

ABSTRACTION AND CONSOLIDATION

ABSTRACT. The framework for this paper is a recently developed theory of abstraction in context. The paper reports on data collected from one student working on tasks concerned with absolute value functions. It examines the relationship between mathematical constructions and abstractions. It argues that an abstraction is a consolidated construction that can be used to create new constructions. Newly formed constructions are fragile entities. In the course of consolidating a construction this student creates connections between the new construction and already established mathematical knowledge and develops a language to describe and guide mathematical actions related to the construction. The resulting abstraction is a more resilient form of the construction and the student is able to justify his assertions. The paper also examines issues in designing tasks to consolidate a construction.

KEY WORDS: absolute value functions, abstraction, activity, consolidation, construction

1. INTRODUCTION

This paper reports on one student's consolidation of constructions he made in tasks concerned with absolute value functions. Abstraction, consolidation and construction are important issues in mathematics education but there are differing interpretations as to what these constructs are and involve. We begin with a consideration of these three terms and their interpretations in the relevant literature. The next section outlines the research project this work arose in and pays particular attention to task design and the pilot study. We then present substantial transcripts from this student's protocols on consolidation tasks with some commentary on the student's development. The paper ends with a discussion.

This paper adopts an activity-theoretic model¹ of abstraction proposed by Hershkowitz et al. (2001). Consolidation is often associated with skill acquisition (Meichenbaum and Biemiller, 1998) and the retention of information (McGaugh, 2000). There is little research on consolidating abstractions. Construction in this paper refers to the emergence of a new structure through constructing actions. Constructing actions are here used as epistemic actions as described by Hershkowitz et al. (2001). This section outlines the model of abstraction of Hershkowitz et al. (2001), reviews the literature on consolidating abstractions and proposes an amended way to view abstraction, construction and consolidation: that it is constructions, not

abstractions, which are consolidated and that consolidation is an essential part of abstraction.

What is a mathematical abstraction? Empiricist accounts dominated Western mathematics education writings for much of the 20th century. Empiricists view abstraction as involving generalisation arising from the recognition of commonalities isolated in a large number of specific instances (see Ohlsson and Lehtinen, 1997 for a critique of this view). A problem with this view is that if someone is to recognise commonalities, then surely that person must already have some understanding (which may be rudimentary) of the abstraction. By the end of the 20th century new ways of viewing abstraction had emerged. A review of these views would lengthen this paper beyond acceptable limits and the reader is referred to Boero et al. (2002) for a summary of several different accounts of abstraction. In this paper we adopt Hershkowitz et al.'s (2001) account of abstraction. This is a dialectical materialist approach which views the genesis of an abstraction as an undeveloped initial entity which develops through the use of mediational means and social interaction. This development is not from the concrete to the abstract but, rather, a dialectical *to and fro* between the concrete and the abstract.

Specifically Hershkowitz et al. (2001) view abstraction as an activity of vertically reorganising previously constructed mathematical knowledge into a new mathematical structure. The term 'activity' is used in the sense of activity theory (Leont'ev, 1981) which stresses that actions occur in a social and historical context. Attainment of the reorganisation of mathematical knowledge requires actions on mental or material objects and such reorganisation is called 'vertical'. Through these actions mathematical elements are combined together, structured, organised and developed into more formal elements. 'Structure' is used in the sense of Davydov (1990), that the development of an abstraction from an undeveloped initial entity involves establishing an internal structure with links that ultimately results in a differentiated and structured entity. The term 'structure' is further used (see Dreyfus and Tsamir, 2004) as a generic term for abstracted structures, methods, strategies and concepts.

Hershkowitz et al.'s (2001) new structures arise in an activity from three epistemic actions: recognising, building-with and constructing (referred to as RBC hereafter). Recognising a familiar mathematical structure occurs when a learner realises a structure which is inherent in a given mathematical situation. Building-with consists of combining existing structural elements to meet a goal such as solving a problem or justifying a statement. Constructing is the rarest but most important action. It consists of assembling knowledge artefacts to produce a new structure which becomes familiar to the learner. Distinguishing features of the constructing and building-with

actions are the 'novelty' and 'motive' driving the activity. In building-with, students are not enriched with novel, more complex structures as these actions respond to an extraneous goal such as justifying a statement and the goal is achieved by using previously acquired structures. On the other hand, constructing actions are related to the reorganisation of the previously acquired structures which brings about emergence of a novel structure. In this process, construction of a new structure is often the goal of the activity; and even if it is not, it is indispensable for the achievement of the goal. The theory claims that RBC actions are dynamically nested: that building-with actions are nested in constructing actions and recognising actions are nested in building-with actions and in constructing actions.

The genesis of an abstraction is seen as passing through three stages: (a) the need for a new structure; (b) the construction of a new abstract entity through nested recognising and building-with epistemic actions with extant structures; and (c) the consolidation of the abstract entity/structure which involves recognising it and building-with it in further activities. Hershkowitz et al. (2001) provide empirical evidence regarding stages (a) and (b) but merely mention the importance of consolidation of the newly emerged structures.

Three other papers consider consolidation with respect to this theory of abstraction. Tabach et al. (2001) and Tabach and Hershkowitz (2002) examine the construction of knowledge and its consolidation. They touch on the importance and necessity of the consolidation after the construction of knowledge and they appear to point to the fragility of new structures but they do not analyse the process of consolidation. A major contribution, in our opinion, to understanding the process of consolidation comes from Dreyfus and Tsamir (2004). They analyse the protocol data of one student working on tasks which involved the comparison of infinite sets and conclude that consolidation is a long-term process in which an abstraction becomes so familiar that it is available to the student in a flexible manner. They identify three modes of thinking that take place in the course of consolidation: building-with, reflecting on the building-with and reflecting. They claim that building-with actions are the most direct and elementary means of consolidation. They characterise the consolidation of an abstraction with the constructs: immediacy, self-evidence, confidence, flexibility and awareness. We will not employ Dreyfus and Tsamir's (*ibid.*) constructs in this study to avoid a narrow line of enquiry in a new area of research. However, some comparative comments between our observations about consolidation and their model can be found in Monaghan and Ozmantar (2004).

These RBC studies are primarily concerned with the process of abstraction, stages (a) and (b), and the process of consolidation rather than the

product of abstraction. Consideration of process and product aspects of abstraction and consolidation has led us to amend RBC theory slightly. We now sketch our amended theory and return to this later, with regard to our data, in the Discussion section. It appears to us that when RBC authors refer to the consolidation of an abstraction or structure, they are actually referring to the consolidation of the construction, seen by them as the final stage of the abstraction. But surely the final stage of an abstraction is when the abstraction is sufficiently familiar and established that the structured knowledge can be used (recognised and built-with) in further activities. In this case we do not refer to the consolidation of an abstraction (as consolidation is part of the abstraction) but to the consolidation of a construction.

2. THE STUDY

The work presented in this paper is part of a wider study examining aspects of scaffolding and social interaction with regard to RBC model of abstraction. The wider study employed an explanatory multiple case study methodology (Yin, 1998). Data were collected from 20 Turkish Year 10 (16–18 years old) students working on four sequential tasks on the graphs of linear absolute value functions, $y = |f(x)|$, $y = f(|x|)$ and $y = |f(|x|)|$.

Application of absolute values in the algebraic domain appears in Year 9 and graphs of absolute value functions are studied in Year 11 of the Turkish National Curriculum. We thus selected Year 10 students for the study. Linear absolute value functions were chosen as an area of mathematics which the students were unlikely to have encountered previously, as a topic which would be intellectually challenging but doable and as an area where sequential tasks could be designed which allowed the students to form abstractions and use these abstractions to form further abstractions. The study was not designed as research into students' understanding of this topic *per se* but, as this was our vehicle for studying abstraction, it is relevant to review studies in this area, which we do in the next paragraph.

We found surprisingly little literature on students' understanding of absolute values and virtually nothing on absolute value functions. Most of what we found was quite old and was more or less equally divided between reports in professional journals for teachers and short research reports. Most of the difficulties reported in the literature are concerned with algebraic aspects of absolute values. For instance, the introduction of the formal definition, i.e. $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$, creates some difficulties for students (Parish, 1992). Students can find it difficult to accept that $|x|$ can be $-x$ (Perrin-Glorian, 1995) and see this as contradictory to their perception of an absolute value as a positive number (Sink, 1979).

Applying the formal definition of an absolute value to solve equations and inequalities involving letters are also reported to create difficulties for some students (Chiarugi et al., 1990). Solutions of these equations or inequalities require students to remove the absolute value and distinguish between the $x \geq 0$ and $x < 0$ cases. Students often make mistakes in breaking equations and inequations down into cases (Arcidiacono, 1983) and in using logical connectives, 'if...then' (Chiarugi et al., 1990) and 'and...or' (Parish, 1992), in solution strategies. The only paper we found which concerns absolute value functions is Eisenberg and Dreyfus (1994) and this is only a minor focus of the paper. They report that their students generally find it easier to transform the graph of $f(x)$ into the graph of $y = |f(x)|$ than into the graph of $y = f(|x|)$. We now return to our study.

To examine the role of scaffolding and social interactions in the formation of students' abstractions we worked with pairs and individual students with and without scaffolded help. The students were encouraged to 'think aloud' (Gilhooly and Green, 1996; Green and Gilhooly, 1996) as they worked on the four tasks (one task per day, 45–60 minutes for each task, all tasks completed within 1 week). The students were selected on two criteria: (1) that they had the prerequisite knowledge necessary to complete the tasks and were thus capable of achieving the intended abstractions; and (2) that they had not previously encountered the topic area and had not formed the intended abstractions. In order to select a sample which met these criteria, a diagnostic test was designed. One hundred thirty-four students completed this test and 20 students were selected. These students were organised into pairs and individuals. Fourteen worked in seven pairs and six worked alone on the tasks. Four pairs and three individuals worked with scaffolded help (from the second author) and the remainder worked with minimal help, aimed only at avoiding student frustration. Due to the space limitations and to allow a detailed consideration of issues this paper presents data from one student working alone with scaffolded help. As the focus of this paper is on abstraction and consolidation we do not explicitly focus on scaffolding.

Data consisted of students' written work during and audiotapes of the sessions. Audio recordings were transcribed verbatim, paying careful attention to accuracy and the speaker (interviewer/student) sequence. Speaker utterances were numbered with each uninterrupted utterance by one speaker being assigned a natural number. Three dots (...) were used to indicate either that the speaker paused or spoke inaudibly. As the students' utterances were connected to their written responses, graphs and computations, these responses were inserted into the transcripts as appropriate. Some comments were inserted in square brackets to allow the reader to follow the conversation between the participants. These transcriptions, originally produced

in Turkish, were later translated into English. Principles guiding translation were: (1) that the English should be clear; and (2) faithfulness to the original intent, e.g. wording as near as possible to the original Turkish.

All data collection and analysis instruments and procedures were piloted prior to the main data collection. Piloting is often simply a check that instruments and procedures fit a study's purpose. In our case piloting located task design faults that were instructive with regard to our understanding of consolidation and abstraction. We thus present the results of our pilot work, with regard to task design,² in some detail in the next sub-section.

2.1. *Piloting and task design*

We outline the initial tasks, comment on problems in pilot work and describe the revised tasks. Four sequential tasks were initially designed. The mathematical focus of the first, second and third tasks were for students to draw/sketch the graphs of the linear absolute value functions $y = |f(x)|$, $y = f(|x|)$ and $y = |f(|x|)|$ respectively (referred to as $|f(x)|$, $f(|x|)$ and $|f(|x|)|$ hereafter). Each of these tasks had five questions. The first three questions of these tasks presented functional equations and asked students to draw or comment on the respective absolute value functions. Question 4 presented Cartesian graphs without functional equations and asked students to draw the respective absolute function graphs. Question 5 asked students to find/describe rules to obtain the absolute function graph from the linear function graph. In the first and second tasks it was expected that students would draw/sketch the graphs of $|f(x)|$ and $f(|x|)$ from their knowledge of the graph of $f(x)$ and of the absolute values of numbers. It was then expected that students would use this new knowledge to construct the graph of $|f(|x|)|$ in the third task. Question 4 of the first, second and third tasks were designed to provide an opportunity for consolidation. Two other opportunities for consolidation were envisaged: the third task was intended to build upon the structures constructed in the first and second tasks and thus consolidate these structures; the fourth task had six questions designed to consolidate all of the constructions in the first, second and third tasks.

The initial tasks were informally tried out on three English students and later formally piloted, i.e. all data collection procedures strictly attended to, on five Turkish students; we report on the Turkish pilot. These five students were selected from 13 volunteers, as they were the only students who met the two diagnostic test criteria described above. Four of these students worked in two pairs and one worked alone on the tasks over four consecutive days. Data obtained from piloting suggested that students working in pairs achieved the intended constructions, $|f(x)|$ and $f(|x|)$, of the first and second tasks. They could not, however, make use of this knowledge

to obtain the construction of $|f(|x|)|$. Further to this, they made virtually no progress on the fourth task, which was prepared for the purpose of consolidation. The tasks and underlying assumptions were revisited. Students' responses and verbalisations drew our attention to the organisation of the tasks and the opportunities for consolidation. Pitfalls in the three opportunities for the consolidation are now discussed.

Prior to piloting it was thought that question 4 of the first, second and third tasks would provide an opportunity for students to consolidate their new construction but protocols demonstrated that this question played an important role in the construction itself. The reason, we believe, is that this question presents graphs without functional equations and this appeared to aid (force) students to develop a rule to obtain the intended graphs by analysing the graphs alone (there is no opportunity to substitute numbers for variables and plot points).

It was expected that the third task concerned with $|f(|x|)|$ would help students consolidate the new constructions, $|f(x)|$ and $f(|x|)$. Students did recognise these constructions but were unable to use them. This suggests that without consolidation these constructions are unlikely to be used in further constructions. This is compatible with the genesis of abstraction proposed by RBC theory of abstraction, that structures are used in further abstractions after consolidation. The fourth task was not successful as a consolidation exercise largely because students were not successful on the third task.

Our analysis of the problems of the initial task with regard to consolidation opportunities suggested that the consolidation task, the fourth task, should become the third task, amended to consolidate only the constructions $|f(x)|$ and $f(|x|)$ and that the initial third task becomes the fourth task. Our analysis of the problems suggested further design issues with regard to the amended third task which we detail below.

Although RBC model of abstraction emphasises the necessity of consolidation after a construction, suggestions as to what activities might lead to consolidation remain imprecise. Dreyfus and Tsamir (2004) claim that consolidation is likely to occur through building-with actions, where students recognise and use the new structure. They also posit that discussions about the new structure, examining it from different perspectives and reflecting on it, can provide opportunities for consolidation. Further to this, the pilot study demonstrated that students became confused about the difference between and the different ways to obtain the graph of $|f(x)|$ and $f(|x|)$. These considerations informed the redesign of the third task.

The revised third task was prepared for the consolidation of the constructions of $|f(x)|$ and $f(|x|)$. In this paper, however, we limit our focus

1. A linear function $f(x)$ is defined as $f(x)=2x-2$. Draw the graphs of $y=|f(x)|$ and $y=f(|x|)$ on the grids given below. NB: Grids are not shown in this paper.
2. $f(x)$ is a linear function in the form of $ax+b$. How would you draw the graph of $|f(x)|$ and $f(|x|)$ by making use of the graph of $f(x)$. Explain your answer with an example.
3. Three friends are talking about how the graph of $f(|x|)$ can be obtained from $f(x)$ which is a linear function.

Aylin says that “The graph of $f(|x|)$ must be over the x -axis. To obtain the graph of $f(|x|)$, one needs to take the symmetry of the negative $f(x)$ values in the x -axis because this function includes absolute value which makes every negative $f(x)$ values positive and positive values exist only over the x -axis”.

Cem disagrees and claims that “There is no difference between the graphs of $f(x)$ and $f(|x|)$ for the positive x values but we cannot say anything about the difference for the negative x values, which depends on the equation of $f(x)$ ”.

Arzu objects to them and argues that “To obtain the graph of $f(|x|)$, one must not change the part of the graph of $f(x)$ at negative x -values and simultaneously the symmetry of this part must be taken in the y -axis”.

As can be seen these students are having some trouble about the graphs of $f(|x|)$. What can you say about each student’s claims and explanations? How would you convince them that your suggestion is correct?

4. Which of the given graphs could be considered as a graph of $|f(x)|$ and/or $f(|x|)$. Explain your reasoning.

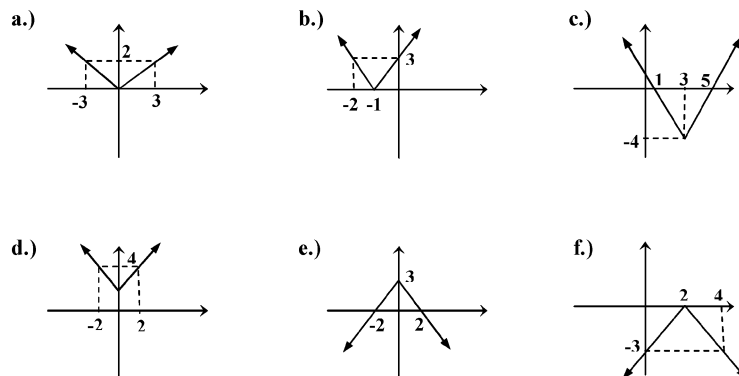


Figure 1. The third task.

to $f(|x|)$ and detail the relevant parts of this task which includes four questions on the consolidation of the construction of $f(|x|)$ (see Figure 1).³ Question 1 presents a linear function and asks students to draw the graph of $|f(x)|$ and $f(|x|)$. Students are free to choose a solution strategy. The aim is to examine the state of the new structures and observe how students obtain the graphs. Question 2 asks students to state how they would obtain the graph $|f(x)|$ and $f(|x|)$ for $f(x) = ax + b$. The aim of this question is to lead students into discussion. Question 3 presents claims from three

imaginary students about how to obtain the graph of $f(|x|)$ from a given graph of $f(x)$. All of the claims in this question are different and incorrect (though they are designed to be ‘intelligently incorrect’!). The aim is to engage students in discussion and for students to justify and clarify their ideas. Question 4 aims to focus on the difference between $|f(x)|$ and $f(|x|)$ from graphic considerations alone.

3. PROTOCOL DATA

Excerpts from one student’s protocol on the revised third task (not the pilot study third task) are presented. We select a single student rather than a pair of students as we feel this helps us focus on consolidation without getting involved in considerations of issues of social interaction.⁴ We call the student Tugay (not his real name). He was one of three scaffolded individuals; all three consolidated the constructions, made in the first and second tasks, in the third task. We have chosen Tugay for this paper because he expressed himself clearly during the task.

We do not present the complete protocol for the third task; this would be very long and would detract from analysis. In selecting protocol excerpts we attempt to present an overview of the work and include all excerpts referred to in the Discussion section. As noted above, our focus is on $f(|x|)$, not $|f(x)|$. However before presenting the protocol excerpts, it is useful to note that Tugay achieved the construction of $f(|x|)$ at the end of the second task. This can be seen in his verbalisations when asked to explain how to obtain the graph of $f(|x|)$ given the graph of $f(x)$. He explained:

Even if we are not given any equation we can obtain the graph of $f(|x|)$. . . To do so err. . . intersection point of the function of $f(x)$ and y err for $x = 0$, through the point of intersection of y, I mean we first find the value of y for $x = 0$. . . then we draw the line of . . . I mean drawing the line passing through this value. . . I mean say $y = a$. . . we draw the line of $y = a$. . . the segment [of $f(x)$] at the negative values of x , up or down. . . I mean we take a symmetry in this line by flipping this part [of $f(x)$ at the negative values of x] up or down and the graph is usually V-shaped. . . umm always V-shaped and this gives us the graph of the function of $f(|x|)$.

These explanations suggest that Tugay, at the end of the second task, constructed a method to sketch the graph of $f(|x|)$ from the graph of $f(x)$.

3.1. Episodes in the protocol

We divide the protocol excerpts into seven episodes. The first two episodes deal with Tugay’s work on Questions 1 and 2. Episodes 3, 4, 5 and 6 deal

with Tugay's work on Question 3. We divide the work on this question into four parts because Tugay's verbalisations display a distinct development in his work on this question. Episode 7 deals with Tugay's work on Question 4. After the episodes we briefly present what Tugay did in the fourth task. This is interesting in itself, in that it shows how he went on to use his abstractions of $|f(x)|$ and $f(|x|)$ in building a further construction, $|f(|x|)|$, and this informs theoretic discussion in the next section. In the following 'I' refers to the interviewer and 'T' refers to Tugay. Each complete utterance is given a new line number. We provide comments after each episode.

3.1.1. Episode 1

2T: [Tugay reads Question 1.] Umm how can we do this?

3I: How are you planning to obtain these graphs?

4T: As far as I remember from the earlier tasks, I can either first draw the graph of $f(x)$ and then take the symmetries accordingly or I can substitute different values of x and then draw the graphs.

5I: OK then, let's talk about them before you proceed.

...

10T: For the graph of $f(|x|)$, we draw a line parallel to y -axis through the intersection point of $f(x)$ and y -axis. Then we take the symmetry of a ray by flipping up and down accordingly. But I am not too sure how! Maybe according to the given graph, I guess.

11I: OK. You told me what you remember about these graphs. I just wonder how you could draw these graphs now?

12T: Umm, I think it'd better if I substitute different values of x and then draw the graphs because I feel more secure in that way. Maybe afterwards I can use what I developed before. . .

Recall that Tugay constructed a valid structure for $f(|x|)$ in the second task. When he was asked to state how to obtain the graph of $f(|x|)$ using $f(x)$, he came up with two ways of doing this: first drawing the graph of $f(x)$ and then 'taking symmetries'⁵ or substituting. Although Tugay remembered elements of how to draw the graph from the graph of $f(x)$ such as taking symmetries, his language was not precise and he also stated that he was not sure about how the symmetries were taken. This uncertainty was apparent in the linguistic forms he used, e.g. "as far as I remember" (4T) and "I am not too sure. . . I guess" (10T). These statements indicate that the knowledge structure he constructed in the second task is rather fragile and needs to be consolidated. He did not use his construction and resorted to substituting because, he said, he felt "more secure in that way" (12T).

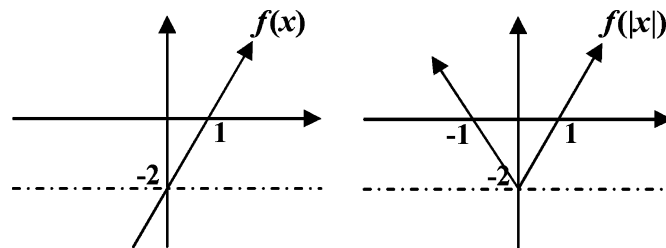


Figure 2. The graphs Tugay considers at 52T.

3.1.2. Episode 2

Tugay completes Question 1 by substituting (14T–42T, not shown) and moves on to Question 2. We present his work related to $f(|x|)$ below.

44T: I think I use the first question for this. Let's see. . . but I need to draw the graph of $f(x)$ first so that I can see what happens. . .

He draws the graph of $f(x)$ and $f(|x|)$ (see Figure 2).

52T: [He looks at the graphs of $f(x)$ and $f(|x|)$]. For the graph of $f(|x|)$, umm, after x becomes negative, I mean. . . umm, first we find the intersection point of $f(x)$ and y -axis. Then we draw a line which passes through this point and which is also parallel to the y -axis. I think I better explain with the above example. We draw the line of $y = -2$ because it is the intersection of $f(x)$ and y -axis. Then the part of $f(x)$ under this line will be flipped up to obtain the graph of $f(|x|)$.

53I: You mean you reflect in the line of $y = -2$?

54T: Hmm yes, yes symmetry but the line. . . I mean the symmetry line changes according to $f(x)$.

A question of interest in this episode is, to what extent was Tugay's use of a specific function necessary to his elaboration? Question 2 asks for a general description explained with an example. Tugay used the example from Question 1 but his explanation appeared to be tied to this example, though he did recognise (54T) that what he calls a 'symmetry line' may change if another function is used. We cannot state for certain but the use of a specific function appears to be intrinsic (if not necessary) to his thoughts at this stage in his development. For example, in 52T he began to articulate how to obtain the graph of $f(|x|)$ in general terms but then returned to his specific graph saying "I think I better explain with the above example". During this episode Tugay did not express his thoughts without recourse to specific examples and appeared to be dependent on them. In this sense his construction of $f(|x|)$ may be said to be 'rigid' at this point in his development.

3.1.3. Episode 3

NB: Between utterances 56T and 107I Tugay's work is concerned with $|f(x)|$ and we omit this part of the protocol. We pick up the discussion as Tugay starts Question 3.

108T: [He reads Aylin's statement] umm, let's see. . .

109I: You may better understand what Aylin says if you illustrate it on a grid.

110T: OK, but I wonder if a graph of $f(|x|)$ can take negative values, I mean under the x -axis! . . . Well, yes, of course it could be. . . I mean. . . let's draw a random graph [he draws a linear graph]. What was I asked? In order to draw the graph of $f(|x|)$, we should draw a line passing through the intersection of the y -axis and $f(x)$. . . and then we take the symmetry in that line.

111I: You took the symmetry of the part of $f(x)$ corresponding to the negative values of x ?

112T: Yeah, (. . .) so the graph of $f(|x|)$ does not have to be over the x -axis all the time.

113I: It does not?

114T: No, it may have some part under the x -axis as well.

115I: You showed that Aylin's reasoning is wrong, but how about her method? She may propose a correct method with the wrong reasoning?

116T: Negative values of $f(x)$

117I: Should take symmetry in the x -axis, she [Aylin] says. . .

118T: No. . . negative values. . . but wait a minute. . . absolute value. . . [He rereads Aylin's statement]

119I: What she [Aylin] says is, briefly, part of $f(x)$ under the x -axis should be taken symmetry in the x -axis.

120T: Yeah I know but it's wrong. Because when the symmetry is taken in the x -axis we obtain a graph like this. . .

121I: So?

122T: So, it will be different from the graph that I drew. Thus it cannot be so. . . I mean I don't agree with Aylin.

In this episode, Tugay convinced himself that the graph of $f(|x|)$ need not necessarily be above the x -axis. Piloting alerted us to a misconception that if a function includes an absolute value sign, then the graph must be above the x -axis (since absolute value signs make negative values positive) and an aim of this question was to allow students to confront this misconception (which Tugay successfully did). However, although he convinced himself he did not, in our opinion, offer a convincing argument or justification that he was right and Aylin was wrong other than noting that Aylin's method was incompatible with his in that they produced different graphs.

3.1.4. *Episode 4*

123I: OK what do you think about Cem's theory?

124T: [He reads Cem's statement.] Are both graphs the same for the positive x values? ... [He looks at the graphs of $f(x)$ and $f(|x|)$ in Figure 2]. ... both graphs appear to be the same. ... for the positive x values. ... [He rereads Cem's statement] "There is no difference". ... yes. ... there is no difference. ...

125I: Do you think he is right?

126T: There seems no difference for now. ... but I mean we have to consider the whole theory to come to a decision. ... [He rereads Cem's statement and reads aloud the second part] (...) yes. ... umm. ... negative x values for the graph of $f(|x|)$ [He reads the second part once again]. I think we can say something about the graph of $f(|x|)$ for the negative x values. ... because only $f(x)$ I mean when the x values are negative, then we take this part symmetrically in a line which parallels to the y -axis. ... so we already obtain the difference between these two graphs. That means that we can say that only a symmetry would be the difference. ... [He looks at the two graphs obtained for the first question]. ... but the difference between $f(x)$ and $f(|x|)$? ... [A long pause]. ... Well, of course, the difference is evident. ... I mean while $f(x)$ is linear, the graph of $f(|x|)$ is something like the shape of V. ... but one arm of the V is symmetric at the negative x values. ...

127I: I am not too sure if I understood you right. So let's go through what Cem says together; he [Cem] says that $f(x)$ does not change for the positive values of x which is this part

128T: Hmm yes that's right

129I: But for the part of $f(x)$ corresponding to the negative x values we cannot say anything, it depends on the equation of $f(x)$; I think he means that one needs to substitute some values of x .

130T: Yeah?

131I: Only then we can find the other part of the graph of $f(|x|)$, he [Cem] says.

132T: But we found [the graph of] $f(|x|)$ without substituting x values. I mean in the last task and for this task we found the same graphs when we substituted and when we took the symmetry. So I think we can obtain the graph without substituting but only taking the appropriate symmetry.

133I: Well, how can we convince Cem that you're right? I mean why Cem isn't right but you are? Is it simply because what you were saying was right for a couple of examples that you solved and you have seen the pattern only for those examples? Maybe Cem also found this theory while working on several examples which contradicted what you are saying?

Obviously we do not have the right to say ‘ok my rule is right simply because it worked on several examples’ while talking about mathematics!

134T: Yes, actually. . . yes you’re right. . . Well, first of all, I remember that for the positive x values the graph of $f(x)$ remains absolutely the same, well I don’t know if I can say ‘absolutely’.

135I: OK.

136T: But for the negative values of x , it was enough to take the symmetry. In fact we made use of analytic geometry for the solutions so. . .but I am not sure if what I am saying is definite. . .I am confused. . .

Three aspects of Tugay’s work in this episode strike us as noteworthy: he was correct; he was not confident when his statements were challenged mathematically; and he did not appear to see how the various mathematical features he has introduced interrelate. Tugay explained his ‘V’ construction well in 126T. 127I–131I re-examined Cem’s statement and Tugay (132T) explained that substitution was not necessary. The interviewer then (133I) questioned the generality of both Cem’s and Tugay’s statements in mathematical terms. Tugay’s responses when challenged (134T and 136T) were hesitant (not “absolutely”, “not. . .definite”, “I am confused”). This hesitancy even applied to his claims about $f(|x|)$ for positive values of x which he had previously drawn without any hesitancy as well as his claim that the graph of $f(|x|)$ for negative values of x could be found from the graph of $f(x)$ alone without the need of an equation. We believe this hesitancy is tied up with the fact that he did not relate his V rule to mathematical structures known to him: features of absolute value, of symmetry and of linear graphs. Although his V rule arose from these known structures, there is no evidence here that he has developed interrelationships amongst these concepts and the structure of $f(|x|)$, i.e. he has not developed a ‘web of meaning’ (Noss and Hoyles, 1996, Chap. 5) and that without this he was not confident when challenged mathematically.

3.1.5. Episode 5

Between 87I–100T (not shown) the interviewer led Tugay to look at the graph of $f(|x|)$ from a different perspective. He returned to the first graph and examined $f(|x|)$ for ± 3 and ± 1 and they noted that $f(|x|)$ has the same value each time.

151I: What would be the reason for this?

152T: I think this is because of the absolute value sign, I mean it is outside of the x .

153I: That means. . .

154T: That means. . . regardless of the sign of the values of x , they will be matching the same value of y .

- 155I: What does this tell us about the symmetry?
- 156T: So it tells us perhaps that all of the graphs of $f(|x|)$ are symmetric in the y-axis. Actually I remember that I told something about it on the second task but I did not realise today though.
- 157I: Perhaps? When you say perhaps I feel [Tugay interrupts].
- 158T: Well yes I mean I need to look at once again! [He examines the graphs]. . . yes, all of the graphs must definitely be symmetric in the y-axis because different values of x with different signs must have the same value of y , which is why it must be symmetric in the y-axis.
- 159I: So if we are talking about the symmetry in the y axis, then what does it tell us about the position of the graph of $f(x)$ in relation to the graph of $f(|x|)$?
- 160T: Yes, umm. . . now I am quite sure that we can say everything clearly about the position of $f(|x|)$ in relation to graph of $f(x)$
- 161I: If we return to Cem's arguments, what would you tell him?
- 162T: Now it is evident that we can say that there is no difference between the graphs of $f(x)$ and $f(|x|)$ at the positive x values. At the same time, we can say surely that the part corresponding to the negative x values must be the symmetry of the ray which is on the right side of the y-axis. So Cem is wrong. I mean we can say how to obtain the graph even without needing an equation.

We return to two of the issues discussed after the previous episode: language and interconnections. There is a noticeable change in the confidence of Tugay's language in this episode compared to the previous one. The language of the previous episode was marked by hesitancy and uncertainty. In this episode, however, especially in 158T, 160T and 162T Tugay used expressions such as "must definitely", "quite sure" and "surely" all of which signify self-confidence. The interviewer's focus on $\pm n$ values did not provide Tugay with new knowledge but enabled him to focus his attention, for the first time in this task, on this symmetric aspect of $f(|x|)$ and this focus allowed him to connect his knowledge about absolute values and symmetry with regard to $f(|x|)$. This interconnection or web of meaning, we believe, prompted his increased mathematical certainty.

3.1.6. Episode 6

Tugay moves on to Arzu's argument. He reads this and immediately responds:

- 164T: No, it is not so. . . I mean when we take the symmetry of the graph [of $f(x)$] at positive values of x in the y-axis we obtain the $f(|x|)$.

The interviewer thrice challenges Tugay but Tugay is sure about his statement.

171I: Well, why does the part of $f(x)$ at positive x values remains unchanged? How do you justify this mathematically?

172T: Why does positive x values not change? Because. . . I mean every value will be positive in the absolute value. . . it does not matter for positive values whether they are in the absolute value sign or not because it is positive anyway so they do not change. But the negative values differ if they are in the absolute value. I mean when they are in the absolute value sign, then they change, they alter into positive. . . and thus result changes. . . so when one substitutes, for example -2 for x in the $f(x)$, then one would obtain a different result from the result of $f(|x|)$ when one substitutes -2 . . . because $|-2|$ is a positive value and this is -2 in the $f(x)$. So they are totally different.

173I: So, for positive values of $f(|x|)$ [Tugay interrupts him]

174T: OK, let me put it another way, in the $f(|x|)$ when we substitute positive values we obtain a result which is the same result as of $f(x)$. But if we substitute negative values in the equation of $f(|x|)$ we get a different result from the result of $f(x)$ when the same negative values are substituted in the $f(x)$.

175I: Which shows that?

176T: Yeah, that proves that the graph of $f(x)$ at the positive x values is exactly the same graph as the graph of $f(|x|)$. On the other hand, as the negative x values change in the $f(|x|)$ so does the graph of $f(x)$ when transformed into the graph of $f(|x|)$. . . I think I made my point, right?

177I: Well, yes, I mean I think so, but I wish to hear from you how we can obtain the graph of $f(|x|)$ from the graph of $f(x)$ once again because I am a little bit confused after so much discussion.

178T: Actually we can obtain the graph of $f(|x|)$ in two different ways. The first one is that umm. . . we can draw a parallel line to the y -axis through the intersection point of $f(x)$ and y -axis. And then for the negative x values we can take the symmetry of that part of graph in this line. Secondly, well. . . I think this is easier. . . that is we can take. . . umm. . . I mean the graph of $f(x)$ at the positive x values. . . remains the same; I mean we can take the symmetry of this part in the y -axis and cancel the part of $f(x)$ at the negative x values. . . and so this is $f(|x|)$. . . yeah. . . yes definitely so.

Tugay quickly and assuredly evaluated Arzu's claim after one reading. He stated his disagreement with Arzu eloquently and precisely and described a correct way to obtain the graph of $f(|x|)$. His quick and assured articulations throughout this episode contrast strongly with his hesitancy at

the beginning of the protocol. Further to this, he made general statements about the graphs of $f(|x|)$ without relying on specific examples (see 178T). Indeed, there is a shift in the relationship between specific examples and general statements. Early in the task, e.g. 52T, he explained through examples but in this episode he used examples to illustrate his already made general statements, e.g. 172T. His structure of $f(|x|)$ was also flexible in the sense that he was able to talk about it in general terms, emphasised different aspects and even defined two different ways to obtain the graph of $f(|x|)$ from the graph of $f(x)$ (174T, 176T and 178T). He was self-confident in his claims and able to defend them when challenged (172T). The substantial difference in the quality of Tugay's arguments and explanations in this episode could be considered a result of his increased familiarity, by virtue of time spent on the task, with the new structure. We see more than this; we see the established interconnections as resulting in a qualitative jump in understanding. In Episode 3, Tugay realised that the graph of $f(|x|)$ can take negative values (can be under the x -axis); in Episode 5, he understood that the graph of $f(x)$ for positive values of x remains unchanged, that this can be reflected in the y -axis to obtain the graph of $f(|x|)$ for negative values of x and that the segment of $f(x)$ at the positive values of x remains the same in the graph of $f(|x|)$. Each of these aspects has undergone a great deal of examination but the product of this work, we believe, evidenced in this protocol is greater than the sum of the parts.

3.1.7. Episode 7

Tugay proceeds to Question 4. He quickly and correctly comments on each of the six graphs. For brevity we only provide protocol excerpts from a and d (see Figure 1). These excerpts, we believe, clearly illustrate the precise and assured nature of his mathematical explanations and indicate that he has consolidated his construction of $f(|x|)$.

188T: [He is talking about the graph a.] ... the part that corresponded to positive values of x is taken symmetrically in the y -axis, which means that it could be the graph of $f(|x|)$.

210T: [He is talking about the graph d.] ... it is the graph of $f(|x|)$ obviously symmetry is on the y -axis, the y values for the positive x 's do not change, and also, $x = -2$ and $x = +2$ take the same value of y . So yes clearly it is the graph of $f(|x|)$.

3.2. Tugay's use of $f(|x|)$ in the fourth task

Although we have focused on the consolidation of $f(|x|)$ it should be noted that Tugay also consolidated his construction of $|f(x)|$. In the fourth task,

Tugay achieved the intended construction of the graph of $|f(|x|)|$ by using the abstractions $f(|x|)$ and $|f(x)|$ in two steps: first by drawing the graph of $f(|x|) = g(x)$ and then the absolute value graph of $f(|x|)$ (i.e. $|g(x)|$). His recognition and use of these two constructions in this task (see below) suggest that the constructions of the first and second tasks have become abstractions (in our sense) which can be used in further constructions. Tugay explained how to obtain the graph of $|f(|x|)|$ from $f(x)$ as follows (NB: The numbering in these excerpts are taken from the fourth task protocols):

148T: OK by making use of $f(x)$. . . umm first of all we take the symmetry of the $f(x)$ at the positive values of x in the y -axis, in so doing we obtain the graph of $f(|x|)$. Then. . . after that if there are any negative values of y in this graph, then we take also the symmetry of these negative values of y . . . like as if drawing the graph of $|f(x)|$ and flip it up over the x -axis. We have to do so because there cannot be any negative values of y in the graph of $|f(|x|)|$. And so, we can obtain the graph of $|f(|x|)|$.

149I: How can you make sure that your formulation is correct?

150T: Well, OK. . . first of all according to this method there are no negative values of y in the graph of $|f(|x|)|$ because the absolute value sign which is outside of the whole expression makes it impossible for any negative values of y to exist, that is the first evidence. In addition, umm. . . we first find $g(x)$ which is $f(|x|)$, and we know that the graph of $f(|x|)$ is symmetric in the y -axis so our rule is correct once again.

4. DISCUSSION

We address five aspects of consolidation and abstraction. We begin by discussing the initial state of the absolute function construction and then changes that were observed during the consolidation process. In the third subsection we put forward the case that what is consolidated is the construction and that consolidation is an integral part of abstraction. The fourth subsection outlines issues in task design. Our final considerations are more speculative and concern interrelations in language development, use of examples and knowledge interconnections.

4.1. *The initial state of the new constructions*

Tugay's new construction was not consolidated when he started the third task in that he was not confident in its validity. He was able to describe how to obtain the graphs of $f(|x|)$ from the graph of $f(x)$ but he expressed feelings of 'insecurity' (12T) and he was not certain (10T) that this construction was a means of obtaining the graphs of $f(|x|)$. A hesitancy in

defending formulated constructions for a considerable period after their construction was common in all the protocols of students who made these constructions. In Tugay's case we can see his uncertainty reappearing as the interviewer probes different aspects of the graphs. In 136T, for example, he stated that he was not sure if his symmetry argument for negative values of x was correct and stated "I am confused".

In the early parts of the protocols of the third task students made extensive use of specific examples and these examples were used as a basis for formulating their ideas. Only students who consolidated the constructions in this task went beyond specific examples and then only in the latter parts of the protocols. This is not surprising but it does draw attention to an apparent need to ground the new constructions in concrete examples. In Tugay's case he stated, (44T), "I need to draw the graph of $f(x)$ first so that I can see what happens". He did not talk about the relationship between the graphs of $f(x)$ and $f(|x|)$ until he had drawn them (see Episode 2). The points and lines he constructed were prefaced with demonstrative adjectives in his discourse: "this point . . . this line" (52T) – he appeared to be unable to formulate his constructions in general mathematical terms free from specific lines and points. We take this, and the uncertainty noted above as evidence that the new constructions are fragile and need to be consolidated.

4.2. *Changes in the course of consolidation*

As Tugay worked on the third task he appeared to consolidate his construction of $f(|x|)$. We focus on: reconstruction of the construction, increased resistance to challenges, developing a language for the construction and greater flexibility.

It appears to us that Tugay reconstructed his construction $f(|x|)$ in the initial stages of the third task. Reconstruction is a process in which constructions are derived as in past constructions, i.e. the constructions are not simply recalled. In the second episode, we see Tugay combining and manipulating various bits of information about absolute values, symmetries and graphs. This process continues throughout the third task. For example, later in the protocol (172T) we see his justification that the graph of $f(|x|)$ is the same as the graph of $f(x)$ for positive values of x by combining bits of information and actively reorganising them. We do not equate reconstruction with consolidation but reconstruction appears to be an important part of consolidation.

The fragility of new constructions, we believe, makes students reluctant to use them to counter challenges. In the course of consolidation, however, students begin to resist challenges by establishing interconnections between

the new constructions and established mathematical knowledge and by reasoning with these constructions.

We believe that interconnections between Tugay's new constructions and his existing knowledge were not sufficiently well established at the beginning of the third task and that this situation contributed to his uncertainty in his claims about his new constructions. However, later in the protocols he established interconnections between the graph of $f(|x|)$, absolute values, symmetry and linear functions (see Episodes 5 and 6). In 172T, for example, Tugay explained why $f(x)$ and $f(|x|)$ were the same for positive values of x by establishing connections between the graph of $f(|x|)$, absolute values and $f(x)$. Shortly after a change in the tone of his assertions can be noted, "that proves . . . I think I made my point" (176T). This change to a confident tone continues from 176T, e.g. compare "yes definitely so" (178T) with "I don't know if I can say absolutely" (134T). This aspect of this construction, that $f(x)$ and $f(|x|)$ are the same for positive values of x , appears to be fully consolidated as he used this to confidently elaborate how to obtain the graph of $f(|x|)$ (174T, 176T and 178T).

Apart from the confidence of Tugay's language, as the construction $f(|x|)$ was consolidated, there was a qualitative shift in the clarity and precision of his language in the course of the third task. We believe that language development (to describe a new construction) has a dialectical relationship with the consolidation of the construction. For example, at the end of the second task (recall Tugay's explanation of $f(|x|)$) and at the beginning of the third task (10T and 52T), Tugay's description of the graph of $f(|x|)$ lacks precision and is slightly ambiguous whereas in 178T his mathematical language is precise and unambiguous. A lack of precision in the initial part of the third task is not surprising, but language development during the task is significant with regard to consolidation in that the language of the new constructions needs time to develop.

Students' use of examples is closely related to this development in their language of the construction. Prior to consolidation, students appear to need concrete examples to formulate their thoughts but after consolidation they appear to use examples to demonstrate assertions. Tugay, for example, in the second task and in 52T, articulated his thoughts by referring to specific properties of graphs. In 172T, however, he used examples to convince the interviewer, to clarify and to justify his assertions.

The use of specific examples to articulate thoughts suggests to us that the new constructions are somewhat inflexible. When Tugay, for example, was asked to give an account of the graphs of $f(|x|)$ (52T) he appeared to begin stating a general rule, "after x becomes negative", but then focused on a specific graph. Later in this protocol (174T), however, he quickly provided an alternative way to view $f(|x|)$ – and did so without recourse

to a specific example. The phrase “let me put it another way” along with the confident and precise way he stated this other way, suggest to us that he has consolidated this construction and was using it flexibly.

4.3. *Consolidating an abstraction or a construction?*

We return to the amendment to RBC theory sketched at the end of the *Introduction* section: that what is consolidated is the construction, not the abstraction, and that consolidation is an integral part of abstraction. Indeed, we have been careful in our use of language throughout this paper to attend to this subtle but important distinction; but on what basis do we propose this?

We see this distinction as more than a pedantic focus on words but as a matter at the heart of this theory of abstraction. RBC theory is based on Davydov’s Marxist theory of thought which essentially proposes a theoretical reproduction of reality.

Theoretical thought can reproduce its objects only through consideration of its *development*. The point is that only then can there be a grasp and a rational expression, not only of the existence of certain things and their properties, but also of their *possibility*, as such, with a subsequent determination of the conditions of their manifestation in a certain form, though necessarily a *general* form (Davydov, 1990, p. 278).

Theoretical thought in the form of an abstraction transforms how we interpret phenomena and our activity. An unconsolidated construction cannot be used to create new theoretical knowledge (because it has not been consolidated!) and so is not, in the Davydovian sense of the term, an abstraction. This is acknowledged by Hershkowitz et al. (2001) and by Dreyfus and Tsamir (2004) but they continue to speak of consolidating an abstraction. There is evidence that an unconsolidated construction cannot be used to create new theoretical knowledge in the pilot work reported on in the second section of this paper. In the initial third task, pilot study students did recognise $f(|x|)$ and $|f(x)|$ but they were not able to use them to construct $|f(|x|)|$; in the main study all scaffolded pairs and individuals consolidated their $f(|x|)$ and $|f(x)|$ structures in the revised third task and used these structures to construct $f(|x|)|$ in the fourth task. This is evident in Tugay’s protocols. At the beginning of the third task the new constructions were fragile; by the end of the third task Tugay was using them flexibly; in the fourth task Tugay used his $f(|x|)$ and $|f(x)|$ structures to construct the graph of $|f(|x|)|$ for a specific $f(x)$ and later (the fourth task: 148T) clearly explained how to do this for an arbitrary linear function. In Davydovian terms Tugay has grasped the structures $f(|x|)$ and $|f(x)|$ and determined the conditions under which they transform $|f(|x|)|$ into

an object of theoretical thought, i.e. $f(|x|)$ and $|f(x)|$ are no longer mere constructions but are abstractions.

4.4. *Task design*

Task design is a matter of critical importance in mathematics education research and in instruction. We have already made a number of comments on the design of tasks for consolidating a construction. In this section we consider two principles for task design in relation to consolidating a construction and note one feature of the task which, we feel, increased its utility as a research task.

Consideration of the problems experienced in the pilot study in the light of Dreyfus and Tsamir's (2004) work led us to believe that tasks which created opportunities for learners to discuss mathematical aspects of the construction would be important in consolidating the construction. This led to the design of Question 3 in the revised third task and the evidence in Tugay's protocol on this question (Episodes 3–6) certainly supports this belief. But the discussion between Tugay and the interviewer was not just any old discussion, it concerned claims that Tugay had to think deeply about in order to evaluate. He had to reason and express his reasons. We consider issues of language further in the next subsection but, for now, conclude that creating opportunities for discussion grounded in mathematical reasoning, to develop a language for the new construction, is an important feature in designing tasks to consolidate a new construction.

Our second consideration on task design for consolidating a new construction is that tasks should be designed to optimise students' progression from undeveloped structures to fully developed structures (in the language of Davydov (1990) that the tasks assist students' 'ascent to the concrete'). This was a feature of our tasks (we consider the third task, see Figure 1). Question 1 presented a specific function but Question 2 withdrew the support this specific function provided. Tugay's protocol in Episode 2 certainly suggests that Tugay was not comfortable with this support being withdrawn but it appears to have forced him to attempt to think beyond specific examples. Question 3, as noted immediately above, was designed to facilitate discussion and the structural interconnections he made in Episodes 3–6 were significant. Beyond issues of language there resides, in Aylin, Cem and Arzu's claims, a need to focus on linking different structural features (symmetry, positive/negative x -axis sub-domains, absolute values). Question 4 was similar to Question 3 in that Tugay had to focus on structural features of his construction, this time expressed graphically instead of written statements. As it happened, Tugay had consolidated his constructions by the time he reached Question 4 and found this question relatively

easy, but if he had not done so, then it is likely that it would have assisted him in making Davydovian real interconnections.

A feature which, in our opinion, increased the utility of the task for drawing $f(|x|)$ for our research purposes is the fact that graphs of $f(|x|)$ can be generated by two different methods: (i) by reflecting $f(x)$, for $x \geq 0$, in the y -axis; (ii) by reflecting the part of $f(x)$, for $x < 0$, in the horizontal line through the y -intercept of $f(x)$ but keeping the part of $f(x)$, for $x \geq 0$, unchanged. This is not the case, in general, for an arbitrary non-linear function. Tugay constructed the second method towards the end of his work on the second task. In considering the claims presented in Question 3 of the third task, however, Tugay focused his attention on mathematical aspects relevant to the first method and, as reported in Episodes 5 and 6, constructed this as a method for drawing $f(|x|)$. We believe that having two valid methods to draw $f(|x|)$ was useful for our research purposes.

4.5. *Language, examples and interconnections*

Our final focus in this section is our perception of a dialectic between the development of a language to describe a new construction, the use of examples and knowledge interconnections. This subsection is more speculative than other subsections but we feel the issues are worth commenting on even if our ability to do this is wanting.

We agree with Vygotsky (1978, pp. 22–24) that speech has an organisational role in the development of knowledge. Our comments on language in the first two subsections above focused on the confidence of Tugay's language but the development of a language to describe Tugay's construction can be discerned. Early on (10T and 52T) we see a mathematically developed language to describe intersection points, parallel lines and equations of lines but not for the newly acquired construction. When the construction is consolidated (178T and thereafter), however, although there are many commonalities with 52T, Tugay's language becomes less procedural and he has new mathematical objects to talk about. In 52T Tugay uses procedural terms ("first we find. . . then we draw. . .") but in 178T he states "we can draw a parallel line. . . we can take the symmetry. . ." which, we feel, indicates that he sees "not only of the existence of certain things and their properties, but also of their *possibility*" (Davydov, 1990, p. 278). Further to this, in 150T (of the fourth task), Tugay's procedures have become a method which he can talk about, ". . . according to this method. . .", and the absolute value sign has become something which can be employed in his reasoning, ". . . because the absolute value sign. . . it is impossible to exist any. . .". Hershkowitz et al. (2001) see language developments such as these as an indication that a construction has become an abstraction, "Generally

if the construction is an abstraction, the learners develop in parallel a language for expressing their new knowledge and using it to explain or justify" (ibid., p. 212).

With regard to reasoning there is, as noted above, a developmental change in Tugay's use of examples in the third task. Examples appear essential to Tugay's explanations in the second task (recall his account of $f(|x|)$ at the end of the second task) and in the early part of the protocol, e.g. 52T; they are a denotational reference for tying meaning to the construction. As the construction is consolidated meaning appears to reside in the structural links established and examples, if used at all, take on the role of illustrating these structural links (e.g. 172T). Tugay's development with regard to examples is parallel to his development with regard to language and we believe they are linked, as Vygotsky (1978) notes with regard to action and speech:

Initially speech follows actions, is provoked and dominated by activity. At a later stage, however, when speech is moved to the starting point of an activity, a new relation between word and action emerges. Now speech guides, determines, and dominates the course of action (ibid., p. 28).

But, as Noss and Hoyles (1996, p. 124) note, "Mathematics is more than action in the sense of activity-with-objects. It is activity with relationships. . ." This brings us back to Davydov (1990, pp. 302–303) and structure, that the "ascent to the concrete, from the undeveloped to the developed" is accomplished "through real interconnections among phenomena". We recognise that Noss and Hoyles' interconnections are wider than Davydov's but strong similarities in the two approaches are present. This similarity was implicitly seen by Hershkowitz et al. (2001) who, while critical of the clarity of Noss and Hoyles' link between webbing and the construction of new knowledge, recognised the similarity of their 'webs of meaning' and 'situated abstraction' constructs to RBC theory. We believe that establishing interconnections between new constructions and existing knowledge structures is a necessary aspect of abstraction (in our sense of the word) and that without this it is impossible to achieve the vertical reorganisation of mathematical knowledge which RBC theory posits. It does not matter (indeed, it is expected) that these interconnections are not initially well-established. Further research and theoretic analysis is clearly required.

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NOTES

1. We follow the authors in calling this a model. We are not sure that this is the best word to describe their creation; it could be argued that 'approach' or 'theory' are better words.
2. We regard the issue of task design, in mathematics education research and in instruction, as an underdeveloped area of major importance (see also Artigue, 2002). It emerged that task design is also crucial in considerations of abstraction and consolidation. We come back to the issue of task design in the Discussion section.
3. This is for reasons of space and because matters relating to consolidation are more apparent in Tugay's work with $f(|x|)$ as he found this more difficult than $|f(x)|$. We refer, at times, to $|f(x)|$ but only to retain the naturalness of the interactions with Tugay (as he was working on both functions).
4. An important consideration that will be the focus of a later paper.
5. 'Taking symmetry of' is the verbatim translation of the Turkish term 'Simetriğini almak'. Students are taught to use this term for mathematical reflections. The term could be translated as 'reflecting' and native English speakers may prefer this translation. However, we prefer the verbatim translation as the term 'reflecting' corresponds to the Turkish word 'yansımak' which is usually used in vernacular speech rather than in mathematical discourse.

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