a lot of information, while the teachers of the reform classes chose to draw information out of students, by presenting problems and asking students

questions" (p. 4). A particularly effective teacher, rather than asking students to present their finished problems, chose to ask students to present their ideas *before* anyone in the class had finished working on the problem, so that students could actively and collaboratively build on each others' ideas. Boaler argued that researchers need to acquire such detailed conceptions of effective teaching if they are to understand and impact practice.

A further example providing detailed descriptions of effective teaching is drawn from the TIMSS-R 1999 video study of eighth-grade algebra lessons around the world (Hiebert et al., 2003). The particular lesson involved a Japanese class (Stigler et al., 2003) where the teacher used a specific problem situation, accompanied by concrete materials to represent it, to help students create meaning for inequalities and for their algebraic form. While the students were working at finding a solution to the problem, the teacher circulated around the class, taking note of the various solution methods being worked at. Afterward, he asked certain students to come to the blackboard to present their solutions in increasing order of mathematical sophistication. The teacher orchestrated the placement of the different student approaches at the board not only so that all the solutions would remain on the board for the class to see and discuss, but also so that he could build upon one method in particular to move ahead with his lesson. The way in which the blackboard was sequentially organized with different representational approaches for the solving of the problem was considered by the teacher to be an important learning tool for the students of his algebra class.

The eclectic nature of the theoretical frameworks of the above studies of algebra teaching reflects the diversity of an emerging field of study where the focus is on what algebra teachers do, rather than on what they know. This diversity can, however, present a challenge in a field where basic knowledge of what constitutes effective algebra teaching is still lacking. The community of researchers of algebra teaching needs to develop a set of appropriate theoretical frameworks for the systematic interpretation of teachers' practice of algebra teaching and their development as learners and teachers of algebra. Although some of the above research has focused on acquiring detailed conceptions of effective teaching practice, the question as to how teachers become effective in their teaching of algebra remains a crucial one. The potential contributing role of professional development and preservice training in this evolutionary process is thus the subject of the next section on research involving algebra teachers in programs of professional or inservice development, as well as in preservice training.

Research Involving the Algebra Teacher Within a Context of Professional Development or Inservice Training

The research that has been conducted within the context of the professional development and inservice training of the algebra teacher has tended to focus on two areas in particular, the integration of new curricula, teaching approaches, and technological environments into the algebra classroom, as well as the nature of the algebra content knowledge and beliefs of teachers involved in these programs.

Integrating New Curricula, Approaches, and Technological Environments Into the Algebra Classroom. The research that has focused on algebra teachers' integration of new curricula into the classroom has highlighted the multiple dimensions involved in studying such phenomena. For instance, Bartolini Bussi and Bazzini (2003) have reported on projects in Italy that include teacher-researchers cooperating with university mathematics educators and with other experts. These projects have typically involved (a) a deep analysis of the nature of the mathematics that is the starting point of the inquiry, (b) the strong involvement of teachers working in their own classrooms, and (c) the observation of both individual and collective learning processes. This study illustrates how, in Italy, teaching and learning have been studied in conjunction one with the other. A related example is the Italian ArAl project (Malara, 2003), an inservice teacher education project that was devoted to the renewal of the teaching of arithmetic and algebra in the sixth to eighth grades. The research explicitly included the classroom teacher as a researcher and emphasized the development of pedagogical content knowledge, the teacher's role in the teaching of mathematics, the impact of his/her personality, as well as social issues within the class group. Chapman (2001) has also pointed to the role played by mathematical belief structures in long-term teacher change involving implementation of new approaches.

Although the practice of situating teachers' professional development within the context of their own classrooms has just been considered above, few studies have also included preservice teachers as additional participants in the learning process. English (2003) has reported on such research, which was based on the four-tiered collaborative model of Lesh and Kelly (2000) and where middle-school teachers and preservice trainees gained powerful insights into students' mathematical development by observing the students as they engaged in modeling problems. Regular meetings between researchers and teachers, which took place both before and after the implementation of

each set of modeling activities, were thought to be essential to the success of the collaborative research.

A study (Hallagan, 2004) implementing the reform-based *Connected Mathematics Project* focused on the development of new awarenesses among eighth-grade teachers as a result of their involvement in the project. More specifically, the use of "Ways of Thinking" sheets helped teachers recognize the multiple ways in which their students interpret given tasks, as well as the role of visual images in facilitating students' work on equivalent-expressions tasks. Within another reform-based project, the *Core-Plus Mathematics Project*, involving 40 teachers and their 1466 students in 2 schools, Schoen, Cebulla, Finn, and Fi (2003) analyzed teacher variables related to student achievement. Generally, teaching behaviors that were consistent with the Standards' recommendations and that reflected high mathematical expectations were positively related to growth in student achievement. In another study related to the use of *Core-Plus* curricular materials, Lloyd and Wilson (1998) observed and interviewed an experienced high school teacher during his first implementation of a 6-week unit on functions. The researchers found that the teacher's well-articulated ideas about features of a variety of relationships in different representations supported meaningful discussions with students. The teacher communicated deep and integrated conceptions of functions, dominated by graphical representations and covariation notions.

The increasing implementation of multirepresentational and CAS technology, and curricula involving related new approaches to school algebra, has been reflected in research addressing teacher adaptation and teacher change. Some studies have, for example, examined teachers' attempts at implementing reform-oriented curricula, and their beliefs and attitudes towards different types of curricula (e.g., Heid, Blume, Zbiek, & Edwards, 1999; Stacey, Kendal, & Pierce, 2002). Some teachers have been found to experience tension between familiar methods and the use of technology, or between the curriculum and their personal educational conceptions (e.g., Chazan, 1999; Haimes, 1996; Lloyd, 1999; Slavit, 1996). Others have been found to need support in thinking about how to use symbolic calculators effectively in class and how to reorganize their use of class time when students are involved in CAS work (Guin & Trouche, 2000). In a study of three teachers' implementation of CAS in the classroom, Kendal and Stacey (2001) found that the teachers made very different uses of the CAS even though they had all been involved in planning an agreed common program. The teachers privileged technology use in different ways that were

consistent with their beliefs about algebra and about the teaching and learning of algebra. On the other hand, in a 4-year technology-implementation study in Mexico (Rojano, 2003) involving 15 schools of seventh, eighth, and ninth graders, Cedillo and Kieran (2003) observed that, when algebra teachers noticed that the introduction of calculator technology led to improved student learning, as well as a more positive attitude toward mathematics among both stronger and weaker students, they altered their teaching styles to more learner-centered approaches.

The studies in this section, which have focused on integrating new curricula, approaches, and technological environments into the algebra classroom, have presented a diversity of findings that range from descriptions of multilevel support programs that facilitate teacher change to the difficulties experienced by teachers in attempting to implement new curricula and technologies. Several studies have suggested the ways in which teachers' beliefs both shape and are shaped by the implementation process. Although each study has provided its own insights, further research is needed on the long-term impact on algebra teaching practice of the various implementation projects. Few studies have extended beyond the initial implementation to explore the evolution in teachers' practice and its relation to the professional development program that triggered it.

Algebra Content Knowledge and Beliefs. Although most of the professional development studies of implementation described above included some initial indicators regarding teacher change, other research that has involved the professional development or inservice training of algebra teachers has rather focused on algebra teachers' content knowledge. The areas that have been investigated include knowledge of conic sections (Hitt, 1998); the algebraic and geometric aspects of slope, scale, and angle (Zaslavsky et al., 2002); symbolic and graphical representations of rational functions (Arcavi, 2002); and number-theoretic ideas (Barkai, Tsamir, Tirosh, & Dreyfus, 2002; Zazkis, 1998). Zehavi (2004) has reported that the way in which the problems were presented and the prevailing atmosphere of the inservice group influenced the resulting mathematical behaviors and problem-solving products. Another dimension that has been studied concerns teachers' beliefs. Menzel and Clarke (1998) observed that the inservice teachers involved in their study were divided in their views as to whether the generalizable verbal description of a mathematical relationship should be regarded as algebraic—a finding that relates to Nathan and Koedinger's (2000b)

results on teachers' beliefs as to what constitutes algebraic activity.

An additional area of inservice teachers' content knowledge that has been researched is that of proof and proving. Knuth (2002) carried out a study with inservice secondary school mathematics teachers (grades 9–12) who were participating in ongoing professional development programs. By means of semistructured interviews with participants in order to examine their conceptions of proof, Knuth found that, although teachers recognized the variety of roles that proof plays in mathematics, the view of proof as a tool for learning mathematics was noticeably absent. With the increasing interest in proof and proving as a potential global/meta-level activity for motivating the work of students with algebraic symbols, the findings of Knuth's study suggest that further teacher development is needed in this area.

The above brief examination of the research involving the algebra teacher within professional development and inservice programs has disclosed that little attention has been paid thus far to the study and development of algebra teachers' pedagogical content knowledge. No studies were found that were similar to the Cognitively Guided Instruction program for elementary school teachers of mathematics (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989). This professional development program with its focus on researched-based evidence regarding children's actual thinking in mathematics has been shown to be effective in changing elementary school teachers' instructional practices. It is suggested that similar programs that focus on the research-based thinking of algebra students be developed. Ironically, for many teachers the need for such knowledge may not be obvious. In a study of 9 high school teachers and their perspectives on what constitutes good mathematics teaching and how it develops, Wilson, Cooney, and Stinson (2005) found that the teachers thought that good teaching requires a sound knowledge of mathematics, promotes mathematical understanding, engages and motivates students, and requires effective management skills. Experience was considered the primary contributor. Nevertheless, many of the studies cited in this chapter and elsewhere (e.g., Krainer, 2005) maintain that knowledge of students' mathematical thinking, in addition to knowledge of mathematics, are necessary preconditions for good teaching.

Research Involving the Preservice Teacher of Algebra

Research involving the future teacher of algebra, although still a relatively undeveloped area of investigation, has been growing steadily. The complexity of the

process of educating students to become secondary level mathematics teachers is reflected in programs (e.g., Bednarz, 2001) that do not attempt simply to “deliver” methods of instruction; instead, emphasis is placed on involving future teachers in a process that leads them to construct the knowledge they will need to grapple with, design, and carry out classroom interventions. However, research that informs as to the nature of this complexity is still quite rare, especially with respect to the teaching of algebra. Recent research involving the preservice teacher of algebra has tended to focus on the following themes: the relation between future teachers’ subject-matter knowledge and their beliefs, the characteristics of their beliefs and attitudes, and the nature of their subject-matter knowledge.

In a study of the role played by prospective teachers’ background knowledge on their beliefs about algebra learning (Nathan & Petrosino, 2003), 48 candidates from an established research and teaching university participated: 35 of them had completed calculus or above and 13 had not. Of the 35 with advanced mathematical knowledge, 16 were in a program for mathematics and science majors and going on to teach mathematics or science at the secondary level; the remaining 19 were teacher education students going on to teach at the elementary level (as were the 13 who had basic mathematical knowledge). The preservice teacher education program in which they were enrolled included the study of subject matter with the use of modern technology, extensive use of the TIMSS video cases, the study of effective classroom interactions, and the development of models of teaching. Participants first ranked six problems in accordance with their expectations of the ease or difficulty that beginning-algebra students would experience when solving them and then responded to a belief survey that broadly addressed issues in mathematical learning, problem solving, and the role of algebra in complex problem solving. The pattern of results, in combination with findings from their previous research, suggested to the researchers that advanced mathematics knowledge, rather than algebra teaching experience, mediates teachers’ views of algebra students’ development and that highly developed subject-matter knowledge can blind teachers to the actual ways of thinking of algebra students.

Other research related to the subject-matter knowledge of the preservice trainee, as well as the beliefs associated with that knowledge, includes a study by Van Dooren, Verschaffel, and Onghena (2002, 2003). In their investigation of preservice teachers’ preferential strategies for solving arithmetic and algebra word problems, they observed that future secondary school

mathematics teachers use algebraic methods for solving even very easy problems that could have been handled more appropriately with arithmetic methods. Half of the primary school preservice teachers were found to switch flexibly between arithmetic and algebraic methods, whereas the other half had difficulty with algebraic methods. The authors also noted that the methods the teachers used as individuals were strongly correlated with what they would expect their future students to use and by which they would evaluate student work. Similar results have been reported by Schmidt and Bednarz (1997). In a study conducted by Zazkis and Liljedahl (2002) involving a group of preservice elementary school teachers, participants’ attempts to generalize a repeating visual number pattern led to the verbal expression of the generality without necessarily being accompanied by algebraic notation. However, participants perceived that their solutions that did not involve algebraic symbolism, although complete and accurate, were inadequate.

Studies documenting the knowledge of algebra and its teaching that are held by preservice teachers include the research by Even (1998), who presented 152 prospective secondary mathematics teachers with questions related to moving from one functional representation to another. Her results illustrated how knowledge about different representations is interconnected with knowledge about different approaches to functions, knowledge about the context of the presentation, and knowledge of underlying notions. Similarly, Sánchez and Llinares (2003) identified a clear link between preservice teachers’ subject-matter knowledge and their pedagogical reasoning for the topic of function and its representations. However, Even and Tirosh (1995) found preservice teachers’ subject-matter knowledge to be insufficient for understanding and explaining students’ reasoning, as was noted as well in the Nathan and Petrosino (2003) study above.

In a study of the nature of preservice trainees’ beliefs and attitudes, Vermeulen (2000) has reported that 1st-year mathematics education students often enter training programs with established concept images in algebra that are not associated with successful mathematics teaching practice. This study, which presented a portrait of the suggested current teaching approaches for eighth-grade algebra in South African schools, contrasted these goals with the orientation of student teachers, many of whom indicated altogether different views on the teaching of algebra and showed their resistance to change. Cooney, Shealy, and Arvold (1998) have noted that preservice programs can seem to have a random effect on the participating teachers being trained, and that such randomness should be ex-

pected until researchers begin to better understand the relation between teacher-education activities and the impact of these activities on teachers' belief systems. Similarly, Proulx (2003) found that preservice secondary mathematics teachers were differentially impacted by the teacher training program they were following and that the major factor affecting this impact was their personal frame of reference, which filtered their interpretation of the preservice experience.

Stump (1999), who investigated the notions of slope held by a group of preservice and inservice teachers, found that few thought of slope as a rate of change involving two variables. When questioned about students' conceptual difficulties regarding slope, several teachers answered rather in terms of procedural aspects, such as being able to compute the slope formula. In a related study, Stump (2001) observed preservice teachers teaching a basic algebra course after they had completed a secondary mathematics methods course. She found that the students experienced difficulty in presenting the notion of slope within real-world situations in the classes they were teaching and suggested that methods courses provide experiences that challenge students to construct connections among geometric, algebraic, and real-world representations of slope (see also Zbiek, 1998).

Bowers and Doerr (2001), who used computer-based activities to challenge prospective mathematics teachers' procedural orientations, noted in their findings the intertwined development of subject-matter insights and specific views of teaching. In their research, Chazan, Larriva, and Sandow (1999) asked, "What kind of mathematical knowledge supports teaching for 'conceptual understanding'?" The researchers used tasks from the existing literature on the solving of equations, and questions about teaching students to solve linear equations, to bring to the surface preservice teachers' substantive knowledge in this area. The authors' lack of satisfaction with the nature of the information obtained led them to raise the question of whether finer distinctions might be necessary for describing teachers' substantive knowledge of the mathematics they teach.

The majority of the above studies indicate that research involving the preservice teacher has begun to focus on the complex relation between subject-matter knowledge and pedagogical content knowledge. However, much remains to be done. The finding that teachers with developed subject-matter knowledge may attempt to flesh out underspecified pedagogical content knowledge by drawing on subject-area knowledge suggests first of all that the field needs to develop adequate descriptions of what is considered to be ap-

propriate pedagogical content knowledge for the domain of algebra. Once this is achieved, a great deal more research is needed on the interaction between content knowledge and pedagogical content knowledge for algebra, as well as on how algebra teachers develop both.

Concluding Remarks on the Research Focusing on Algebra Teaching and the Algebra Teacher

Although we now know relatively more about the algebra teacher than we did in the early 1990s, a great deal remains to be researched. There are major areas that have only begun to be studied, for example, the practice of algebra teaching and how teachers learn to be effective teachers of algebra. The above review of research related to studies on algebra teaching and the algebra teacher point to four areas in particular as being in need of focused study.

The first concerns the lack of appropriate models for the observation and analysis of the practice of algebra teaching. The eclectic collection of theoretical frameworks currently being used in research on algebra teaching may work against the systematic building up of a body of coherent knowledge in that area. Second, ways need to be found to incorporate the large body of research findings regarding the learner of algebra into the professional development and preservice and inservice training of algebra teachers. Teachers' insufficient knowledge of students' ways of thinking in algebra is, in part, a reflection of the lack of attention paid to the research base that exists regarding the algebra learner in many programs of teacher training and professional development. Research needs to be conducted on the role played by the gradual integration of this research base into the development of the pedagogical content knowledge of algebra teachers and their practice. Third, the interaction between teachers' content knowledge and pedagogical content knowledge, in conjunction with their understanding of how students' subject-matter-specific knowledge develops, requires major research efforts. Related to this suggestion is the recommendation that research attention focus simultaneously on both the teaching and learning of algebra, and the relation between the two. Fourth, with the steady increase in the use of technology as a tool for algebra learning, considerably more research needs to be done on the importance of the role played by the teacher in maximizing its benefits. Several studies have emphasized that even though technology is a powerful addition to the algebra classroom, it can only become a real learning tool when supported by appropriate

teacher intervention. The nature of such intervention remains an understudied area of research.

Two further areas in need of attention have been pointed to by Doerr (2004). One concerns the lack of shared methodologies in the field, which can hinder progress. While the majority of studies use questionnaires, interviews, and observations, and others use concept maps (e.g., Leikin, Chazan, & Yerushalmy, 2001) and varying forms of action research (e.g., Raymond & Leinenbach, 2000), the methods for analyzing the data are almost as varied as the number of researchers. This latter diversity was noted above with respect to theoretical frameworks for the study of algebra teaching. The other point raised by Doerr concerns the fact that

much of the work of teacher educators, in preparing teachers for practice, is not part of the larger body of research, but rather resides as the uncoded collective wisdom of practitioners and is not subject to the scrutiny of research. (2004, p. 283)

Attention to both of these areas could, according to Doerr, result in the establishment of a professional knowledge base for investigating the teaching of algebra.

To close this section, I return to Tirosh et al. (1998), who remarked:

The mathematics education literature does not offer enough information and discussion regarding the impact of various approaches and teaching methods . . . it is important for teachers to be acquainted with various teaching methods and to be aware of their pros and cons in different contexts with different teaching aims, and with different students. (pp. 62–63)

We researchers have built a large base with respect to the learner of algebra and have developed an extensive understanding of the nature of algebra learning. We have yet to develop the same understanding of algebra teaching and of the kinds of practice that are effective in bringing about such learning in algebra students.

IN CONCLUSION: WHAT DOES THIS BODY OF RESEARCH TELL US?

This chapter has focused especially on the more recent research in the learning and the teaching of algebra, dating from the early 1990s onward. Nevertheless, explicit links were made with the research that was carried out prior. Students ranging in age from about 11 years and upward, that is to say, from about the

sixth grade of middle school up through secondary school and into college, were the subject of the learning research that was examined. Studies involving practicing algebra teachers, as well as those following programs of professional development or inservice or preservice courses, were included in the research that was surveyed regarding the algebra teacher.

The chapter was divided into two parts. The first part looked at the two main competing views of school algebra, at the primary sources of meaning from which the algebra student draws in making sense of the objects and processes of algebra, and at the various models for conceptualizing algebraic activity. The GTG model with its threefold categorization according to the generational, transformational, and global/meta-level activity of algebra served as the framework for presenting the research findings on the learning of algebra within the second part of the chapter. Research on the teaching and teacher of algebra completed the second part.

As the vision of school algebra has widened considerably over the decades—moving from a letter-symbolic and symbol-manipulation view to one that encompasses multiple representations, realistic problem settings, and the use of technological tools—so too has the vision of how algebra is learned. The body of research that was surveyed in this chapter has provided compelling evidence that students derive meaning for algebraic objects from multiple sources. A complex interplay occurs within meaning building among the letter-symbolic and graphical forms, problem situations, and linguistic and gestural activity, as well as past lived experience.

With respect to the manipulative/transformational activity of algebra, the research focus has also expanded. Whereas symbol manipulation has in the past been regarded as a primarily rule-based activity, more recent research has been examining the importance of students' coming to view algebraic transformations as a theory-laden activity. The research involving symbol-manipulation technology has similarly been exploring whether the use of this technology can lead students to embrace a conceptual perspective on symbol manipulation, at the same time that they are developing algebraic techniques. Although this research has begun to show some of the ways in which students think about algebra as theory-based activity, the affordances of such approaches require a great deal more study.

A comparison of the research related to the transformational activity of algebra with that related to the generational activity suggests nevertheless a gap between the two. Within generational activity, students draw upon a wide variety of sources of meaning for algebraic symbols, including multiple representations,

word problem situations, and physical and linguistic activity. However, transformational activity narrows the focus primarily to the letter-symbolic. Little is known about how the meaning derived from multiple sources evolves in students and is transformed from embodied and contextual meaning to an understanding of symbols that are subject to formal manipulation. This is an area where further research is especially needed.

Another area where future research might fruitfully be carried out concerns the interaction between functional and nonfunctional approaches to algebra that are included within given programs of instruction. As noted within the chapter, such hybrid programs, which may in fact constitute the majority of algebra programs at the present time, can be a source of confusion to students with respect to the articulation between generational and transformational activity. Teaching experiments designed to assist students in distinguishing function approaches from equation approaches, and in meaningfully integrating them, are strongly recommended.

Research that has investigated the transformational activity of algebra in technological environments such as CAS has signaled the importance of the quality of both the tasks and the classroom discussions during which the teacher attempts to draw out the conceptual aspects underpinning the transformational activity. This research has emphasized that the insertion of such technology into the algebra classroom does not remove the need for paper-and-pencil algebraic techniques; in fact, just the opposite seems true. Further studies that examine the articulation between machine techniques and paper-and-pencil algebraic techniques, as well as the nature of the tasks that enhance the emergence of both conceptual and technical expertise within symbol-manipulation-technology environments, are however needed. CAS research has also noted that the use of this technology as a didactical tool can occasion mathematical discussions that do not normally occur in algebra classrooms. However, for these discussions to be fruitful, the role of the teacher was found to be of crucial importance. A great deal more research is needed concerning the impact of classroom discussion on algebraic growth: what kinds of discussions are most beneficial, how best to encourage students to present their ideas, with what particular kinds of tasks, with which students, at what specific moments, and so on.

The role of word problems in algebra instruction remains somewhat problematic. Although some research reported the advantages of experience with realistic word problems and modeling situations in algebraic activity, others reported the opposite. Furthermore, research on the representation of word-problem

situations by algebraic equations continued to show the difficulties that students have with such activity. However, it is not the case that students cannot solve these problems by a variety of arithmetic methods; in fact, they are more likely to solve correctly word problems than either similar-equation or word-equation formats of questions. Research suggests however that word-problem solving, as it is currently engaged in, may not be the most productive route to the generation of equations, particularly with the younger student.

Although generating symbolic representations of problem situations was shown to be fraught with obstacles, the use of graphical representations for both solving problems and interpreting the letter-symbolic was found in many studies to be an effective approach. This is an area where graphing technology has played a significant role. Research has shown that students sometimes require several years to effectively generate equations to model problem situations, but that when that happens, the letter-symbolic becomes a more preferred mode of representation than the graphical. Research that can help to clarify the factors involved in the successful transition to a symbolic mode of representation for problem situations would be most useful.

Research with the older student, which has tended to focus more on the within-mathematical sources of meaning than on the sources of meaning outside of mathematics, indicates that a majority of students continue to experience difficulty in “seeing structure.” Just as the younger adolescent needed a great deal of time in coming to see how to model situations with equations, so too has the older student experienced difficulty with what has proved to be another time-demanding phase of symbolic learning, that is, seeing form and structure in those symbols. Although the *Principles and Standards for School Mathematics* (NCTM, 2000) recommends that students in Grades 6 to 8 begin to “represent and analyze mathematical situations and structures using algebraic symbols” (p. 222), the existing evidence suggests that few sixth graders in the United States are actually called upon to engage in symbolic work. Yet, in other countries (e.g., China, Russia, Singapore, and South Korea), students begin a formal study of algebra that involves not only the development of algebraic reasoning and generalization, but also the use of algebraic symbols and the solving of equations as early as the fourth grade (Cai et al., 2005). If, as the findings discussed earlier suggest, students need a great deal of time in becoming comfortable with algebraic symbols and in acquiring the fluency and power that symbols can provide, then serious consideration ought to be given to the notion of having students begin the process at an earlier age.

With respect to the use of technology in algebra learning, some have argued that tools such as computer algebra systems can be made to do all the symbolic work, thus removing the need for students to engage in such learning themselves. However, research has shown that the conceptual does not evolve in the algebra learner if the technical aspects are neglected. The use of technology does not lighten the burden of algebraic work if students have not developed machine techniques to obtain what they want from the machine and conceptual tools for interpreting the machine output. Furthermore, these conceptual tools for interpreting machine output seem to emerge from a complex interplay of paper-and-pencil and machine techniques.

The last section of this chapter involved the algebra teacher and the teaching of algebra. This is a field of study that has been growing during the past 20 years. However, researchers still know relatively little about what makes for effective algebra teaching and how algebra teachers learn to develop their craft. Yet, the current body of research has yielded some notable findings that in turn suggest areas in need of further study. Research has shown that teachers' knowledge of students' actual algebraic thinking may be quite limited. Students can reason beyond what many teachers think they are capable of. In addition, some studies have revealed that the weaknesses that exist in teachers' pedagogical content knowledge may be compensated for by more highly developed subject-matter knowledge that inappropriately maps the structure of the algebraic domain onto students' learning trajectories. This suggests the need for research into the ways in which teachers' knowledge of students' thinking can be further developed, by for example tapping into the research base that exists with respect to the learning of algebra. As emphasized by Stacey and MacGregor (2001),

Awareness of how students think about . . . relationships and of what they are likely to perceive in mathematical situations will help teachers to decide what aspects of presentation need special emphasis and what follow-up will be necessary, leading to better teaching and better outcomes. (p. 152)

A related component of algebra teachers' pedagogical content knowledge that has been found to be in need of development, especially among beginning algebra teachers, concerns the knowledge of students' misconceptions and the teaching approaches that could address these misconceptions. Researchers in this area have argued that the existing research literature has yet to evolve regarding descriptions not only

of teaching approaches that work, but also of those that have been found not to work. It has been proposed that research needs to go beyond documenting what students think and how they interpret various objects and processes to include more information on the advantages and disadvantages of different teaching approaches, the contexts in which they were used, and the nature of the students with whom the approaches were tried. This suggests a closer connection between the study of the learning of algebra and the study of the teaching of algebra.

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AUTHOR NOTE

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