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JUSTIFYING AND PROVING IN THE CABRI ENVIRONMENT

ABSTRACT. This paper describes a long term teaching experiment carried out with students from the 9th–10th grades. Geometrical constructions in the Cabri environment were selected as a specific field of experience, within which the sense of theory may emerge. The idea of construction constitutes the key to accessing the idea of theorem, moving from a generic idea of justification towards the idea of validating within a geometrical system. The study aims at clarifying the role of the Cabri environment in this teaching-learning processes: analysis of protocols shows the possible evolution of a justification into a proof but at the same time indicates that this evolution is not expected to be simple and spontaneous.

KEY WORDS: Cabri environment, geometrical constructions teaching experiment

INTRODUCTION

Among the rich collection of didactic software flourishing in the recent past, dynamic geometry softwares certainly take a central position. In the research projects devoted to studying their role in supporting geometrical reasoning, the general consensus is that they provide a revolutionary means for developing geometrical understanding and heuristics (Goldenberg and Cuoco, 1996; Laborde and Capponi, 1994; Hoelz, 1992). In particular, this software seems to make the exploration of geometrical configurations and the identification of meaningful conjectures more accessible to pupils (see also Healy and Hoyles, this issue).

On the other hand, there is no general agreement about the contribution of dynamic geometry in the development of theoretical thinking, and especially in the construction of a meaning for proof. The creation of dynamic geometry systems has caused concern amongst some educators (see the discussion of Chazan, 1993), who worry that the facilities, like “measure” and “dragging”, offered by the software could lead to the “further dilution of the role of proof in the high school geometry” (p. 359). Although we agree with some of the arguments supporting this opinion, we claim that a specific use of the software’s tools, as can be accomplished by the teacher in school practice, may reverse this tendency.



This paper concerns a teaching experiment which focused on the use of Cabri- Géomètre as a mediator of the idea of mathematical proof. Cabri- Géomètre was chosen for many different reasons, but especially because of its flexibility. As will be explained in the following sections, Cabri allows the teacher to adapt the microworld according to specific requirements of their educational objectives.

A TEACHING EXPERIMENT

The research project described in this paper began some years ago. It was conducted in the framework of a long-term teaching experiment, located in the 'research for innovation' paradigm, in which action in the classroom is both a means and a result of the evolution of research analysis (Bartolini Bussi, 1994, 1996). One of the main objectives was to investigate the feasibility of a teaching approach centered on the use of the microworld Cabri-Géomètre and aimed at developing theoretical thinking in geometry. Our hypothesis was that the teaching/learning process associated with this development can be expected to be gradual, thus the experiment involved following students through two years of study, corresponding to the 9th and 10th grades of schooling.

The research study was carried out through a strict collaboration between researchers and school teachers and the teaching experiment set up on the basis of a sequence of activities designed by the whole group. These activities were realized in the classroom by the teacher as a regular development of mathematics class, i.e. the experiment was included in the regular curriculum. The content of the geometry curriculum was not upset, but the general approach changed dramatically. The approach developed involved the integration of the software Cabri-Géomètre into classroom activity, not only as a didactic support, but as an essential part of the teaching/learning process. Three regular classes at the upper secondary school level, and from different schools, participated in the project. At the end of the first two years, a group of teachers decided to continue following the project, which of course has also developed in other directions. The analysis is carried out on the complex of information collected: notes from direct observation, pupils' protocols and transcripts of collective discussions.

Although it is difficult to describe the many aspects involved in a long-term experiment, the following section intends to present a general overview. The aim is to give a flavor of the classroom activities and an idea of the associated 'results' in relation to the particular issue of this paper: students' introduction to proof.

CHARACTERISTICS OF THE TEACHING EXPERIMENT

As far as the theoretical framework is concerned, the main elements characterising the teaching experiment are the following:

- classroom activities are organised within the '*field of experience*' (Boero et al., 1995) of *geometrical constructions* in the Cabri environment.
- The evolution of the field of experience is realized through the social activities of the class, aimed to a social construction of knowledge. The core of social activities is constituted by '*mathematical discussions*' (Bartolini Bussi, 1996).

The Field of Experience of Geometrical Construction

Since antiquity, geometrical constructions have had a fundamental theoretical importance (Heath, 1956, p. 124), clearly illustrated by the history of the classic impossible problems, which puzzled the Greek geometers so much (Henry, 1994). From the perspective of classic geometry, drawing tools, despite their empirical counterpart, may be conceived as theoretical tools defining a particular geometry. In this sense, classic Euclidean geometry has been traditionally called '*ruler-and-compass geometry*', referring to both the origin and limitations of its objects.

Actually, the theoretical meaning of geometrical constructions, i. e. the relationship between a geometrical construction and the theorem which validates it, is very complex and, as Schoenfeld (1985) describes, certainly does not seem to be immediate for students. Schoenfeld also argues that "many of the counterproductive behaviours we see in students are learned as unintended by-products of their mathematics instruction" (p. 374). Mariotti (1996) presents confirming evidence from a very different school context, and suggests that the very nature of the construction problem seems to makes it difficult for students to take a theoretical perspective.

The practical meaning, related to the possibility of the concrete realisation of a drawing, may critically interfere with the adoption of a theoretical perspective. In fact, any geometrical construction may be utilised to obtain a drawing with a certain guarantee of efficiency, but it is also true that '*impossible*' constructions, despite their theoretical impossibility, can be realised with arbitrarily chosen precision (take for instance, the case of the trisection of an angle, Henry, 1994, p. 104).

In spite of their long tradition, geometrical constructions have lost their centrality and, at least in Italy, completely disappeared from the geometry curriculum. Rarely can one find any reference to '*drawing tools*' when

geometrical axioms are stated, while geometrical constructions do not belong to the set of problems proposed in the textbooks.

The appearance of dynamic geometry software seems to be spurring a renewed interest for constructions, with their basic role brought on the scene by the instrumental approach related to the use of graphic tools. Cabri-Géomètre offers a microworld which embodies Euclidean geometry, with its elements and its properties (points, line, circles, but also midpoint, angle bisector, perpendicularity, parallelism, . . .); but, in particular, Cabri refers to the classic world of geometrical constructions. Any Cabri-figure is the result of a construction process, it is obtained after the repeated use of tools chosen from amongst those available in the 'tool bar'. Moreover, the effect of most of the Cabri tools corresponds to the effect of the classic geometric tools: a Cabri-figure is obtained by intersecting lines and circles, constructing perpendicular or parallel lines and the like.

But the construction facility is not the only characteristic which makes Cabri so interesting. There is also the possibility of direct manipulation of its figures, a manipulation conceived in terms of the embedded logic system, Euclidean geometry. Cabri-figures possess an intrinsic logic, the logic of their construction. The elements of a figure are related in a hierarchy of relationships, corresponding to the procedure by which they were built. These relationships can be made evident by the use of the dragging mode. Variations in the basic points (elements), that is those that can be dragged, constitute the data of the construction and what cannot be dragged constitute its results. The Cabri-figure is the composite of these elements, incorporating various relationships which can be differently related to definitions and theorems of Geometry.

There is also something more. The presence of the dragging mode introduces a specific criterion of validation for the solution of construction problems: a solution is valid if and only if the figure on the screen is stable under the dragging test. Because the dynamic system of Cabri-figures embodies a system of relationships, consistent in the broad system of a geometrical theory, solving construction problems in Cabri means not only accepting all the facilities of the software, but also accepting a logic system within which to make sense of them.

The Construction Task

From our considerations of the notion of geometrical construction in relation to the Cabri environment, we formulated the hypotheses that it may serve as a *key to accessing* the meaning of proof. In other words, we choose the 'construction task' as the core of all activities in our teaching experiment aimed to introduce students to proof.

Let us analyze the characteristics of the construction task, as it is presented to the students. When students are asked to construct a geometrical figure, two distinct requests are made:

1. a procedure aimed to obtain a specific drawing/figure (for instance a Cabri-figure);
2. a justification of such a procedure, explaining the reasons for its correctness.

The two requests correspond to two distinct parts in the expected written answers.

In the Cabri environment, the construction activity is integrated with the dragging function, that is, the construction of a figure can be associated with control by dragging. In this case, the necessity of justifying the solution comes from the need of validating one's own construction, in order to explain why it works and/or foresee that it will function. Of course, dragging the figure may be sufficient to convince one of the correctness of the solution, but at this point the second component of the teaching/learning activities comes into play. Construction problems also become part of a social interchange, where different solutions are reported and compared.

Collective Discussions

Alongside Cabri-construction activities, the introduction of '*mathematical discussions*' (Bartolini Bussi, 1998) forms a critical element of our experiment. These collective discussions play an essential part in the teaching/learning process, with specific aims of both cognitive (construction of knowledge) and metacognitive (construction of attitudes towards learning mathematics) natures.

In studies related to students' understandings of proof, the role of discussion has been analyzed and the difference between an argumentation and a proof has been clearly described (Balacheff, 1987; Duval, 1992–1993). Our proposal refers to a specific type of discussion, '*mathematical discussion*' defined as "a polyphony of articulated voices on a mathematical object" (Bartolini Bussi, 1996).

Mathematical discussion is not simply comparison of different points of view, its main characteristic is the cognitive dialectics between different personal senses and the general meaning (the terms are used according to Leont'ev, 1981) which is introduced and promoted by the teacher.

In the teaching experiment described in this paper, the cognitive dialectics concern the sense of justification and the general meaning of mathematical proof. The motive of the discussion activity relates to the evolution of the meaning of justification, associated with the problem of

construction. In other words, the approach to the construction problem within the Cabri environment is expected to introduce first, the need to justify a solution, then the need to negotiate the adequacy of this justification. The dragging test has a fundamental role in this process of justification. The fact that students can see for themselves when and which geometrical properties remain constant under dragging can help them to understand when they have produced something that merits justification (see also, Hoyles and Healy, 1999; Healy and Hoyles, this volume). In the subsequent mathematical discussions, it will become necessary to negotiate and introduce explicit criteria for the acceptability of the justification itself. Because Cabri constructions can be validated within the theory of Euclidean Geometry, our teaching experiment aimed to make the idea of construction evolve into the idea of theorem, i.e. the idea that the construction procedure must be validated within a geometrical theory.

According to our theoretical hypothesis, the teacher plays an important role in both stimulating a need to justify and the introduction of explicit criteria for acceptability. In fact, the dialectics which arise during collective discussion of different points of view, and correspond to different personal senses attributed to the idea of justification, could not spontaneously converge to a meaning of proof which is mathematically consistent. The teacher brings the 'voice' of mathematics and has the role of fostering the evolution of pupils' personal senses towards the mathematical meaning.

In this evolution, a basic role is played by Cabri. Its contribution cannot be understood, however, without an accurate discussion of the role of the teacher in the educational process. In the Vygotskian perspective, the role of the teacher is not limited to the organisation of the teaching environment (for instance, selecting the task to be proposed), the teacher's activities are also central in the process of meaning evolution which is expected to take place. Within the Cabri environment, the teacher can find specific tools of semiotic mediation (Vygotsky, 1978) which can be used in the discussion in order to guide students' personal senses towards the geometrical meaning of a construction. That is to say, the development of the teaching experiment is based on the process of semiotic mediation, accomplished by the teacher through the use of specific elements of the software.

Semiotic Mediation

In terms of Vygotskian theory, the figures and commands of Cabri may be thought as external signs of geometry theory, and as such they may become *instruments of semiotic mediation* (Vygotsky, 1978). This involves their use by the teacher in the concrete realisation of classroom activity and according to the motive of introducing students to theoretical thinking.

In the design of the teaching experiment, the sequence of activities developed in a structured manner, with activities within the microworld (construction tasks) alternating with activities of collective discussions. The idea is that, under the guidance of the teacher, students would construct a parallel between the world of Cabri constructions and geometry as a theoretical system. Let us describe how this parallel is conceived.

The Construction of a Theory

When a deductive system is concerned, there are two interwoven aspects: the idea of proof and the idea of theoretical system (both local and global theorization may be considered). In Mariotti et al. (1997), we referred to a *mathematical theorem* as the unity of a statement, a proof and a theory of reference. A theorem involves both the introduction of the idea of validation and the statement of the rules of such a validation. The acceptance of validation depends upon these rules and their meanings.

When students start work with Cabri, they enter into a geometrical system with specific rules and their meanings. When the whole Cabri menu is used, the whole Euclidean geometry is available; thus the theoretical system is a highly complex one. In fact, because of the richness of the 'geometrical tools' available, it is difficult to state what is given (axioms or 'old' theorems) and what must be proved ('new' theorems). The richness of the environment may emphasize the ambiguity about intuitive facts and theorems and may constitute an obstacle to the choice of correct elements of the deductive chain of a proof. In other terms, there is the risk that pupils are not able to control the relationship between what is given and what is to be deduced. For this reason, we decided to take advantage of the flexibility of the Cabri microworld and adapt the menu to our educational objective.

Thus, at the beginning, an empty Cabri menu is presented and the choice of commands discussed, according to specific statements selected as axioms. In this way, a double process is started, concerning, on the one hand, the enlargement of the Cabri menu and on the other, the enlargement of the theoretical system. New constructions are achieved in the microworld and, in parallel, the corresponding theorems are added to the theory: new elements are introduced by theorems and definitions, new commands are introduced in the menu.

Table I contains a short summary of the sequence, based on the parallel development of geometry theory and Cabri environment. On one side, one finds the theoretical elements discussed and enclosed in the shared theoretical system, on the other, one finds the new elements enclosed in the Cabri menu.

TABLE I

The parallel between Geometry and Cabri tools

GEOMETRY THEORY	CABRI MENU
- Definition of the primitive elements - Criteria of congruence in terms of construction and in the classic way	- Intersection of two objects (Creation) line, circle, segment, ... (Construction) Compass
Transport of an angle (Theorem)	Report Angle
Construction of the angle bisector (Theorem)	(Construction) Angle Bisector
- Definition of perpendicular line as angle bisector of a straight angle - Construction of the Perpendicular line from a point to a given line $P \in r$ $P \notin r$ (Theorem)	(Construction) Perpendicular line
- Definition of parallel lines - Construction of a parallel line to a given line (Theorem) - Axiom of parallel lines $r // s \iff$ internal alternate angles are equal.	(Construction) Parallel line
Note: What has to be included in the theory and what has to be added to the Cabri menu is always discussed collectively.	

Instead of proposing an already-made Euclidean axiomatisation, pupils were directly involved in the construction of both the Cabri menu and the corresponding geometry system. In this way, the system was built up, step by step, with a slowly increasing complexity which can be managed by pupils. According to our hypothesis, participating to that process is fundamental for the evolution of the meaning of theory.¹

EXAMPLES FROM THE CONSTRUCTION TASKS

There is insufficient space to enter into the details of the analysis of the protocols. Nonetheless, the following examples aim to give an idea of the evolution of meaning that was described in the previous section, from the initial approach to the construction task to a more elaborated approach to a geometric problem. The examples are drawn from the data of the same class, but they can be considered as exemplars of the results found in different classes. They show the complexity that students face in making their first steps in the theoretical world of geometry: even after students have abandoned an empirical attitude, there is evidence that a correct answer still presents difficulties.

As explained above, students begin working with empty Cabri menus that are gradually enlarged as a result of collective discussion. When the students began work on the first task to be considered, the menus included, besides the primitives of the creation menu, the construction commands 'Intersection of objects', 'Compass' (i.e. 'Report of length') and the 'Report of angle'.² From the theoretical point of view, this situation corresponds to the three criteria of congruence for triangles; something that students had already incorporated into their theoretical system and that they could refer to in their justifications. The following task is presented to the pupils.

Construct the Bisector of an Angle. Describe and Geometrically Justify your Solution

Alex and Gio (9th grade) described two attempts to solve the task (see Figures 1(a), 1(b)):

First Attempt

We took two points and we made a line pass through them, then we took another point C, which does not belong to the first line. We joined the point which doesn't belong to r1 with a second line, in so doing we determined an angle.

We transferred (*abbiamo riportato*) a segment AB, belonging to r2 and we transferred the same segment on r1 ($AB=AC$); we drew two circles (centre, point) center in C and point A and center in B and point A (*puntando in C e apertura AC e puntando in B con apertura AB*).

We joined A and D (line through two points). We took the intersection between the circle and the line, but . . .

FAILED!

Second Attempt

We drew an angle as we had in the first attempt. We drew a circle (centre/point), taking a point belonging to r1.

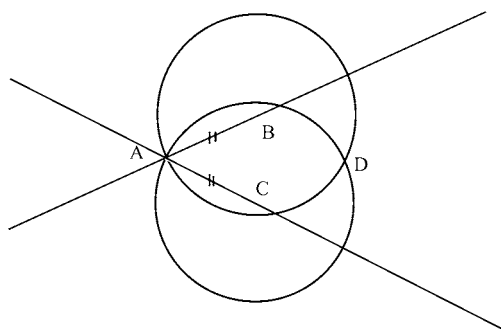


Figure 1(a). First attempt.

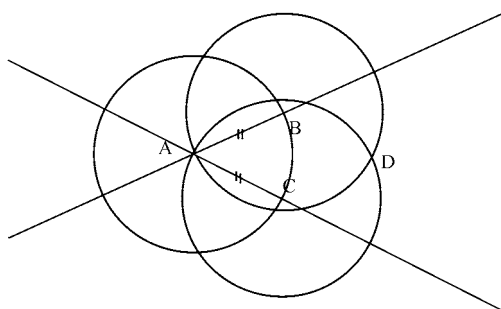


Figure 1(b). Second attempt.

This circle gave us the segments AB and AC belonging to r_1 and r_2 , which are equal because they are radii of the same circle. We drew two circles (centre B and C point A) using the intersection of two objects (of the two circles) we found the point D that we joined with A determining the angle bisector.

In the first attempt, something happened which led the construction to fail; despite what they wrote, the students made the two segments AB and AC equal ‘by eye’. The pair realised the construction had failed when it did not pass the dragging test; dragging had been accepted as means of verification, and they started again. In the second part, the text of the description is more accurate, as if, after the first failure, the students had felt the need of being more attentive to their construction procedure. At the same time, it presents a first rudimentary trace of a justification: “This circle gave us two segments AB and CD belonging to r_1 and r_2 , which are equal because they are radii of the same circle”. This sentence, inserted within the description of the procedure, cannot be considered a ‘proof’, nevertheless, it shows that the students had accepted the request for justification and tried to use the validating principles coming from the Cabri command that they used.

As explained above, as the sequence of activities progresses, theory is slowly enlarged; new constructions have become part of the theory. In general, at the point at which students encounter a new construction task, the theory contains some more elements. For example, after the construction of the angle bisector, perpendicularity was introduced as follows:

Line t is perpendicular to line s if and only if t is the angle bisector of a straight angle with the vertex on s .

Then the following theorem for the isosceles triangle was proved.

In any isosceles triangle, the angle bisector of the angle at the vertex is perpendicular to the opposite base.

The following task was then posed to the students:

Given a straight line r and a point P not belonging to it. Construct the perpendicular to r passing through P .

Alex and Gio's description of their solution to this task is presented below (see Figure 2):

We took a line r , passing through points A e B , then we took a point $C \notin r$.
Then we drew a circle (centre, point) having AC as its radius and then we traced it, centering in A . We drew a circle (centre, point) having radius BA and centre B .
We then determined points C e D with 'intersection of two objects' and we joined C and D .
 $CD \perp AB$

Proof

Let us consider the triangles ABC and ABD , which are equal by the 3rd criterion $\rightarrow$ AB in common,

$AC = AD$ because they are radii of the same circle

$DB = CB$ because radii of the same circle

Equal angles are opposed to equal sides, therefore the equal angles $\angle ABC$ and $\angle ABD$ are opposed to sides CA e AD . Angles CAB and BAD are opposed to sides CB and BD . We know that angles $\angle BCD$ and $\angle BDC$ are equal because they are at the basis of an isosceles triangle and also angles $\angle ACD$ and $\angle ADC$ are equal.

Triangles DOA and AOC [point O is marked on the drawing but it was not explicitly defined] are equal according to the 2nd criterion, namely equal angles are opposed to equal sides, therefore angles COA and DOA are equal and right angles ... angles AOC and BOD are equal because they are vertically opposite angles.

This protocol shows a good theoretical control of the figure, i.e. the image on the screen and its construction. The students clearly separate

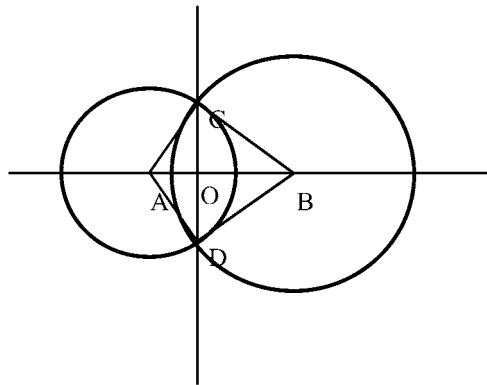


Figure 2. The construction of the perpendicular line.

the description of the construction and its justification and the justification correctly refers to the theory available. This is not the most common construction, and actually, the use of the two points (A and B) determining the given line r , produces two circles with different rays, this makes the validation more complicated, requiring a delicate analysis of the figure overcoming intuitive evidence. Nevertheless, Alex and Gio succeed in finding a justification, correctly related to the construction process, i.e. taking into account only the relations coming from the construction. The theoretical control of Alex and Gio developed as they progressed through the Cabri construction tasks.

It is also possible to observe an evolution of the sense of proof related to the idea of construction, in the following excerpts from Cathy's work with the software.

In the task of constructing the angle bisector, Cathy (from the same 9th grade class as Alex and Gio) describes two attempts. In the first, the description of the construction was very rough; Cathy referred to the construction of two circles, but she did not explain the points defined by them. Then she went on:

they (*the two circle*) intersect, resulting in a common point; from here we then trace the line which passes through the vertex and then through α (*she has indicated the angle with an α*).

The justification given looks more like an affirmation:

It is obvious that, as a result of this procedure, angle α will be divided into two equal parts.

In her second attempt, even if the solution was not correct, Cathy seemed to feel the need to build the construction made starting from the constructions available and to produce a justification. As a matter of fact, having taken

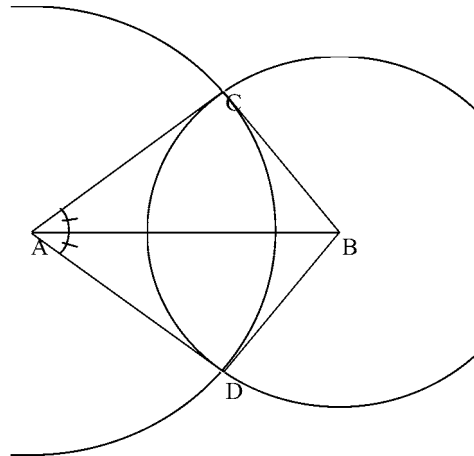


Figure 3. Cath's construction.

an angle (given by two segments), and using the tool 'Report of angle', Cathy doubled it, she wrote:

transporting the angles, I construct an angle β equal to α .

Once the 'desired' figure had been obtained, that is, an angle divided into two equal parts, Cathy did not seem to realise that her procedure solved a problem of construction different from the one she had been given and she tried, in any case, to give a 'proof'. Starting from the figure produced, she wrote down her proof proving that the two triangles constructed are equal. Cathy writes:

Now for the 3rd criterion of equality I can prove that triangle ABC is equal to triangle ADB ... Now AB represents our bisector since it divides an angle that I will call γ into two equal angles ($\alpha = \beta$).

It is clear that Cathy's attention was focused on producing a drawing representing an angle divided into two equal parts, rather than on constructing the geometrical object, angle bisector (see Figure 3). Nevertheless, she had a general idea that the justification must be found in the stated criteria. A first trace of development appeared in her solution of the task of constructing the perpendicular line, in which we can observe a greater care than in the case of the construction of the angle bisector, with regard both to description of the procedure and its justification.

- When $P \in r$.

Cathy's first attempt was performed 'by eye'.

I trace a line passing through two points (I go in 'Creation')
 I draw a point on it (I go in 'Construction')
 Through this point I pass another line, through two points, which looks perpendicular to me.
 RESULT: the point does not remain still. FAILURE
 I cancel everything [...] WHITE SCREEN

The second attempt presented a construction that is correct and accompanied by a justification referring both to the construction and to the definition of perpendicular.

I trace a line passing through two points: I draw a point which remains on the line forever (I select Construction and then 'point on an object').
 At this point I trace the ANGLE BISECTOR passing through P.
 POSITIVE ATTEMPT: I save the figure.
 Proof: I have followed the definition of PERPENDICULAR to a line.

- *When $P \notin r$*

For this second case, Cathy described the exploration that led her to the identification of the construction: after constructing a line and a point P, and after taking two points S and Q on the line, Cathy constructs the bisector of angle SPQ (see Figure 4), she drags the figure until the triangle SPQ becomes an isosceles triangle so that the bisector becomes perpendicular to line r (see Figure 5).

At the angle P, I mark the bisector. With the 'little hand' I adjust the triangle SPQ until it becomes isosceles or equilateral so that the bisector becomes $\perp$ to r.

I think I've managed to do it. I determine point R, in the point in which the bisector intersects r. I check, marking the angle (*she refers to the use of the tool 'mark an angle'*) whether or not PRQ and PRS are right.

After building this exploratory construction, Cathy did not describe and perhaps did not repeat a construction over again, but she correctly articulated the hypotheses relative to a correct construction and writes a proof of the thesis:

Hp. $SP = PQ$
 PR bisector

And the thesis to be proved:
 Ts. $PR \perp r$

Proof

I consider the triangles PRS and PRQ:
 They have

- PR in common
- CH bisector for Hp.
- $SP = PQ$ for Hp.

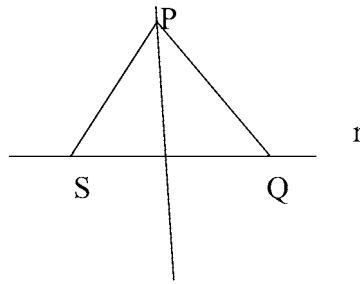


Figure 4. Cathy's first drawing.

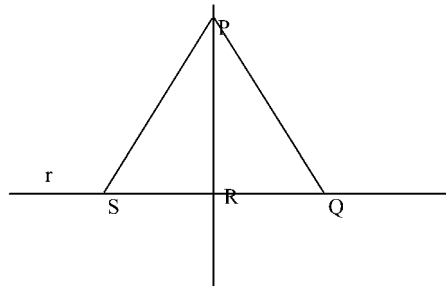


Figure 5. Cathy's second drawing.

- $RPS = RPQ$ for Hp.
- SRP and PRQ are equal for the 1st criterion and because PR is the bisector of the straight angle $\Rightarrow PR \perp r$.

The key element was the bisector of the angle at P, which in the case of an isosceles triangle is perpendicular to the opposite side. It is interesting to observe that dynamic exploration has played a key role for the identification of the solution, at the same time the result of the exploration has been correctly interpreted within the theoretical frame.

Finally, let us see the solution proposed by Cathy to an open ended problem presented a few weeks later. The problem is the following:

Given a parallelogram, render one of its angles a right angle: make your observations.

This is no longer a problem of construction, but we can observe how Cathy used a construction in order to fix the hypothesis of a formulation and to guide the exploration aimed at producing a theorem. She constructed (describing the construction in words) the parallelogram: she traced two pairs of parallel lines and identified the vertices, using the command 'intersection of two objects'. In Figure 6(a) we see that Cathy expresses, alongside the drawing she produces, the hypotheses corresponding to the

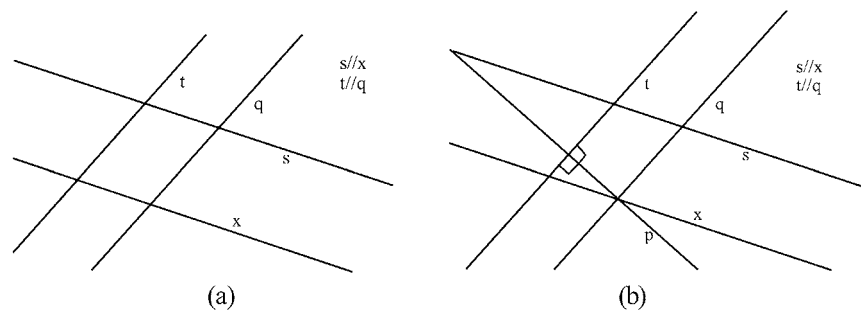


Figure 6. The first set of drawings provided by Cathy.

construction made with Cabri ($s//x$ and $t//q$). She could have dragged this until one of the angles became a right angle, but instead she then construct a perpendicular (p) to one of the lines (t). She describes her next step as follows:

I move with the 'little hand' until the perpendicular overlaps with line x ; in this way I obtain a right angle.

The resulting configuration is shown in Figure 6(b). As she did this she sees that the angle between line p and line q is a right angle. She was than ready to draw a new figure (Figure 7(a)) which corresponded to the hypotheses of the statement, from which she elaborated her proof:

Knowing that angle α is supplementary to β because they are internal angles of parralel lines, I discover that all the angles are right angles.

I know that α is a right angle because $p \perp t$. Since α and β are internal angles of parralel lines, in other words $180 - 90 = 90$, β is also a right angle. According to the property of the parallelograms for which adjacent angles are supplementary, all the angles are right angles.

The dynamics of the figure led Cathy to identify the solution, but of particular interest is the use of the construction of the perpendicular p line, as a reference element which is 'constructed' and therefore stable. This element embodies the hypothesis of the problem (only one angle is a right angle) and is used to keep the control on the figure, in terms of what is given and what is to be proved (for a more complete discussion of this phenomenon see Mariotti, in press). Cathy seems to have achieved the capacity of controlling the figure, both for exploring and for proving, and this control seems directly derived from the idea of construction and its theoretical meaning.

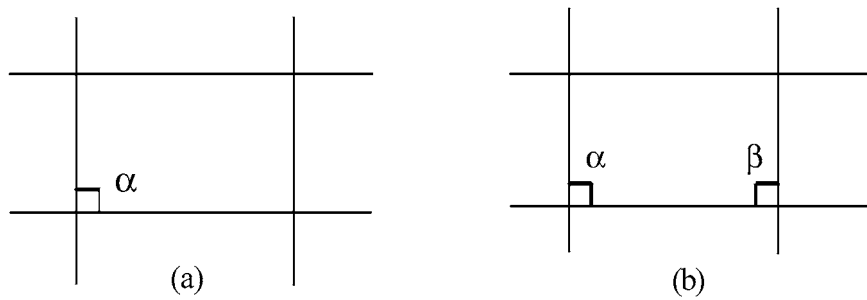


Figure 7. The second set of drawings provided by Cathy.

FROM COMPUTER CONSTRUCTIONS TO COLLECTIVE DISCUSSIONS

A very particular element of the class activity involved the creation of personal notebooks; each student has his/her own notebook, where all the new achievements in the theory were recorded and described: axioms and definitions, new constructions and related theorems. The teacher periodically revised these notebooks, but most of the time this revision was made as a social activity. The following excerpts are taken from a collective discussion devoted to the first collective revision of the notebooks and provide an interesting perspective on students' views of the relationship between Cabri tools and geometry theory.

The discussions occurred outside the computer lab, after few weeks from the beginning of the Cabri work and aimed at revising pupils' notebooks. During this collective work, the small pieces of theory, systematised in the notebooks were analysed, checking for consistency and reflecting on the status of each statement in respect to the whole system. In particular, the discussion focussed upon the different roles played by the elements within the geometry theory: axioms, definitions and theorems. Despite the absence of computers during these discussions, references to Cabri emerge, sometimes in students' contributions, sometimes in teacher's intervention, showing how the software, its tools and their functioning, may contribute to the construction of meanings.

The first part of the collective discussion involved the consideration of 'basic geometric elements' and their correspondence with particular Cabri tools. The first excerpt begins at the point in which the element, triangle comes into focus:

Excerpt 1

67. BAZZ: you can **construct a triangle** . . .

68. Teacher: what does it [*the notebook*] say then?

69. MOR: **axiom 1** triangular inequality ... given three segments **it is possible to construct a triangle** which has these segments sides, if the sum of the minor segment is larger or equal to the larger segment.
70. Teacher: it gives me a rule to construct MOR: the triangles. Teacher: with the segments that I have already defined ...
71. MOR: this is an axiom, not a theorem ...
72. Teacher: why is it an axiom? Could we give a demonstration of this?
73. CHORUS: no ...
74. BERN: I could make a drawing with three segments ... but it agrees with the other three ...
75. MOR: but they are infinite ...
76. BAZZ: but **how can we explain that ... I mean, prove that it is true?** ...
77. MAS: the axiom includes all the cases, also of the segments with which we can't construct the triangle ...

In this excerpt, **axiom 1** was recalled as one of the elements introduced in the theory in relation to the possibility of **constructing a triangle**. The distinction between axioms and theorems was correctly explained in terms of proof, while it was stressed that a drawing mode may be used to verify, but not to **explain ... to prove that it is true** (76). The last remark seems particularly interesting, showing the deep understanding reached by that student.

During the second excerpt, the teachers moves the discussion from **axioms to theorems**, at which point a different element, the report of an angle, is introduced.

Excerpt 2

138. Teacher: [...] But we are still dealing with axioms ... we have seen a primitive entity, we have seen a definition, we have seen an axiom ... what is still missing among our representatives of geometry?
139. PIER: a theorem ...
140. MOR: now there is '**report of an angle**' ...
141. Teacher: let's try to find out what the 'report of an angle' is ... A primitive entity, a definition, and an axiom or a theorem?
142. MOR: it's a theorem because it can be proved ...

When the teacher asked for a theorem, a student responded by mentioning a Cabri tool (139–140), correctly stating, in response to further probing by the teacher, the theoretical status of the Cabri construction.

The third more extensive excerpt from the discussion concerns the role of construction in the production of proofs.

Excerpt 3

228. Teacher: at this point ... we have defined the angle and we shall define a thing which is the bisector of the angle. ... We are using a great many geometrical objects that

- you are familiar with ... now we are slowly reconstructing them [...] the bisector of the angle ... who wants to read the bisector of the angle? Bern *you* read it ...
229. BERN: the bisector of the angle is that ray which starts in the vertex of the angle and divides the angle into two equal parts ...
230. Teacher: let's repeat a moment ... [*BERN reads it again*] ...
231. Teacher: [...] do I have to construct them ...? From the definition I must construct a ray whose origin is in the vertex of the angle and which divides the angle in two equal parts ... who feels like reproducing the construction again on the blackboard? Someone else, not Bazz. We want to check all the construction and in particular we want to see if the one we saw with Cabri can also apply to the straight angle ... because in this way what will the bisector of the straight angle become?
232. BAZZ: the perpendicular to the straight line ...
233. MOR: from an external point ...
234. BAZZ: no ... from a point of the line ...
235. Teacher: ok MOR go and construct the angle bisector ...
236. BAZZ: but there has to be the description of the construction of the bisector (he means in his note-book)? [...]
240. BAZZ: I haven't got it ... I must write it there ... can I stick it there because it doesn't fit here. ...
241. Teacher: yes of course ... then
242. MOR: [*he describes the construction of the angle bisector, which exploits, for the two equal circles, the segments determined by points the 1 and 2, obtained as the intersection of the first circle with the sides of the angle ...*]
243. Teacher: I'll ask you a question now ... let's suppose we are in Cabri ... the two equal circles constructed afterwards in your drawing also pass through V ... so my question then is ... is it just a case or do they have to pass there? ...
[...]
251. Teacher: . Ok ... let's see the PROOF ... the hypothesis and the thesis ... we have to put these first otherwise we don't know either where we are starting from or where we want to go. ...
252. BAZZ: our hypothesis is the construction ... shall I write that as well?
253. Teacher: of course, if you want to ...
254. [*Hypotheses and thesis are written on the blackboard, using the letters and number as labels on the figure. MOR describes the Proof*].
255. BAZZ: with all the circles which are equal ... I don't remember if it was ok with Cabri ... I seem to remember that at a certain point the drawing disappeared. ...
256. Teacher: let's try to understand if that was really the case and why ... [*The discussion continues regarding the different constructions which had been made previously*]

In this excerpt, the reference to previous experience in Cabri was recurrent and took different forms:

- Constructions correspond to theorems. Thus, for any construction, a theorem and the descriptions of the construction should be present in the notebook.
- Constructions had a fundamental part in the evolution of the theory, and the discussion of an acceptable construction was explicitly stressed by the teacher (231).

- Cabri was evoked in order to question the generality of a construction (243).
- The contribution of Cabri in the evolution of the meaning of *hypothesis* and *thesis* of a statement clearly appeared in the exchange between the teacher and the student BAZZ (251–253). During this exchange, the teacher (251) intentionally mentioned the distinction between hypothesis and thesis and Bazz reacted by remembering that the construction process incorporated the hypotheses, wondering if this observation was important enough to be written in his notebook.

These three excerpts were drawn from a discussion which occurred while the ‘theory’ being constructed by students was still in its early stages. Nonetheless, it seems that, at least for some of the students, understanding has been achieved about the meaning of constructing a theory and proving within it the validity of a statement.

The work of collective revision revealed itself very useful from two different points of view. It allowed an ordering to be imposed on the sequence of axioms, definitions and theorems and it enabled a reflection on the theoretical status of each element of the sequence. In this activity of revision, the relationship between theory (axioms, definitions and theorems) and Cabri (the use of different tools) emerged: tools and construction processes were recalled in different occasions.

Cabri tools correspond to different theoretical elements, introduced and utilized with different aims and different modalities. Tools corresponding to primitive entities, basic definitions, and the first set of axioms were introduced at the very beginning, when the menu was empty. New tools, on the other hand, were introduced only after the corresponding new theorems were validated within the theory (for instance, it was only after the production and the validation of its construction, that the tool ‘angle bisector’ was introduced in the Cabri menu).

The following basic aspects emerged from this combination of tools and theory:

- The instrumental aspect that axioms and theorems have in respect to validating new elements of the theory emerged from the instrumental use of the corresponding tools in order to solve a construction task.
- The different status of hypothesis and thesis in a theorem became clearly expressed by the different elements occurring in the solution of a task.

In fact, the process of construction, i.e. the sequence of commands used to build a particular figure, incorporates the hypothesis. The resulting Cabri figure incorporates the link between hypothesis and thesis: the properties

expressed by the thesis emerge as a consequence of the properties stated by the construction. Hence construction, in its twofold meaning, a sequence of commands executed and a figure produced, provides a instrument of mediation for the interlaced meanings of hypothesis, thesis and proof.

Further supporting evidence to our claim can be found in the individual reports, produced after the revision activity described above. The report was organised around the following questions:

1. What type of relationship do you think there is between the Cabri commands and the Euclidean Geometry?
2. a) Is it true that each Cabri command corresponds to an axiom or to a definition?
b) Give some examples and motivate your reply.
3. How do you think movement of the Cabri figures can be interpreted in Euclidean geometry?

The analysis of the replies obtained has shown how it was possible to construct a relation of meaning between the Cabri commands and geometry, with respect to the aspects in which we are most interested, in other words the meanings of theorem and of theory. In particular, it should be observed that only two pupils out of fourteen were misled by question 2a); the great majority correctly stated that the Cabri commands variously correspond to axioms, definitions or theorems. The following two responses are presented as examples; it is interesting to remark that Ann, in spite of the first positive answer to question 2a), correctly distinguishes between axioms and theorems in the examples.

Gian

1. Each Cabri command allows us to apply the theorems or the axioms of Euclidean geometry.
2. a) No
b) Because some commands allow us to use theorems which must be proved before use.
3. The movement of the Cabri figures is very important to see whether the construction of a figure is right or wrong.

Ann

1. In my opinion the Cabri commands are based on Euclidean geometry, and this means that each element of Euclidean geometry is included in the buttons of the Cabri Program.
2. Yes. The following are some examples of axioms and theorems and definitions.
 - Perpendicular line (definition)
 - Bisector (theorem)
 - Angle (definition)
 - Point (axiom, which is part of the primitive entities)

- Line (axiom, which also belongs to the primitive entities)
 - Circle (definition)
 - Triangle (definition)
3. The movement of the constructions or Cabri figures serves to check the correctness of the construction. We used this advantage a lot when we constructed the bisector . . .

As a final remark, a particular tool among those available in Cabri, which plays a very important role in the process of meaning construction should be mentioned: the History command. When activated, this command provides the sequence of operations realizing the construction; it represents an external sign of the construction procedure, allowing one to run over the single steps again, and making possible the observation and analysis of the sequence of relations constructed between elements of a figure. The presence of this tool allows the teacher to refer to a decontextualised, depersonalized and detemporalized copy of the construction with the aim of shifting pupils' discourse from the drawing produced to the relationships stated through the construction, i.e. it can function as an *instrument of semiotic mediation* (for a more detailed discussion see, Mariotti and Bartolini Bussi, 1998).

THE DEVELOPMENT OF THE SENSE OF JUSTIFICATION: A GENERAL OVERVIEW

The general results of our experience can be summarised as follows:

- they confirm the general hypothesis concerning the emergence of a theoretical sense of a construction task;
- they provide evidence of the evolution of the idea of justification, towards the idea of a proof, as justification strictly related to the specific principles shared and stated by the class community.

More precisely, our experience shows that this evolution may be articulated into a sequence of three steps:

- a) Explanation of reasoning adopted during the solution process. Students are invited to report all their attempts, even the wrong ones (the solution given will be the subject of a collective discussion). At this point, the basic aim is that of communicating to the teacher and classmates one's own reasoning, i.e. the main objective is that of being *understood*.
- b) After the first discussions, when the verbal reports are used to discuss the 'correctness' of the constructions, a new aim arises: making one's own construction *acceptable*. This means that it is no longer sufficient

to be intelligible (clear, complete and precise), it becomes necessary to state a number of rules to be respected; in other terms an agreement must be negotiated in the class about the acceptable operations.

It is this step that can be considered the genesis of a theoretical perspective: looking for a justification within the system of rules introduces students to a theoretical status of justification.

- c) Once students accept the principle that a construction must be validated, they try to adapt their justification to this new request, the main question becomes how to defend one's own construction; an agreement must be negotiated about the rules that the whole class has to accept.

CONCLUDING REMARKS

Our exploratory study points out a direction of research that, in our opinion, seems very promising. The basic aim concerned the pupils' introduction to 'theoretical' geometry, as a system of relations among statements, validated by proof. Our main hypothesis claimed the possibility to use construction task in the Cabri environment as *a key to accessing* the theoretical world of geometry.

The sense of construction, deeply rooted in the experience of the Cabri environment, is explicitly related to geometry theory. The descriptions of the procedure, explicitly required by the task, show an evolution: arguments, accompanying the description of a construction, slowly approaches the status of theorems. This means that the justifications provided by the students assume the form of a statement and a proof: the hypothesis drawn from the construction are correctly related to the thesis, while the justification explicitly refers to the system of principles, shared and stated within the class community.

The theoretical perspective achieved in the solution of construction problems, provides a key to accessing the general meaning of theory. Justification does not evolve in a proof spontaneously; on the contrary the evolution reveals its complexity, but the specificity of the environment to which the arguments are related, and the direct guidance of the teacher may determine this evolution. The teacher may use different means in order to accomplish her/his task, but, in the case of collective discussion on Cabri constructions, in addition to the standard strategies, used by the teacher to manage discussions in a whichever context, there are strategies that are specific for the Cabri environment. Put another way, Cabri offers specific tools of semiotic mediation (Vygotsky, 1978) which can be used by the teacher according the particular educational goal.

This paper did not present a detailed analysis of the teacher's role during collective discussions. Rather, our intention was to give a general overview of a very complex research project, clarifying its goals, its basic hypotheses and describing what we consider very successful results. Further investigations are in progress, aiming to refine the analysis of the process of semiotic mediation accomplished by the teacher through the use of semiotic tools available in Cabri.

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NOTES

¹ See also Mariotti and Fischbein (1997) about the case of definitions.

² These are macros to copy respectively a segment and an angle; they were available in Cabri version 1 in use at that time.

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