

STUDENTS' UNDERSTANDING OF THE NOTION OF FUNCTION IN DYNAMIC GEOMETRY ENVIRONMENTS

ABSTRACT. To make optimal use of computational environments, one must understand how students interact with the environments and how students' mathematical thinking is reflected and affected by their use of the environments. Similarly, to make sense of research on students' thinking and learning, one must understand how the environments and contexts used in the research may affect the conclusions one derives.

The research on students' learning of functions has approached the topic in terms of symbols and graphs (see, for example, Leinhardt et al. (1990) for a review of work up to that point; Harel and Dubinsky (1992) for a collection of research; and Dugdale et al. (1995), for some recent thinking about implications for curriculum reform using technology). Dynamic geometry environments (DGEs) like Cabri Geometry or Geometer's Sketchpad, offer us an opportunity to get a new perspective on these old and important issues. DGEs let students build geometrical constructions and then drag certain objects around the screen in a continuous manner while observing how the entire construction responds dynamically. In this way DGEs model functional relationships that are not specified by symbols or represented by graphs.

Based on interviews with undergraduate mathematics majors, this paper presents preliminary observations that confirm some old results and raise some new questions about students' notions of function.

1. INTRODUCTION

Dynamic geometry environments (DGEs)¹ are microworlds² in which learners create geometric constructions, specify mathematical relationships (like perpendicularity or congruence), and then adjust the figure by dragging certain points (or other objects) around on the screen. As the points move, segments may stretch, shrink, or rotate as the entire figure responds dynamically. In some cases, a movement of a point causes a rigid transformation only: a rotation or translation of the entire figure. In other cases, one sees non-rigid transformations: certain relationships among the figure's components may change while others remain invariant. With few and specialized exceptions, the dragging transformation preserves only those relationships that were explicitly specified by the user in the construction, and relationships that are direct mathematical consequences of these. Incidental features change freely. This gives students a

special opportunity to study the *essential* properties of their construction – for example, properties that are true of *all* triangles, regardless of their shape – and to investigate the logical implications of a set of constraints.

It is perhaps little surprise that these new features change how students experience geometry. But, despite the explicit intent of DGE designers to change only the mode by which one investigates the familiar domain of Euclidean geometry, even some aspects of the *geometry itself* are altered from what they are in paper-and-pencil work. Laborde (1995) expands this idea: “[*Cabri* geometry] may be very sophisticated in comparison with a paper and pencil situation. . . . [Hence,] designing a problem situation in a computer based environment requires a new analysis of the mathematical objects, operations and didactical feedback involved in the situation” (p. 37).

In at least two ways, geometry software with or without the dynamic feature (used prudently) can help students focus their mathematical thinking on bigger and often more abstract mathematical ideas than is common in paper-and-pencil work. For one thing, such environments allow students to manipulate and investigate constructions without being mired in the technical aspects of drawing them, as can sometimes happen if one must sketch on paper. Furthermore, construction on paper virtually forces one’s attention (at least initially) on the actual lengths and positions of objects. Geometry software designed for experimentation renders these features changeable, and leaves one to focus on more abstract relationships, and invariant (often qualitative) properties of geometric entities – properties that are independent of the (often quantitative) particularities that one cannot easily avoid in a paper work.

The addition of dynamic dragging can shift one’s attention in yet another way. By giving one a strong feeling of operating a smoothly functioning mechanical device – one in which the mathematical objects behave as if they were physical, obeying the laws of mechanics and conforming to our intuitions about movement in a continuous space – DGEs evoke some of the curiosity people have about how things “work,” and can also provide flexible opportunities to investigate the world of continuous functions.³

Although the literature on DGEs contains thoughtful work on topics such as conic sections (Scher, 1996) or ideas from trigonometry (Capponi, 1992), most of it discusses the learning of classical topics in geometry, with relatively less attention to other mathematical domains. Little of it is about functions; where functions are mentioned, the topic is either the properties of specific functions, or methods by which specific properties of certain classes of functions can be investigated. For example, Capponi (1992) describes how the dynamics of *Cabri* enables students to gather

data for exploring the invariance of the cosine function; Laborde and Laborde (1995) look at geometry as a modeling tool in numeric or algebraic domains.

We do *not* make the assumption that the “Idea of Function” is a single, internally consistent thing. Like so many other ideas, the idea of function tends to have a distinct face in distinct environments. Despite the fact that the concept of function has a formal mathematical definition, the ideas and connections that come to a person’s mind – even when that person is an expert – are not controlled by the definition alone, but are sensitive to the context in which the person is working and reasoning. In this paper, we attempt to describe how work in a DGE can contribute to the understanding of Function in two ways:

1. As a context within which students may explore functions without reference to algebraic language or graphs, it can help broaden the picture that *researchers* have of students’ understanding of function. Where do students see functional relationships in their DGE experiments? What properties of these functions do they recognize? What do they treat as the variables of these functions?
2. For the same reasons, it can also broaden students’ ideas of function by extending what Vinner (1983) calls students’ “concept image.”

Section 2 of the paper gives an example of how DGEs can shed new light on students’ thinking about functions. Section 3 discusses how the notion of function can be viewed from a DGE perspective. Section 4 outlines our research methods – the constructions the students investigated, the questions we asked, and the way the data were analyzed. Section 5 presents the results, including an analysis of four specific ideas central to a robust understanding of functions: the nature of the independent variable, constant function, undefined function, continuous function.

2. HOW THE FUNCTION CONCEPTION CAN BE VIEWED FROM A DYNAMIC GEOMETRY PERSPECTIVE

An example will help to illustrate the ideas that we will develop further in Sections 3 and 5 of this paper. It shows some of the differences between the thinking about functions one sees in an algebraic context and how one might interpret the notion of function in a DGE.

Asked for his ideas about *function*, Arvin, an undergraduate mathematics major who is planning to teach, described a function in terms of a “function machine.”

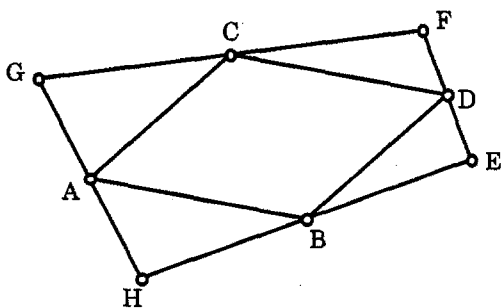


Figure 1a. A quadrilateral based on a parallelogram – The original figure.

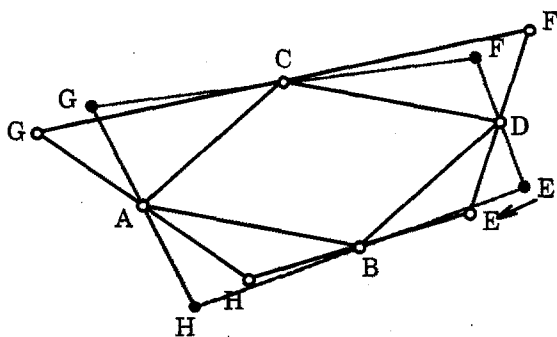


Figure 1b. A quadrilateral based on a parallelogram – The effect of dragging E.

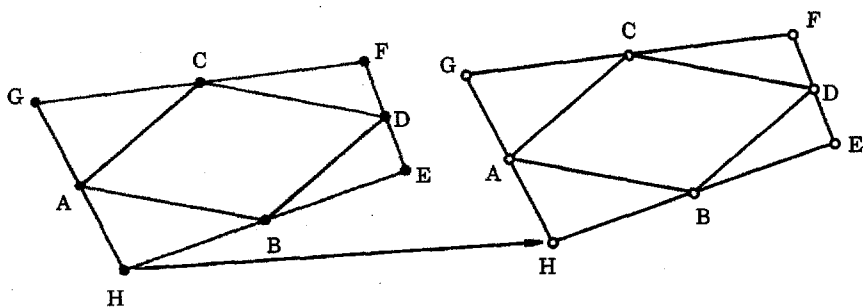


Figure 1c. A quadrilateral based on a parallelogram – The effect of dragging H.

Arvin: I always remember it as what I was taught in seventh grade. It is a machine that you throw in an x and you get out something which is a y normally, and it does something to it. [...] We drew elaborate pictures of these machines that chug and chug until it would spit out something that was different, or maybe the same.

In a DGE, there are no x s or y s to invoke, and so one must rethink functions in order to *find* the independent and dependent variables and the “machine” that relates them. The following excerpts illustrate this idea. They are taken from a discussion of a figure in which a parallelogram $ABDC$ is taken as a basis, point E is an arbitrary point, and quadrilateral $EFGH$ is constructed by reflecting E through D to generate F , then F through C to get G and so on (see Figure 1a).

Because E is arbitrary, it can be moved freely and the effects are well defined to affect only F , G , and H . Figure 1b shows what Arvin sees when he moves E a little bit. By contrast, as a consequence of the way the figure was constructed, F , G , and H are treated as fixed with respect to the other points in the figure. In Cabri, such points cannot be moved at all. In Sketchpad, the assumption is that the entire figure is free with respect to the plane, and so movements of F , G , or H translate the entire figure. Figure 1c shows what Arvin sees when he moves H . Arvin notes this effect and then moves E again to see the figure's change. The following excerpt describes his thoughts about these two movements:

Inter: Can you see a function in this [dragging E], or what prevents this movement from being a function?

Arvin: It's a function because the movement of E changes the position of F , G , and H . It changes the angles of F , G , and H . It changes the distances. So there is a definite function to be seen. [...]

A characteristic of a function, as I see it, is that if you change your x you're going to get a different y . If you change your initial condition – if you change what you are putting in – your output should differ. . . . What I'm calling the output is the positions of F , G , and H , and they change. Whereas, when I move H , nothing changes in respect to each other so if you want to say it in terms of a function, the only function involved with H is the position of the field [= “the position with respect to the plane”] which probably doesn't have anything to do with anything. Whereas, [...] it [E] has to do with the position with respect to each other at least for the other three points that are

moving. There is a dependency. F , G , and H are dependent on E but nothing is dependent on H .

Inter: So when you drag H you do not see any function here?

Arvin: Right. Unless you would qualify a function as a movement of the figure in the field, but that is not what's going on with the rest of the picture.

Inter: Can you describe the conditions when you can see a function?

Arvin: [...] *That at least one thing on the screen is dependent upon the movement that you are making or the change that you are making*, just like when you have an independent axis and a dependent axis. The y changes because the x changes. [emphasis added]

There is really a lot to think about in this one excerpt. For one thing, Arvin is telling us that his concept *image* of function (despite the definition which he stated earlier, which includes “or maybe the same”) requires change (“[When I move E], the positions of F , G , and H , [...] change, [so there is a definite function]. Whereas, when I move H , nothing changes ...”). In addition, his concept image of function requires dependency and, even more than that, causality (“the y changes *because* the x changes”).

It is also clear that he is imposing an intrinsic perspective (“position *with respect to each other*”). He recognizes that when he moves H there is a change, but only with respect to “the field” – an extrinsic perspective. But because the change is not *with respect to* any parts of the figure itself, he rejects that extrinsic perspective, saying that “the position of the field ... probably doesn’t have anything to do with anything.”⁴

The fact that DGEs may evoke ideas like the ones that Arvin expresses (e.g., the dependency in change and a tendency to view from an intrinsic perspective), has pedagogical importance. Such perspectives contain the seeds of a variety of mathematical ideas, and hence may invite one to look more deeply into the mathematics involved and can form a basis of very stimulating classroom discussions.

3. HOW FUNCTIONS CAN BE VIEWED FROM A DYNAMIC GEOMETRY PERSPECTIVE

There are several transformational tools available in the various DG implementations – free dragging of points, rotations, reflections, translations by specified vectors, and so on. Of all these, we will consider in this section

only that first tool, the feature that defines DGEs: the ability to drag points freely while watching the construction responds dynamically.

Dynamic change in a construction can be interpreted as a function in at least two distinct ways.

From one perspective, the movement of the user's hand from a specific point in $\mathbf{R}^2$ to another specific point in $\mathbf{R}^2$ along a particular path is the function that transforms the object on the screen from one state (pre-image) to another state (image). The input of a function of this kind is a construction (complete with its placement in $\mathbf{R}^2$) – one element drawn from the space of all possible constructions that could be on the screen.

From another point of view, the rules that determine a dependency, captured in a construction, defines a function. Such functions relate values in $\mathbf{R}^2$ – a position of a point – to other values that may also be in $\mathbf{R}^2$ (the position of another point), or in $\mathbf{R}$ (a measurement performed on some feature), or even in higher dimensions (e.g., the position, slope, and length of a segment). The output varies as the input is dragged, according to the dependencies determined as the construction was built. Other points of the figure can be considered as parameters: changes in their position define related members of a family of functions.

These two perspectives are summarized in the table below.

	View 1	View 2
Function/ transformation	The movement of the mouse, dragging a point (or other object).	A construction that relates values in $\mathbf{R}^2$ to values in $\mathbf{R}^n$ ($n \in \mathbf{N}$, $n \geq 1$).
Defined on	"Construction space," in which each element is a construction (and its placement in $\mathbf{R}^2$). The construction before dragging is the preimage.	$\mathbf{R}^2$
Taking values in	"Construction space." The construction after dragging is the image under the transformation.	$\mathbf{R}^n$ ($n \in \mathbf{N}$, $n \geq 1$)

These perspectives already suggest some ways in which DGE may alter the way students can explore, manipulate, and discuss functions. For one thing, what is meant by the "behavior" of a function is no longer a verbal label that one abstracts analytically from various static forms (symbols, tables, or graphs): it has the active nature that we associate with the word "behavior" in other realms, and it is directly observable and open

for experimentation. Moreover, DGEs can expand the nature of the sets that functions connect: students can meet, for example, a function whose domain is a set of points in $\mathbf{R}^2$ and a function whose range is a set of squares. Such examples also help disconnect the notion of function from a set of algebraic symbols. The latter is *one* way that *some* functions can be expressed by, but it does not exhaust the possibilities. In addition, DGEs can expand students' understanding of certain *properties* of functions. For example, ideas about continuous functions from $\mathbf{R}$ to $\mathbf{R}$ can be extended to include continuous functions defined on plane figures.

4. RESEARCH BACKGROUND

The observations reported in this paper represent preliminary findings on students' conception of the notion of function in DGEs, a component of a broader research project on students' thinking in DGEs. Among the questions we had about the influence of DGEs on student thinking, the following four topics guided our research design on the understanding of functions.

- Functions** To what extent do the students recognize functions in these geometric situations? How do students see and come to see the various functional relationships among elementary geometric objects?
- Behavior** In dynamic geometry, two figures that look the same and specify the same geometric relationships among their components may still have very different behavior as their parts are moved. How do students recognize, describe, and think about behavior?
- Entity's role** In paper-and-pencil geometry, all objects are static, but in dynamic geometry, they have behaviors. How do students (come to) see the different functional roles (independent and dependent variable, for example) that elementary geometric objects might play?
- Domain** How do students come to understand restrictions in the domain of a geometric function if at all?

We designed three figures that the students were asked to explore (these figures will be described in detail below.) To focus students' attention on the mathematical behavior, each figure was composed of relatively few objects. For the initial research, we prepared the figures in advance so that the students would not be distracted with the details of their construction.⁵

a. Methodology

Guided by our intention to explore the territory broadly enough to select the most promising locales for deeper investigation, we wanted to see what features of the environment *students* attend to, care about, or find surprising, and to hear how they describe what they see. Consequently, we thought that at the first stage of the research, described in this paper, it is preferable to work with students who have the mathematical background and sophistication to field a variety of questions, and are articulate and confident enough to tell us what they think. Accordingly, we selected three bright and successful undergraduate mathematics majors in their last semester before graduation. All three were considering a future in teaching.

Each interview started with a short demonstration of one of the DGEs, in which students became familiar enough with the basic tools of the software to build simple constructions. Then, we invited the students to explore the figures we had already prepared. For each figure we were interested in what the student observed *prior* to dragging – what objects, configurations, and relationships stood out. Before dragging, we would ask for conjectures about what *might* happen. While many kinds of behavior were possible – and the students were aware that almost anything *could* be – it was useful to know what students expected or were capable of imagining. Our general questions, as students dragged objects around, focused on what students were seeing and thinking about; what varied; what objects, measurements, or relationships remained invariant; how the relationships might be explained in terms of the geometry; and where their observations might be described in terms of functions.

Even while the students investigated the constructions that we had already built, the discussion often led to situations in which students asked (or were encouraged) to construct other figures, to add elements to the figures, or to measure various components of the figure under discussion, in order to understand better what they were looking at.

The interviews were videotaped from two directions – one camera facing the computer screen and the other facing the interviewee to capture expression and gesture. Three tapes resulted: a tape of each camera's image for maximum resolution, and a combined tape to provide a synchronized view of the screen and the interviewee's actions and responses.

After the audio portions of the interviews were transcribed, we asked the students themselves to go over these transcripts with the composite videotape and refine the transcript in two ways. We invited them to comment on their own thinking in any way they liked in a separate column, without changing the original transcript. Also, so that the transcripts could be more easily understood without further reference to the video, we asked them

to insert the essential visual information that was missing from the audio (e.g., figures and references to locations in the figures that were expressed only with words like *this* or *there*).

b. Figures

The first construction (Figure 2) was designed to focus on the role of geometric Entities. It consisted of a line through two “free” points *A* and *B*, and a second line parallel to it through the free point *C*. A fourth point, *D*, was placed on the line through *C*, freely movable along that line, but constrained to remain upon that line. Movements of *A* or *B* can change the slope of the line and consequently of $\overline{CD}$, which is constrained to pass through *C* and remains parallel to $\overline{AB}$. But movements of the other two points have different effects. Movements of *C* can change the distance between the two lines, but affect neither their slope nor the distance between *D* and *C*. Movements of *D* only change the distance between *D* and *C*. One may see a variety of functional relationships in this figure: for example, the slope of $\overline{AB}$ determines the slope of $\overline{CD}$; dragging a point moves a line.

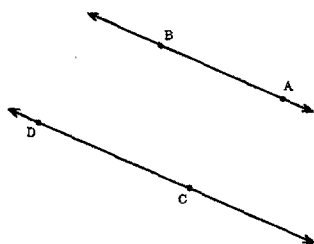


Figure 2. Construction 1 – Two parallel lines.

On the surface, this figure may seem quite simple, but its behavior is different than that of a similar construction on paper. On paper, one feels a symmetry between the two lines. Even the term “parallel lines” reflects this symmetry. Likewise, all points seem to have the same role. However, in a DGE, components that seem quite alike may behave differently. The different roles surprised the students, and afforded a good opportunity to talk about the relationships among the mathematical objects involved.

The second construction (Figure 3) was designed primarily to investigate the **Behavior** question.

Each of constructions 2 and 2a, adjusted to look the same when first encountered, shows a quadrilateral with midlines that form the sides of a parallelogram. In the left-hand construction, the vertices of quadrilateral *ABCD* are completely arbitrary and can be moved freely, while the other four points are determined as the midpoints of *ABCD*’s sides.⁶ In the other

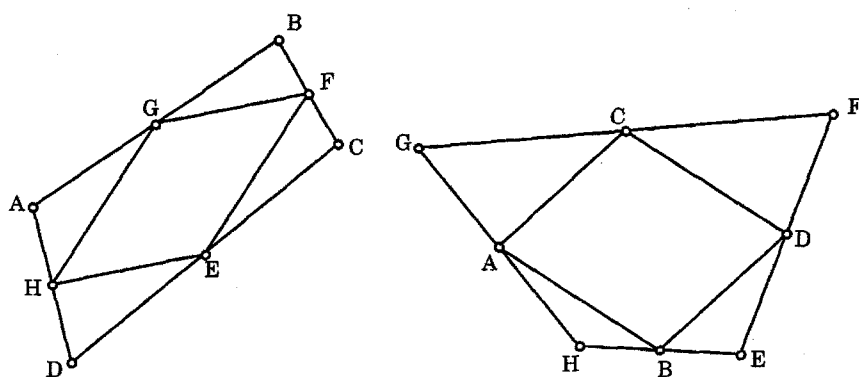


Figure 3. Constructions 2, 2a – Parallelogram inscribed in a quadrilateral.

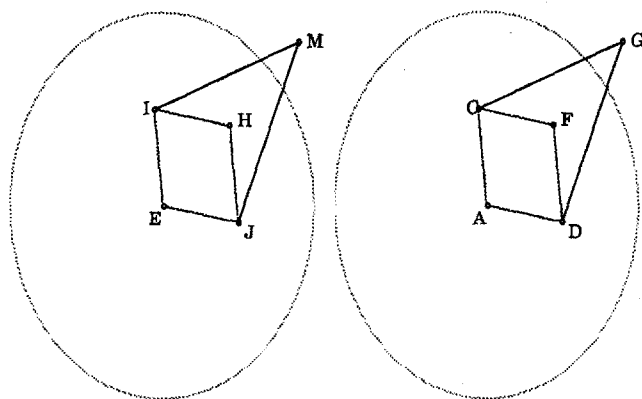


Figure 4. Constructions 3, 3a – Collinearity in a rhombus, a kite, and a circle.

case, the parallelogram is determined first. Three of its vertices (A , B , and C) are completely free; the fourth is then constrained by the restriction that $ABDC$ be a parallelogram. One vertex of the “circumscribed” quadrilateral (E) is also free, but the locations of the other three vertices (F , G , and H) are then completely determined by $ABCD$ and the location of E .

Our **Behavior** question was straightforward. While these two figures do not differ in their appearance or even the static relationship among their parts, they do differ in the behaviors they exhibit when points are moved. We wanted to know how students would explain the differences, what would they attend to and how they would describe their observations.

The third figure was designed to investigate the **Behavior** and the **Domain** questions.

In constructions 3 and 3a (Figure 4), as in constructions 2 and 2a, we presented the students with look-alike figures with nearly identical geometrical relationships, but different behaviors under dragging. Each consists of a kite and rhombus sharing two sides. This construction ensures that three points are collinear along the kite's axis of symmetry.

In the left construction, the sidelength of the rhombus is fixed at k units, and the two longer kite sides are fixed at $2k$. Points E and H can be moved freely (staying collinear with M). Within limits, this reshapes the rhombus, but if E and H are more than $2k$ apart, this creates a contradiction. I and J cannot then exist, and so everything that depends on their existence vanishes. In effect, while H can be moved anywhere, certain "functions" of H (measurements or relationships one might watch as H is moved) are defined only over the domain consisting of the set of points P such that $EP < 2k$. A light-colored, dashed circle of radius $2k$ around E bounds that domain. No explanation is given, but the student can observe that if either E or H is moved in such a way that H falls outside of the circle, everything suddenly disappears except for the circle and the two points E and H .

In the right-hand construction, the location of A determines the position of the center of a hidden circle, and also the size of that circle. Points C and D can be moved freely along the circumference of that circle. As no conflict is generated by this situation, there is never a disappearance.

5. STUDENTS' THINKING ABOUT FUNCTIONS IN A DYNAMIC GEOMETRY ENVIRONMENT

There is an extensive literature on students' conceptions about the notion of function (Breidenbach et al., 1992; Cuoco, 1993; Goldenberg, 1988; Harel and Dubinsky, 1992; Leinhardt et al., 1990; Piaget, Grize, Szeminska and Bang, 1977; Sfard, 1992). A common finding is that in solving problems about functions, students are at their best in performing numerical evaluation or prescribed algebraic manipulation, but tend to encounter difficulty in analyzing or even describing function behavior.

This difficulty seems to fade when functions (even **R**-to-**R** functions) are represented dynamically (Goldenberg, Lewis, and O'Keefe, 1992). Especially when numerical information is suppressed, the most salient remaining feature of the display is the functions' behavior. When movements of the student's hand (computer mouse) represent (control) movements of the function variable, and when the function value is observable as a separate coordinated movement, students spontaneously notice and invent ways of talking about and classifying different kinds of change, and even change-per-change. They invent informal ways of talking about qualitative

(topological and transformational) features of function behavior, and are receptive to more precise vocabulary for these features.

Dynamic representation of functionally related change, relatively free of numbers, is essentially the situation in DGEs. As far as we know, none of the research on student learning with geometrical software in general or DGEs in particular has especially focused on understanding of the concept of function. Interviews with the developers of DGEs show that their design decisions were not influenced by thoughts of how the environment might affect students' learning of ideas about functions. Despite this lack of deliberate intent, the developers of these software tools provided an unusually rich environment in which students may expand their notions of function, and may develop and explore variety of related ideas.

The many differences between algebraic (or graphical or tabular) and dynamic-geometric representations of function provide opportunities for alternative experiences that can, with appropriate reflection, greatly expand students' ideas of function. We shall organize these ideas around the following four issues:

- a. The nature of the independent variable
- b. Constant function
- c. Undefined function
- d. Continuous function

a. The Nature of the Independent Variable

One idea central to an understanding of function is the role of dependent and independent variables. Trying to explain what the "independent variable" means in mathematical language, Arvin uses a metaphor of power and government to describe construction 1. To help himself "understand the dynamics behind what points and lines *govern* other points and lines" (emphasis added), he begins by constructing yet one more line parallel to the two we had provided (Figure 5).

Arvin: B dictates the movement of all the lines because they have to all remain parallel.

Int: Why does B dictate movement, while D does not?

Arvin: Maybe AB was just the first line that was drawn and so everything is dependent on that line because we drew a line parallel to AB and then draw a line parallel to DC which has to be parallel to BA . It [line BA] seems to be like the leader.

When Arvin reviewed his own transcript, he maintained the metaphor of "govern," "dictate," and "leader," and provided the additional explanation

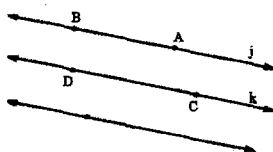


Figure 5. Arvin investigates Construction 1 in terms of functions.

that D didn't "have the same authority as B ." It is interesting to note how this metaphor fits the aim for which the figure was designed, i.e., to focus on the role of geometric entities.

Arvin's metaphor shows us that the language a student uses while talking about mathematical issues may reflect associations not solely mathematical. In fact, even when students use mathematical terms, their use may be colored by everyday meanings, as can be seen in the following excerpts.

Arvin: [*Describes construction 2a*] I'm assuming that $ABCD$ is independent of $EFGH$ meaning that any time I move E , F , G , or H , it doesn't affect $ABCD$ at all but E is the only one that I moved that it affects the others at all.

Arvin's use of the word "dependent" in the earlier excerpt could be interpreted in its everyday sense or as a reference to the "dependent variable" – the common and technical meanings did not clash in this case. But his use of "independent" in this new excerpt cannot be taken in both ways. Here, "independent" refers to the function *value/output* – what we regard as the *dependent* variable – and means that it is constant, not dependent, unchanged regardless of changes in the independent variable.

Tammy expresses a similar sense in discussing construction 2:

Tammy: GH and EF are the same length. Let's see if they stay the same. [*She experiments.*] Yeah. They're always the same length.

Int: And what does it mean?

Tammy: It means that I can't change the length of one, independent of the other. If I change the length of GH it changes the length of FE .

Her use of "independent" is, again, not to refer to the independent variable, but to the linkage between the values of two different functions whose independent variable happens to be the same.

Besides the meaning captured in the term "independent variable," "independence" has other meanings even within mathematics. The students are

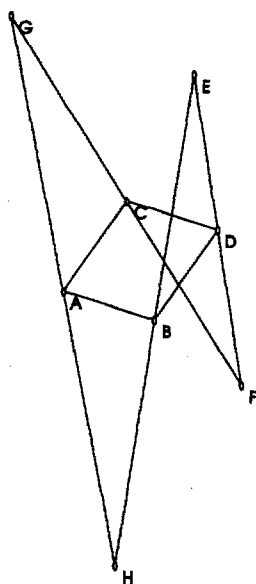


Figure 6. Tammy investigates Construction 2a in terms of functions.

not misusing the word, nor are they necessarily confused about variables. But, these observations do show that one must listen closely to know whether students' correct uses of technical vocabulary indicates knowledge of the technical meanings. In particular, in casual classroom discourse, and in the absence of the kind of evidence provided by Arvin's (correct) use of "independent," his (correct) use of "dependent" might have been taken as sufficient indication that he understands the dependent/independent variable distinction. While his use of "independent" does not show that he *lacks* such understanding, it does show that *we* lack the evidence.

Where the above examples might just be a matter of acquiring language and clarifying potential ambiguities, Tammy, in the excerpt below, expresses concerns about the very nature of the independent variable. She is quite comfortable seeing the behavior of construction 2a in terms of a function, and attempts to describe how what she calls "the position of *F*" is determined by *D*. Recall that in this construction, movements of *D* translate *ABCD* rigidly but leave *E* fixed. As a result, movements of *D* (which is to say, all of *ABCD*) determine the positions of *F*, *G*, and *H* in accord with the requirements of the construction. Figure 6 shows what one sees when *D* is dragged.

This entire mechanism was completely clear for Tammy. In particular, she knew that *DF* is equal to *DE* and is, in that sense, dependent on *DE*

and so a function of DE . But, in attempting to describe what to “put in” the function, she is uncertain whether to think of that independent variable as D ’s *position* or as *its distance from* some reference object:

Tammy: Let’s take the function between the . . . – it’s not so much the *length* between D and anything that matters, but the *position* of D . Well, I guess that *is* the length. E is the fixed point, so [the variable has] got to be the length. [. . .] Well, I don’t know whether I should talk about position or length. If I talk about length, the input is the length of DE , and the output is the length of DF . So that’s one of those functions where you just put in one number, you get out the – it’s $x = y$ function, $y = x$.

Tammy seems reluctant to treat *position* as the independent variable. This may be a result of a lack of a clear enough definition of position, or perhaps because “position” isn’t close enough to her perception of what an independent variable may be.

b. Constant Function

The sense of “independence” that Arvin and Tammy identified (in the previous sub-section) appears to be a factor in determining where students recognize functional relationships. In this sub-section we elaborate more on this recognition of functional relationships (one of the topics that guided our design of the figures) and extend what is discussed in Section 2: How the function conception can be viewed from a DGE perspective.

In the following excerpt Julia discusses construction 1 (Figure 2). She manages to rethink and rework her ideas about which-is-a-function-of-which easily in a DGE context where none of the objects are numbers.

Int: Okay, D is a function of A . Do you see another function of A ?

Julia: The placement of the lines are a function of A . B and C remain fixed so I’d say that B and C are not functions of A . Nothing changes with B and C , I don’t think. No, they stay the same, but yes, the placement of the lines are a function of A .

She further seems quite comfortable even with vaguely defined function values as “the placement of the lines.” Yet, by saying “ B and C remain fixed so I’d say that B and C are not functions of A ,” she appears not to include constant functions of A as functions.

On the one hand, this may simply reflect common, non-mathematical usage: what mathematics calls “a constant function” is, in common parlance, a situation in which the outcome “is not a function of” what

is being varied. But we have observed another phenomenon that suggests that the issue here is not just one of language. For some students, it may be that there is a two-part qualification for them to perceive functional relationships.

For some object B to be a function of A , not only must B change in response to changes in A , but it must also change *its relation to* the independently varied A .

Julia illustrates the first half of this qualification. In her example, B remains fixed as A moves, and thus B 's position *in relation to* A is altered as A moves. But, because B does not move itself, " B " (presumably meaning " B 's position") does not change, it is not considered as a function of A .

In the following excerpt, Arvin illustrates the second half of this qualification. Recall that in Construction 3 (Figure 4, left) H is a freely movable point, while attempts to move I , J , or M translate the entire figure.

Arvin: The only time that I see a function, or an interesting function, is when I move H because any other point that I move *just moves*. . . . When I move I or J or M , it moves the entire picture.

He says "just moves" as if to indicate that the point does nothing else: it just moves. In fact, of course, it moves *everything* with it, but in such a way that all spatial relationships within the figure (and in particular, to the dragged point) are preserved. By saying "or an interesting function," Arvin indicates that he accepts the idea that one *might* describe functions of I , J , or M , but he treats this merely as a grudging second thought, which he then proceeds to disregard.

To consider a particular dragging as a function, Julia requires the first condition (the dependent object moving) and Arvin requires the second (the dependent object changing relationship with the independent object). For Julia, there is no function when the dependent object does not move, even though its relationship with the dragged point changes; for Arvin, there is no interesting function when the relationship does not change, even though the dependent object moves.

There is a second typical confusion about constant functions upon which work in a DGE can shed some light. Typically, when students are asked to define a constant function, they give examples like $y = 5$, but they also commonly confuse it with the identity function, giving the example $y = x$. In many cases, both answers are given without understanding why they are examples of constant functions.⁷

In analyzing Construction 1 (Figure 2), Julia demonstrates these two interpretations of “constant function”: $f(x) = x$ or $f(x) = a$.

Int: When you move $D \dots A$ doesn’t move. Can you see here a function?

Julia: I guess no, not really. Now I am trying to think of other functions [...], in what kind of functions nothing happens. Well, $f(x) = x$. [...] Something does happen still, though. I mean, it still changes. Yes, so I would say D is not a function.

Julia’s reference to functions in which “nothing happens” helps to explain why students might confuse constant functions with the identity function.

In a *constant function*, the value of the independent variable does not affect the value of the dependent variable. The analogue in a DGE is that the movement of the mouse (no matter what it drags with it) does not affect the position of whatever point or other measurement is being considered as the dependent variable.

In the case of the *identity function*, the values of the independent and dependent variables are the same. In a DGE, this amounts to two objects that share the same location: as one is moved, the other moves with it.⁸

What our students seem to be showing us is that:

1. The patterns $y = b$ and $y = x$, though mathematically different, are conceived in a similar way – both seem to “do nothing.”
2. In a DGE, but *not* in algebraic environments, relationships that might be expressed algebraically in the form $y = x + b$ are also considered as do-nothing functions, or, as the students put it, *not* functions.

Let us review this second observation. In the relationship $y = x + b$, the value of the dependent variable is changed in the same way as the value of the independent variable: Regardless of the value of the parameter b , if you change the value of x by b , then y will also change by b . The analogue can be seen in a DGE when two points are related in the (vectorial) relationship $B = A + u$. As the value (position) of the independent variable A changes, the position of the dependent variable B changes in the same way – if the mouse moves A by a vector v , then (regardless of v) B will translate by the same vector v . The result is that the position of B *in relation to* A does not change.

Usually, when students are asked to give *algebraic* examples of *functions*, they tend to present examples like $y = x + b$. The particular mathematical analysis we suggested above gives the geometric situation a parallel structure, i.e., $B = A + u$. Nevertheless, when our students encoun-

tered geometric relationships of such a $B = A + u$ form, they regarded them as “do-nothing” *non*-functions. Why would students treat these two situations so differently?

An obvious answer is that they are not imposing the same mathematical analysis on the visual display. It is possible to create symbolic representations in the same form for the two situations, but the students are not facing the symbols: they face the value-pairs that the symbols represent. When one faces a table of numbers, the constant difference between y and x is not apparent until one performs a computation, and so one's attention is drawn to the *process* that changes an x value into a y value (the addition of b). It is that numerical transformation that students see as a function. In the geometric case, both A and B have an independent existence, and B shadows every movement of A . The relationship between their values is *visually apparent* and one does not have to perform any computation to discern the relationship nor does one sense any *process* that “transforms” A to B .⁹ This explanation implies that the students' different approach towards the similar algebraic and the geometric relationships depends on the level of activity required of them in order to get the function to produce outputs.

So, at least in some DGE constructions, situations that might be represented algebraically as $f(x) = b$, $f(x) = x$, and $f(x) = x + b$ appear to be perceived in a similar way: “nothing happens” or “nothing interesting happens.”

c. Undefined Function

We designed Construction 3 (Figure 4, left) to raise discussions both about behavior and about the domain over which a function is defined. We did so by designing the figure in such a way that almost all of the figure vanishes when H meets the circle (Figure 7, a–c).

Despite some confusion, Julia succeeds in connecting this phenomenon with an undefined function:

Int: Can you describe this phenomenon in terms of a function of H ?

Julia: Where H meets the circle, outside edge – gosh. At that point, at that point, when they meet the circle, well J becomes I , [. . .] I think they become the same point. And I'm not sure how to describe it as a function. . . . I can't sort it out. The J and the I – they become the same. And from that point on, they are the same, I believe. I can't see them, but I think they're the same. But then J , I and M disappear.

Int: Mm-hm. So what is happening there?

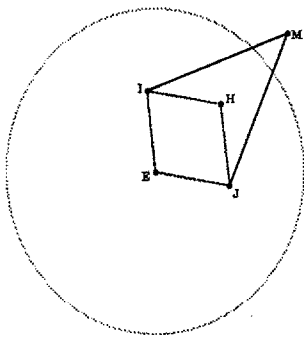


Figure 7a. The initial position.

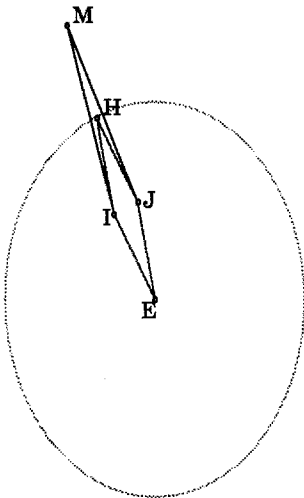


Figure 7b. A “moment” before H meets the circle.

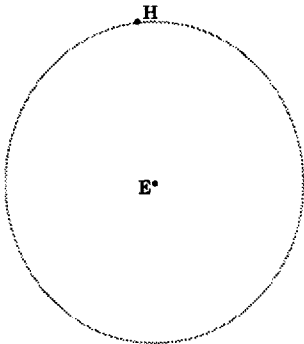


Figure 7c. When H meets the circle.

Julia: Well, I guess the function is undefined. The func – when, I guess you could describe it as when H is outside of the circle the function, the – J , I and M are somehow undefined. They don't exist. Not sure why. . . .

Int: Hm. That's interesting. That's like . . . you gave an example of that with just numbers before, but I'm trying to remember the example.

Julia: The square root of x ?

Int: Yes, that's it.

Julia: Square root of x when it's a negative number. The function is not defined. OK, so in that example there are values of x for which the function's not defined. In this example there are values of H , for example when H is placed outside of the circle, for which the points J , I and M are not defined.

Julia continued developing the analogy between the function $f(x) = \sqrt{x}$ and the movement of H . We believe that when students and teachers take the time to analyze properties of mathematical objects or ideas that are being compared, and to examine them from various points of view, then there is a real opportunity for learning. In particular, the act of discerning what two superficially dissimilar situations have in common – in this case, examining common features of the algebraic function $f(x) = \sqrt{x}$ and the geometric construction – is an act of abstraction. If done thoughtfully, it can serve to exercise and strengthen students' abilities and inclinations to perform such mathematical abstractions, and can increase their familiarity and comfort with higher levels of mathematical abstraction (Hazzan, 1996).

d. Continuous Function

Although the idea that DGEs might make important contributions toward students' thinking about functions was not apparently on the minds of the developers of DGEs, ideas about continuity were. For example, Jean-Marie Laborde, one of the developers of Cabri, is clearly applying the epsilon/delta reasoning to dynamic pictures when he says:

We want to allow mappings to have properties of continuity. If you make a small variation [to get from one configuration to another] then, if it's possible, you should just have a small variation of the image. As a consequence, the two configurations seem, to the user, to be "close": a small variation (of the dragged object) gives small deformation (of the entire picture).

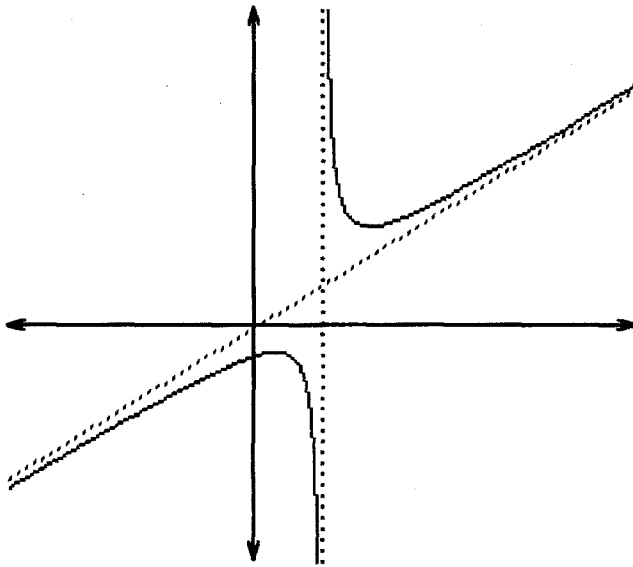


Figure 8. A rip as a sign for a discontinuity in a Cartesian graph.

However, it seems that how one perceives continuous change depends very much on the tools and environment by which one investigates the change.¹⁰

Students' perception of a function like $f(x) = x + \frac{4}{x-3}$ will help us to discuss what students perceive continuity to be.¹¹ The usual Cartesian graph shows the rip at $x = 3$ that is likely to be perceived as a discontinuity (Figure 8).

By contrast, imagine the same function displayed dynamically as a pair of markers (Figure 9). One marker can be moved independently (controlled by mouse movements) along a line that represents the real value x , while the other of which slides in a coordinated way along a second line so that it always indicates the real value $f(x)$. When students investigate this representation, moving their x -marker from far left to far right, they see the $f(x)$ marker tracking closely at first. As the x marker continues to the right, the $f(x)$ marker increasingly lags behind x until $x = 1$, at which point the $f(x)$ marker stops following x and begins to race quickly back to the left. Moments later it zooms back in from the right to catch up with x (at around $x = 5$), at which point it dramatically slows down, reverses its direction again, and begins to stick closely with x again as they both march off to the right.

Students describe this kind of motion as continuous (Goldenberg, Lewis, and O'Keefe, 1992). One first-year high school student said that the

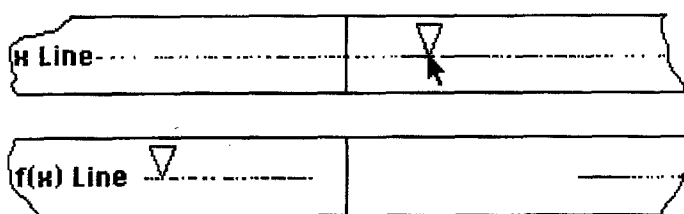


Figure 9. A dynamic display of a function as a pair of markers.

$f(x)$ marker “shot off this side [left end of the number line], went around back of my head, and came back in on this [right end] side.” This informal description, spontaneously invented by a young student, contains within it the germ of a sophisticated mathematical idea: the notion of the real line as a closed-loop – the “one-point compactification of the real line.” By adjoining to the reals a single point at infinity (subsuming what are sometimes called “positive infinity” and “negative infinity”), and defining the function as a mapping from $\mathbf{R} \cup \infty$ to $\mathbf{R} \cup \infty$, this idea renders the function continuous (p. 216).

The following geometric construction (Figure 10) exhibits the same behavior. F is constructed as the intersection point of the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$, and so is also the center of a circle that passes through points A , B , and C . As C is moved along a trajectory that passes between A and B , such as the curved path shown in Figure 10, one can observe F moving southwest along the perpendicular bisector of $\overline{AB}$ with increasing rapidity, flying out of the picture at high speed, then rushing back in from the northeast, and eventually slowing down again.

There are at least two reasonable ways of interpreting what becomes of F at the instant in this journey when C is collinear with A and B . One way is to note that at such a time, the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ are parallel (or identical), and so F , their point of intersection, is undefined.

Another way starts with the observation that paths through all three points are *always* possible. One’s sense of continuity leads one to think of the special case as a “circle of infinite radius,” placing the center F infinitely far away (again, in either direction) along the perpendicular bisector of $\overline{AB}$.

The sense of point F as an object – a real thing that may approach, retreat, or hide, but whose very existence is constant – is very strong. Students tend to affirm its existence even when pressed. To illustrate this point we chose an excerpt taken from an interview conducted in the second phase of the research, in which we are working with high-school students. Here is David’s notion of where F is.

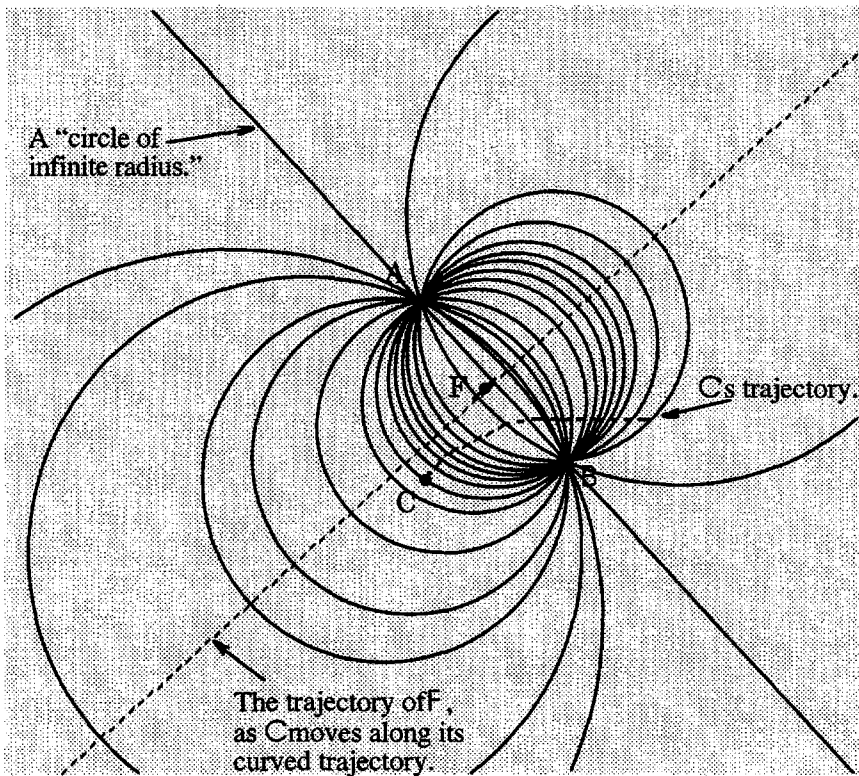


Figure 10. A journey of F , the center of a circle through points A , B , and C .

Int: [Asking about the case in which A , B , and C are collinear.] So about how far away is F now?

David: What comes into my head is that once it's gone [off the southwest corner], it's already here [just waiting outside the northeast corner]. It's unseen. It's on its way back.

Whether or not it is *good* for students to think in this way depends, as with any mathematical comparison, on the level of analysis they can give it. If done thoughtlessly, identification of circles and lines, and invocations of a point (or points) at infinity may be no more than ad hoc attempts to explain uninformed perceptions. But if students *can* and *do* give adequate thought to the suggestions they make, peculiar phenomena like this can be the starting point for relative advanced mathematical ideas, such as infinity and continuity. The more subtle issue of the point at infinity – anything dealing with infinity is not trivial – is harder to interpret cleanly in a single case. But kindred cases (e.g., the real-line-as-circle, parabola as ellipse

with one focus “at infinity,” and so on) can be treated as seeds for ideas to be developed more fully at later times.

6. DISCUSSION

Although what we observed is plausible even in a broader population than the one we interviewed, we do not make any statistical claims: our interviewees were too few and were deliberately chosen in a way that is not representative. The purpose of the paper is, instead, to promote a general principle of research, to suggest a new medium for the study of functions (and the study of students' conceptions of function), and to share a few insights about students' conceptions of function.

The general principle, not original with us, is that context and medium matter. If we want to understand how students think about functions, it is not sufficient to probe only within contexts, media, and language that match the ones in which they've acquired their ideas.

Perhaps even more importantly, if we want *students* to probe their understanding of functions – and in reflecting upon their observations, improve that understanding – they must have an opportunity to test their ideas in contexts that require a certain amount of reinterpretation.

The DGE is a medium that exhibits many of the important ideas we want students to construct in relation to functions, but does so without any of the familiar trappings: no algebra, no graphs, no numbers. The geometric context may provides enough contrast with algebraic contexts to allow essential aspects of the important ideas to be distinguished from features of the representation (as in Section 5c, the interpretation of the notion of undefined function). Moreover, while providing a novel territory within which to reexamine many ideas acquired through other means, it may also have special advantages for learning (or examining the learning of) ideas about *change* and *behavior* that tend not to be easy for students in static environments. Working in such environments may lead students to discuss function-related ideas in dynamic as well as static ways.

Students apparently perceive functional relationships in DGEs quite naturally. Their own words show that, without any explicit preparation, they quickly assimilated the idea of function to their observations of the effects of dragging objects around on the screen. DGEs may thus serve as a good tool to introduce, explore, and discuss the concept of function.

Finally, we tried to show how students' descriptions in the context of a DGE helped expand *our* understanding of their understanding of function. Specifically, we asked where students see functions in a DGE figure, how they think about a figure's behavior, how they interpret entities'

roles in a DGE, and how they understand restrictions in the domain of a geometric function. The answers shed some light on student's notions of the nature of the independent variable, constant function, undefined function, and continuous function. The observations are particular to the DGE, but suggest broader application to students' conceptions about these aspects of function in general.

7. EPILOGUE

The source of the ideas presented in this paper were interviews with undergraduate students. We are now in the process of interviewing high school students. Based on the responses of these two groups, our intention is to draw, in the future, a theoretical framework for the epistemology involved while working in DGEs. We believe that such a theoretical framework can help one analyze students' understanding while working in DGEs, and can also guide the preparation of curriculum material.

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NOTES

¹ There are many implementations, including: Geometer's Sketchpad (Jackiw, 1991, 1995); Cabri (Baulac, Bellemain, and Laborde, 1990, 1994); Cabri II (Laborde, 1995); Geometry Inventor (Brock et al., 1994); SuperSupposer (Schwartz and Yerushalmy, 1992); Euklid (Mechling, 1994); GEOLÓG (Holland, 1993); Thales (Kadunz and Kautschitsch, 1994).

² Papert (1980) discusses the idea of microworld in relation to Logo. For a common formulation of a microworld see Laborde and Straßer (1990).

³ See also Goldenberg and Cuoco (1996) for an analysis of some of the differences between geometry software environments without dragging and DGEs, both in terms of students' experience of the mathematics and in the mathematics itself.

⁴ This phenomenon may be viewed as another kind of evidence for Papert's claim that intrinsic geometries are more "natural" to students (Papert, 1980, pp. 63–68).

⁵ While much has been learned using pre-made constructions, there remain some research issues that are hard to settle in this way because suppressing the details involved in a construction can influence students' interpretation of what they are seeing. Follow up research, still in progress, took the alternative approach of having the students construct everything.

⁶ In Geometer's Sketchpad, attempts to move the midpoints in this construction result in a (rigid) translation of the corresponding segment. In Cabri, a similar construction would not permit movement of the midpoints. See Holzl (1996) for a further discussion about differences between points that can move and points that cannot.

⁷ This statement is based on the authors' teaching experience.

⁸ How one thinks about this depends somewhat on the representation one is using. One "natural" way to think about it on a DGE screen is two essentially indistinguishable objects. A different way is to imagine these objects represented each in its own space. In $\mathbf{R}$ to $\mathbf{R}$ functions, one might see x and y represented on their own separate axes – two distinct objects that are *not* in the same place, but have the same "value" as measured in their own spaces. The $\mathbf{R}^2$ to $\mathbf{R}^2$ analogue requires two separate planes, one for the independent variable and one for its image.

⁹ Of course, there is more that must be investigated here. The figures that Julia and Arvin were working with were complex, and perhaps the gestalt of the entirety of such a complex figure translating across the screen seemed more rigid and unchanging than would, for example, a figure consisting of only the two points A and B related as $B = A + u$. Background is also important. A figure consisting of only three points, in which two move in the relationship $B = A + u$, while C remains fixed, presents a comparison of the constant function with the translation function, and also provides the "with respect to" that seems to influence students' ability to perceive change, or what they regard as "real movement," on the screen.

¹⁰ In what follows we focus on situations that were not discussed by the three undergraduate interviewees. However, as continuity is an important property of functions that can be investigated naturally in DGEs, we find it appropriate to present the following analysis in this section.

¹¹ In our discussion here, we are not dealing with the mathematical issue of whether, in the neighborhood of $x = 3$, this function is to be considered continuous or not – a matter that is treated variously in different textbooks. Instead, we are dealing with what students perceive continuity to be.

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