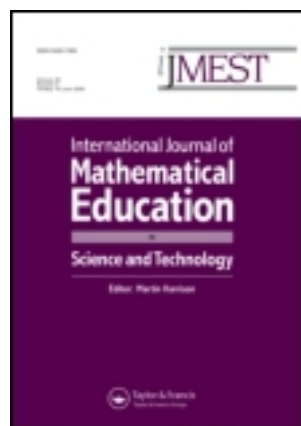


This article was downloaded by: [Nat and Kapodistrian Univ of Athens]

On: 27 March 2013, At: 04:44

Publisher: Taylor & Francis

Informa Ltd Registered in England and Wales Registered Number: 1072954 Registered office: Mortimer House, 37-41 Mortimer Street, London W1T 3JH, UK



International Journal of Mathematical Education in Science and Technology

Publication details, including instructions for authors and subscription information:

<http://www.tandfonline.com/loi/tmes20>

Dragging as a conceptual tool in dynamic geometry environments

F. Lopez-Real^a & A. Leung^a

^a Faculty of Education, The University of Hong Kong, Pokfulam Road, Hong Kong S.A.R., China

Version of record first published: 20 Feb 2007.

To cite this article: F. Lopez-Real & A. Leung (2006): Dragging as a conceptual tool in dynamic geometry environments, International Journal of Mathematical Education in Science and Technology, 37:6, 665-679

To link to this article: <http://dx.doi.org/10.1080/00207390600712539>

PLEASE SCROLL DOWN FOR ARTICLE

Full terms and conditions of use: <http://www.tandfonline.com/page/terms-and-conditions>

This article may be used for research, teaching, and private study purposes. Any substantial or systematic reproduction, redistribution, reselling, loan, sub-licensing, systematic supply, or distribution in any form to anyone is expressly forbidden.

The publisher does not give any warranty express or implied or make any representation that the contents will be complete or accurate or up to date. The accuracy of any instructions, formulae, and drug doses should be independently verified with primary sources. The publisher shall not be liable for any loss, actions, claims, proceedings, demand, or costs or damages whatsoever or howsoever caused arising directly or indirectly in connection with or arising out of the use of this material.

Dragging as a conceptual tool in dynamic geometry environments

F. LOPEZ-REAL* and A. LEUNG

Faculty of Education, The University of Hong Kong, Pokfulam Road,
Hong Kong S.A.R., China

(Received 16 March 2005)

Some ‘drag-to-fit’ solutions given by student teachers to three geometric construction problems in a dynamic geometry environment (DGE) are analysed. The responses of a group of experienced mathematics teachers to the question whether or not such solutions can be considered ‘legitimate’ are then discussed. This raises fundamental questions concerning the concept of legitimacy, the relationship between DGEs and Formal Axiomatic Euclidean Geometry, the nature of ‘conceptual tools’ in different geometric environments, and the functions of dragging in DGEs. The authors argue that, if dragging is viewed as a conceptual tool, then certain drag-to-fit solutions, although soft constructions, may still be considered as conceptually legitimate and therefore valid. Finally, some important questions are raised concerning the impact that teachers’ different attitudes towards legitimacy might have on students’ learning through DGEs.

1. Introduction

A Dynamic Geometry Environment (DGE) is a computer microworld with Euclidean geometry as the embedded infrastructure. In this computational environment, a person can evoke geometrical figures and interact with them [1]. It is a virtual mathematical reality where abstract concepts and constructions can be visually reified. In particular, the traditional deductive logic and linguistic-based representation of geometrical knowledge can be re-interpreted, or even redefined, in DGE as dynamic and process-based interactive ‘motion pictures’ in real time. One of the issues that researchers in DGE often attempt to address is how a learner interacting with this ‘new geometry’ can ‘learn’ the traditional formal axiomatic Euclidean geometry. There appears to be a tension (rooted in the Euclidean view of what geometry is) pulling DGE research towards the direction of bridging an experimental–theoretical gap that seems to exist between the computational microworld and the formal abstract conceptual world. For example, research has been done on investigating how students’ reasoning process transits from conjecture-making in DGE to the production of formal rigorous deductive proof after the Euclidean tradition. Underlying this line of thought is an implicit proclivity that

*Corresponding author. Email: lopezjf@hkucc.hku.hk

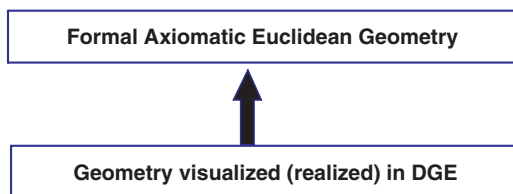


Figure 1. Hierarchical relationship between FAEG and DGE.

there is a hierarchical order of significance between DGE and the formal axiomatic Euclidean geometry.

The upward pointing arrow in figure 1 represents an ascent (or a transition) in ‘rigorous levels’ of geometrical knowledge. Formal Axiomatic Euclidean Geometry (FAEG) sits at the top of this hierarchy because it is the praxis of what is agreed as rigorous mathematics. DGE is often regarded as a milieu for geometrical experiments and for verification of geometrical theorems. There is still lacking a common consent in the DGE community on what ‘rigour’ and ‘proof’ mean in DGE. Furthermore, FAEG is often envisaged as the Ideal Geometry (in the sense of pure logic and reason) that has no compatible counterpart in physical reality. With this presupposition, the upward pointing arrow can literally denote a pointer in the sense of a signifier pointing to the signified. There are no solid findings yet on exactly how this bridging process between the experimental and the theoretical works, though often there seems to be a quantum leap from the DGE domain to the abstract formal domain. In all cases, this transitional leap is the result of dragging activity in DGE (see [2–5]).

Dragging in DGE is a powerful dynamic tool to acquire mathematical knowledge. It serves as a kind of interactive amplification tool for the learner to see global behaviour via variation [6]. Studying dragging modalities employed by students has been a major undertaking in DGE research (see [6–9]). In Leung and Lopez-Real [10, 11] different utilization schemes of a drag-to-fit strategy were discussed and the ideas of potential solution and potential proof were put forward. It seems that dragging in DGE can open up some kind of semantic space (meaning potential) for mathematical concept formation in which dragging modalities (strategies) are temporal-dynamic semiotic mediation instruments that can create mathematical meanings, that is, a window to enter into a new semiotic environment of how geometry can be re-presented (re-shaped). Hence it is reasonable to ask: what is an acceptable ‘grammar’ that could construe mathematical meanings in this visual and motion-based virtual system? If such a grammar can be agreed on, it should obviously be structured on dragging modalities. Dragging modalities can construe qualitatively different experiences for the same geometrical phenomenon in DGE. Arzarello *et al.* identified different types of dragging strategy (wandering, bound, guided, dummy locus, line, linked and dragging test) ‘which students use according to their different purposes during the solution process of open problems’ [12]. The process of making meaning, hence learning, is a semiotic process and dragging can be regarded as a prototypical form of it in DGE. As we learn how to mean via dragging, a meaning potential in DGE is constituted. This constitution of meaning potential can be realized in the way a ‘grammar’ of dragging is constructed in DGE,

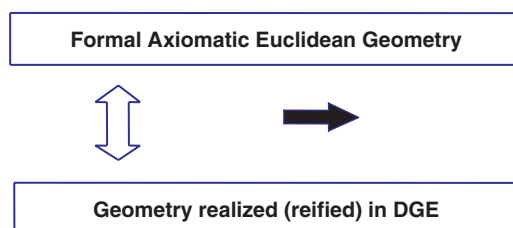


Figure 2. Parallel relationship between FAEG and DGE.

that is, a semantic system that makes sense of and captures geometrical reality in DGE.

Furthermore, the question of whether the geometry realized in DGE is ‘legitimate’ deserves deeper consideration. Balacheff [13] discussed the idea of computational transposition where, in a computational environment, mathematical knowledge is transformed partly ‘due to the specificity and constraints of sophisticated representation systems both at the interface and ‘inside’ the machine’ (p. 147). Hence in DGE, traditional Euclidean geometry may be transformed into a new situated geometry that might not just change the rules of the game, but the game itself. There is an implicit tendency by researchers to treat geometrical experiences in DGE as partial understanding that needs to be made whole in the traditional formal axiomatic context, thus giving rise to the experimental-theoretical gap problem. We often try to ‘legitimize’ the geometry in DGE by imposing axiomatic rigidity on evocative computational objects; for example, objects constructed in DGE need to be robust and should pass the drag test as described by Hoyles and Healy [2]. However, for a learner immersed in DGE, it might be more natural for him or her to develop a new geometrical sense that is indigenous to the computer environment in the process of concept formation. The learner’s prior knowledge in formal axiomatic geometry can fuse with the dragging instrument in DGE to form a network of knowledge systems, whose components exchange meanings with each other rather than insisting on having one conforming or converging to the other.

In figure 2, FAEG and DGE are ‘parallel’ systems that are situated in different semiotic phenomena. In Balacheff’s terms [13], they are epistemological domains of validity of two different knowledge representations. FAEG is the static and symbolic realm of rigid logical structure. The vertical two-way arrow denotes the connection (networking) that enables an exchange of meaning between the systems. The horizontal arrow stands for a concurrent mediation process that signifies some kind of mathematical reality. This perspective embraces the idea that dragging in DGE is a semiotic tool (or a conceptual tool) that helps learners to form mathematical concepts, rather than just a tool for experimentation and conjecture making that doesn’t seem to match the ‘logical rigour’ in FAEG. In particular, a drag-to-fit strategy may act as a kind of ‘transformational logic’ that bridges the two semiotic systems (the vertical two-way arrow) in the sense that dragging brings about concept formation (the signified object) in both FAEG and DGE. A mathematical concept can be thought of as a ‘virtual object’ that can be represented by different semiotic sign systems (the horizontal arrow). Winsløw [14] proposed that one can ‘think of (mathematical) objects as representations modulo *object preserving transformations*’. Object preserving transformations are strict rules of processing and conversions that

may be carried out to change from one representation to another while preserving the reference to a common mathematical object [14]. In light of this, dragging strategies have the potential to become such object preserving transformations that can play a significant role in the ‘creation’ (in Winslow’s sense) of geometrical objects.

Lopez-Real and Leung [15] initiated a discussion, via illustrative examples that contrast ‘traditional’ and DGE solutions to construction problems, on the idea that dragging in DGE is a conceptual tool that may legitimize certain constructions in DGE. The following is an expansion of that discussion, further illustrated by the views and comments of a group of in-service mathematics teachers. The last section of the paper continues a discussion on some implications of this alternative perspective and their pedagogical significance.

2. Background

In recent years, during sessions on DGEs with Post-graduate Certificate of Education and Bachelor of Education students, we have become aware of some interesting differences of opinion regarding the validity of solutions for certain construction problems. While the students were able to appreciate the difference between robust and soft constructions in general, nevertheless there were certain problems where the distinction became somewhat blurred and arguments arose. One such example is the following problem:

Draw any line segment AB . Devise a construction that will trisect AB . (That is, determine two points C, D on AB such that $AC = CD = DB$).

Apart from ‘traditional’ Euclidean solutions to the problem, a DGE-specific solution was produced by many students and this is illustrated in figure 3.

The students first constructed a line segment AB and then two equal intersecting circles starting with an arbitrary centre C on AB and radius CA . The first circle intersects the line segment AB at D and DC was used as the radius of the second circle centred at D intersecting the line segment AB at P . By dragging C until P visually coincides with B they claimed that they had solved the problem. At first, we had no doubts about this. It seemed a rather simple example of the difference between ‘robust’ and ‘soft’ constructions and it is clear that only robust constructions are legitimate solutions. Nevertheless, regarding this particular problem the students remained rather unconvinced. This was in part due to a curious feature of their construction. In the usual situation for a ‘soft’ or unstable construction, although points may be dragged to positions where it appears the

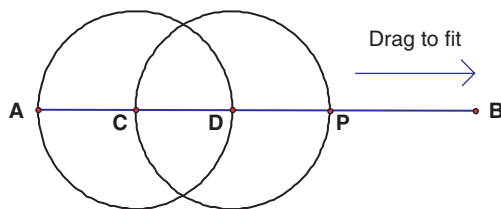


Figure 3. ‘Drag to fit’ solution for the segment trisection problem.

construction is solved, the diagram ‘falls apart’ again when other points are dragged. Thus, we may drag a point to coincide with another point in the diagram but it does not ‘lock on’ to the other point. In contrast to this, when C is dragged in figure 3 so that P coincides with B, the two points do lock together. That is, if we now drag this coincident point, the line segment AB *varies* and points P and B remain locked. However, with yet another curious quirk, if point C is now dragged again, P ‘pulls away’ from B and the two are unlocked once again. It appears that we have a construction here that is something of a ‘halfway house’ between a robust and a soft construction.

Other constructions were also solved by a ‘drag-to-fit’ strategy and again disputes arose about their legitimacy. As a follow-up to these occurrences, we decided to run a discussion session with our part-time Master of Education students where such solutions would be presented for debate. These students were all serving and experienced teachers of mathematics in both primary and secondary schools. Three problems were given to these teachers and they had the opportunity to solve them first. Only a short time was given for the solving activity since its main purpose was simply familiarization with the problems prior to subsequent discussion. In the event, many of the teachers produced precisely the drag-to-fit type of solution we wished to discuss. The session was videotaped and sections later transcribed.

3. The problems and teachers’ comments

3.1. Trisection of a segment

The first problem was in fact the one we have previously illustrated in figure 3. Very quickly in the discussion two opposing viewpoints emerged:

To me this is not legitimate. I think it is just a clever way of finding the approximate position to trisect the line. This is an approximation, it is not the sense of what a usual ... a standard geometric construction should mean. (Teacher A)

This is a way that we can’t do by compasses. By dragging the point C you can adjust the radius of the circle. So this method I think can be considered legitimate in the Sketchpad environment. (Teacher B)

There are layers of implicit meaning in the phrases ‘standard ... construction’ and ‘legitimate in ... Sketchpad’ that other teachers were perhaps struggling with too. For example:

From my point of view ... when we drag until the circumference touches the other end, the problem is, is it really touching the end point? We don’t know. (Teacher C)

Teacher A went on to enlarge his argument against legitimacy as follows:

I think whether this way is possible depends on whether you are dragging in a continuous way. The circle must be increasing continuously. However, as I have mentioned, the mouse is a discontinuous machine ... so you may have gaps between the points, you never know.

It quickly became clear that there was no disagreement about the property of three equal segments being generated by the intersection of the two circles. The point of contention was how this property was being used to produce a construction. Teacher D tried to sum up these difficulties:

I think the key question is about operations. Because we have some concepts, or theories. For example ... without this software ... even (if) we have the property of the circles we can't operate on this concept to trisect this line. But after we have such software we can operate on such concept to do something ... I think it is a problem of how we see operational concepts. For example, if we say the mouse is discrete while the line is continuous ... if when someday the computer can do something continuously then this problem may be solved.

The implied suggestion in the final comment is that some 'improved' computer version of DGE might legitimize the construction. We shall return to the issues raised here later.

3.2. The square and the triangle

The second problem was the following:

Draw any triangle. Construct a square inside the triangle such that two vertices of the square lie on one side of the triangle and each of the other vertices lie on the other two sides of the triangle.

This problem was discussed over 50 years ago by Polya in his seminal work *How to Solve It* [16]. There are two interesting aspects about the problem. First, it is still quite a challenging construction problem for students encountering it for the first time. Second, Polya's approach to the problem, using the strategy of 'loosening constraints', leads to a solution that lends itself perfectly to a dynamic geometry environment, the main difference being that the paper-and-pencil approach requires a considerable 'investment of time and effort' as Straesser puts it [9].

Three possible solutions were discussed. In the first solution (figure 4) a square DEFG is first drawn satisfying the conditions for 3 of its vertices and then a ray is drawn from A through F to cut BC at J. This determines the position where the required square HIJK can easily be constructed. With the help of dragging and the

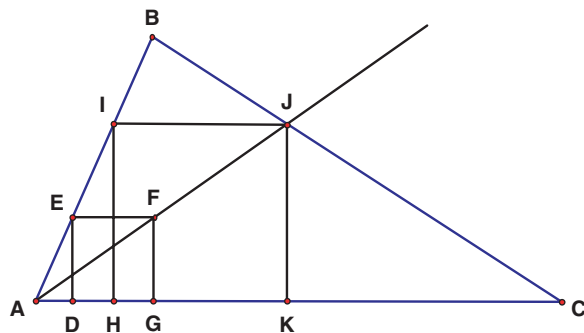


Figure 4. 'Polya' solution for the square-triangle problem.

Trace function in Sketchpad (the DGE that students were engaging in), some of the teachers found this solution. In the second solution (figure 5), square DEFG is also drawn but this time the point E is dragged along AB until vertex F lies on BC. Again, a number of the teachers did this. A third solution was also discussed based on drawing a rectangle first and then dragging and measuring until it becomes a square. However, *all* the teachers agreed that this was not a legitimate construction since it involved measurement and hence could never be exact. In the discussion concerning the first two solutions, there was agreement that the first is robust whereas the second is not. However, some of the teachers argued that this was not the crucial point. As teacher E put it:

To me this solution (dragging) is very convincing. If I have the right version of Sketchpad that vertex (F) will snap onto the line.

The ‘right’ version is presumably one that will allow such a ‘snap function’. Two other teachers continued the argument:

In mathematical proof we use logic to argue whether our proof is correct or not. So I don’t focus on the approximation, I focus on the logic of the argument. (Teacher D)

I think it would be better to distinguish two things, the principle behind and the way you do it. I would agree that in principle *if*, remember if, if the point is lying on the last side that would be a legitimate square. But I would argue against how you could possibly claim that, in an environment like this. (Teacher A)

3.3. Trisecting an angle

The third problem was the famous trisection problem of Ancient Greece, that is to find a construction that will trisect any *angle*. This is particularly interesting, first because all the teachers ‘knew’ it was impossible, and second because of its relationship to the first problem. Not surprisingly some of the teachers quickly came up with an analogous dragging solution as shown in figure 6.

Given $\angle AOB$ to trisect, from OA we can easily construct three equal angles $\angle AOC$, $\angle COD$, $\angle DOP$ (in the same way that we constructed AC, CD, DP previously). We now drag the independent point C until P coincides with B and we have the solution. There is one interesting difference between this and the segment dragging solution.

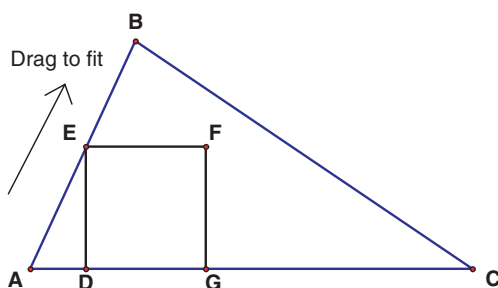


Figure 5. ‘Drag to fit’ solution for the square-triangle problem.

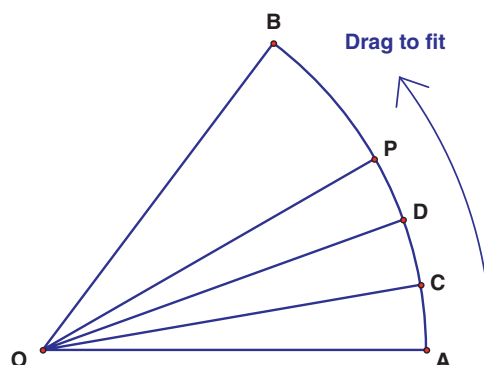


Figure 6. 'Drag to fit' solution for the angle trisection problem.

In the line segment trisection we found that there was a very real sense in which P appeared to lock on to B. However, this time, although the construction follows a precisely analogous sequence, we find that after P coincides with B, if we then drag this coincident point the line segment OB is pulled away from P. There does not seem to be a 'partially robust' construction as we had before. The teachers' comments concerning this solution were quite similar to those illustrated earlier but in particular, those who were happy with the 'drag-to-fit' solutions were undeterred by the fact that this was a famous 'impossible' construction and argued that it *was* a legitimate construction in a DGE environment. The concerns of the others are best expressed by this teacher's comment:

DGE is just to realize or to make Euclidean construction more convenient not just enhancing it so that it makes something impossible possible. Trisecting an angle in Sketchpad is still impossible if you are obeying the rules of Euclidean geometry unless suppose the programmer had a construction called 'snapping point'. Of course, the programmer can design the software in this way but then...it is not of any pedagogical value. If we want to teach students geometric construction in the Euclidean system then we won't use this software. (Teacher F)

His comment concerning pedagogical value suggests that he considers a 'snapping function' would solve all construction problems without the need for any underlying theoretical knowledge. However, this may be a serious oversimplification. We now discuss some of the important issues raised by these constructions and the teachers' comments.

4. Conceptually exact solutions

The teachers were asked to comment on the legitimacy of certain constructions but what precisely is meant by a legitimate solution? Perhaps first we should distinguish between 'conceptually exact' solutions and 'practical' solutions. For example, in Euclidean geometry there is a very simple construction for bisecting an angle. However, if we actually use this construction with real tools (i.e. compass and straight edge) it is clear that our solution can *never* be exact. We can only construct

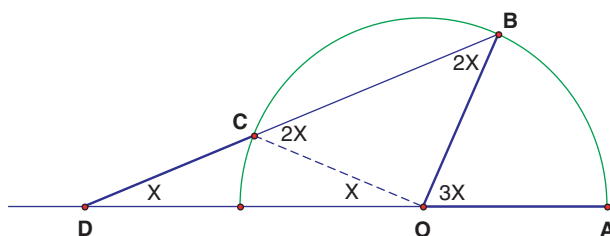


Figure 7. Archimedes' conceptual solution for the angle trisection problem.

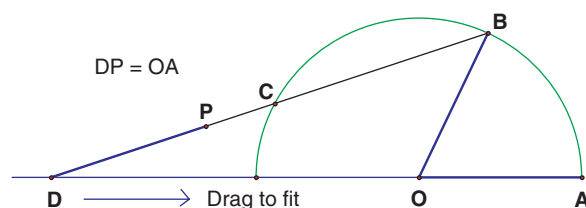


Figure 8. 'Drag to fit' version of Archimedes' solution.

the bisected angle to a particular degree of accuracy that depends on the actual tools we are using. In this sense, the Euclidean construction represents a conceptually exact solution performed with 'conceptual tools'. In fact, the conceptual tools of Euclidean geometry (or the 'rules of the game' as it were) are just two: a compass and an *unmarked* straight edge. The latter tool is of particular interest here. We may gain some insight by returning to the problem of trisecting an angle.

Historical attempts to solve the trisection problem may be divided into two categories. First, a solution that does indeed produce a *conceptually* exact trisection, but does not conform to all the conditions of the problem. Second, a solution that does conform to the conditions but gives only an approximate trisection of the angle. An example of the first type, attributed to Archimedes, is shown in figure 7.

$\angle AOB$ is the angle to be trisected. If a line BCD can be constructed such that $CD = OA$ then $\angle ADB$ is the required trisection. Unfortunately, the construction requires the use of a *marked* straight edge in order to construct BCD and this contravenes the original conditions that only compass and unmarked straight edge may be used. As reported in Eves [17] this approach changes the original problem to one that the ancient Greeks described as a *verging* or *insertion* problem. It is important to note that the ancient Greeks rejected this solution simply because it required a marked straight edge. If such a 'conceptual tool' had been accepted then equally this construction would have been accepted as a legitimate solution. Now, the notion of 'verging' is highly relevant to a dynamic geometry environment because, in a sense, verging can be considered as synonymous with dragging. In fact, we can quite easily set up the construction illustrated in figure 7 within DGE, as shown in figure 8. The required trisection is found by simply dragging the independent point D until P coincides with C (DP is first constructed equal to OA). An alternative drag-to-fit solution to this problem is discussed in Lopez-Real [18].

5. The conceptual tools of DGE

The legitimacy of a geometric construction then is crucially dependent on whether the operations involved in the construction conform to the ‘rules of the game’. This in turn depends on the conceptual tools that enact those rules. So what are the conceptual tools of a dynamic geometry environment? Since Sketchpad (and other DGE software such as Cabri) was designed expressly to facilitate the learning and exploring of Euclidean geometry, it is hardly surprising that the basic tools are Circle (equivalent to Compass) and Segment/Ray/Line (equivalent to Unmarked Straight Edge). In addition we have Point (which simply enables us to place one of the fundamental ‘objects’ of Euclidean geometry) and Pointer (which crucially enables us to drag objects). It is this last tool that is of most importance to us here and indeed is the one that gives dynamic geometries their power.

What is the function of dragging in DGE? We have previously mentioned the range of dragging modalities that are identified in the literature but, taking a broader perspective, we can make a classification based on just two quite distinct functions. The first we can describe as ‘confirmatory’ and the second as ‘exploratory’. In Euclidean constructions the confirmation is established through deductive proof. In dynamic geometries, such as Sketchpad, confirmation comes inductively via dragging to check the ‘robustness’ of the construction. That is, certain elements of the construction are constrained by being dependent on other elements. Dragging the independent elements enables us to see whether or not the constrained relationships remain intact. The exploratory function of dragging is a little different. Here we constrain certain elements of a constructed figure and then drag independent elements in order to see whether we can determine properties of the figure that remain invariant. Many of the tasks designed for pupils using dynamic geometry software are of this nature.

Does the drag-to-fit function that we have illustrated in the solutions belong to either of these categories? Perhaps it does have an exploratory element, but it also has a different element which is implicit in its description ‘drag-to-fit’. That is, it *completes* an action or construction to provide a solution. Let us consider the dragging function as a conceptual tool. It is a tool that enables us to drag a point (or other object) through all possible positions subject to given constraints that we may have imposed. In the particular case of the square-triangle problem (figure 5), when we drag point E along AB, point F also moves subject to the constraint that it is the fourth vertex of the square we have constructed. Our dragging can take point F from a position *inside* the triangle to a position *outside* the triangle. Conceptually then, point F must lie exactly on the line BC during this process. This is not a question of measurement; it is a conceptual reality (in fact, existence of a point). This differs fundamentally from the third solution to this problem that we mentioned earlier.

In figure 9, the teachers argued that it was very simple to construct a rectangle DEFG under the given constraints. By dragging point E along AB it is clear that at some point the rectangle must become a square. Again, this is a conceptually correct argument. But the teachers readily conceded that this was not a satisfactory construction. The problem here is to know at what point this happens. Since the measurements could never be exact, even if the ‘snap function’ were available to us (as described by Teacher F), E has nothing to ‘lock onto’ that would give a conceptually legitimate dragging solution. It is important to emphasize this point

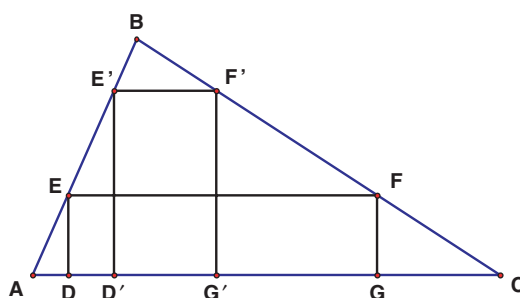


Figure 9. Dragging rectangle approach to the square-triangle problem.

because this is what distinguishes the drag-to-fit solutions we have discussed from other ‘soft’ solutions to construction problems.

If we consider the dragging function in Sketchpad to be a *conceptual tool*, in the way described above, then the drag-to-fit solution to the square-triangle problem (and to the trisection problems) is surely a legitimate one. Of course, if we wish to convert that conceptual construction to a practical one, then clearly the solution will only be as accurate as the design of the software allows. But this is no different from a conceptually accurate Euclidean bisection of an angle being converted to a practical situation. Now, unlike Euclidean geometry, dynamic geometry software does not lay down the ‘rules of the game’ in such an explicit manner. It may be that its *intention* is that the conceptual tools should be equivalent to those of the Euclidean environment. However, in this connection it is worth asking *why* the Euclidean tools of straight edge and compass were chosen in the first place. It could simply be that these were the practical geometrical tools already in use and hence their conceptual counterparts would naturally be chosen. On the other hand, it could be argued that these conceptual tools are already ‘embedded’ in Euclid’s first three postulates concerning straight lines and circles. By analogy, we might accept the dragging tool in DGE as a conceptual tool, simply because it is an existing practical tool in that environment. But we might also consider that DGEs contain a ‘hidden’ postulate that could be expressed as: *a point may move continuously along a line whilst preserving ratios of segments within the line*. Again, if such a position were taken this would legitimize many construction solutions via dragging and it is worth remembering that the ancient Greeks themselves (and other cultures throughout history) did much valuable geometry that did not remain within FAEG.

Our experience with students’ attempted solutions to construction problems, and with the discussion of our MEd teachers reported here, suggests that many of them *implicitly* think of the dragging tool as a conceptual tool. And, of course, the dragging feature is such a powerful and pervasive aspect of dynamic geometries, this is hardly surprising. Much of the discussion here about dragging solutions arose from the rather strange quirk we found in the solution to the segment trisection. One can easily envisage how this could in fact be made a deliberate feature of the software. Clearly one would not want a point (or any other object) to ‘snap on’ to another object whenever they coincide. The feature would need to be similar to the ‘trace option’ which could be activated whenever required. Such a software would make the conceptual dragging solutions discussed above an explicit feature which in

turn would give rise to robust constructions. Within this new environment, would such solutions not be clearly legitimized? But such speculation about a new version of the software begs an important question, which is this: Within the present dynamic geometry environments, should such dragging solutions be legitimate? For those of us brought up in the Euclidean experience there is a natural tendency to reject the legitimacy. However, this may be due to our inability to fully accept the consequences of the dynamic environments. For students today, and perhaps more so in the future, the dragging feature is such an integral element of the geometry that they may well (either consciously or unconsciously) consider it to be a conceptual tool. With that perception it might well appear bizarre to the students that their teachers reject such solutions to construction problems.

6. Discussion

6.1. Theoretical perspectives

We have so far suggested that if we accept dragging in DGE as a conceptual tool (or a tool that can bring about structures and patterns), then certain drag-to-fit constructions in DGE can be regarded as 'legitimate'. Hence, by including a 'pointer' (a proto object that allows dragging) as a fundamental geometrical concept (like that of point, circle), we have new 'rules of the game', or even a new game, for experiencing geometry. Since these drag-to-fit constructions (e.g. figures 3, 5 and 6) in DGE can sometimes lead learners to discover a robust construction or formal proof [10, 11], they can play the role of a semiotic mediator that carries meaning exchange between the two epistemological domains DGE and FAEG (figure 2). However, they are dynamic objects that need 'completion' and thus are transitory in nature. This suggests a refinement of the concept of 'soft construction' in DGE to something that we might call *transient semiotic construction*. Referring to the discussion in the Introduction to this paper, these transient semiotic constructions may also be those object-preserving transformations in Winslow's idea of signification of mathematical objects.

In a discussion on internalization of dragging as theoretical control, Mariotti [19] commented on conjecture making as a result of dragging this way:

... the interpretation of perceptive invariants in terms of a geometric conjecture requires the interpretation of the dragging control in terms of theoretical control. One must be aware that the hierarchy of a construction realizes a logical relationship relating given properties perceived as invariants in the dragging mode. In other words, the use of Cabri in the generation of conjecture is based on the internalization of the dragging function as a logic control, capable of transforming perceptual data into a conditional relationship between hypotheses and thesis.

The notion of dragging function as theoretical/logic control aligns quite well with the idea that dragging is a conceptual tool, in that dragging is a semiotic instrument that can *empower* (enhance the capability of) a learner to transform visual information into logical arguments. This empowerment is rooted in a hierarchy of DGE construction that follows some kind of 'dragging logic', for example, a transient semiotic construction that can be 'activated' by a drag-to-fit strategy. The work of Arzarello *et al.* [12] also suggested a hierarchy of dragging modalities

mentioned in the Introduction (i.e. wandering, bound, guided, dummy locus, line, linked and dragging test) under the auspices of cognitive ascending and descending processes between drawing and theory:

Ascending and descending processes shown by dragging practices in Cabri reveal cognitive shift from the perceptual level to the theoretical one and back in students' mathematical activity.

In particular, 'the transition (cognitive shift) is ruled by *abduction*' [12]. Magnani [20] describes abduction as a kind of reasoning that expresses likelihood:

... theoretical abduction is the process of *inferring* certain facts and/or laws and hypotheses ... that *explain* or *discover* some (eventually new) phenomenon or observation; it is the process of reasoning in which explanatory hypotheses are formed and evaluated. (pp. 17, 18)

This type of '*inference to the best explanation*' ([20] p. 19) fits well in the study of dragging in DGE. In fact, 'dragging logic' can be thought of as a kind of 'manipulative reasoning ... that embody tacit inferences in procedures involving visual and tactile performances' ([20] p. 54). Thus, a 'theory' of dragging could be a kind of interpretive process in which inferences are made from visual dynamic data to hypotheses that best explain the visual dynamic data, that is, one should not anticipate an absolute and perfect explanation of dragging behaviour.

Other than looking for a pragmatic theory of dragging (e.g. investigating dragging strategies), we can also consider meta-theory that classifies dragging strategies. As illustrated and discussed in previous sections, we propose to regard dragging as a conceptual tool that can facilitate the formation of theoretical concepts rather than merely a tool for confirmation and exploration. This comes about when we can visually experience patterns of variation via dragging in DGE on the computer screen. The experienced variations (for example, contrast and generalization) enable us to discern the geometrical properties or concepts that are realized in dragging. In Leung [6], an attempt was made to categorize different dragging strategies under different patterns of variation in a specific DGE problem-solving scenario in which, for example, wandering dragging was for making contrast (whether the dragged object satisfies certain condition or not) and dummy locus dragging was for generalization (the traced locus is a *locus of validity* under certain dragging procedure). Thus studying the relationship between the pattern of variation and dragging strategies may open a door to the construal of a semantic system of dragging in which the geometrical reality in DGE can be 'encoded'.

The above discussion can initiate a direction of investigation to further develop an elaborated understanding of dragging strategies and indeed a grammar for dragging in DGE that can foster a process of internalization under which geometrical meanings can be shaped. If such a semantic of dragging can be established, then learning dragging strategies in DGE may be conceived as learning about geometry in DGE and dragging as a conceptual tool in DGE would have a firmer legitimate status.

6.2. Pedagogical perspectives

Of equal, or perhaps greater, importance than the problem of establishing firm theoretical foundations to the questions raised in section 6.1, is the problem of how

these ideas are perceived in the classroom. After all, this is where the geometrical learning of students largely takes place and where the perspectives and attitudes of the teachers can have a significant impact. As we have seen from the teachers' comments in section 3, the views of this group of experienced teachers were dichotomized between those rejecting the 'drag-to-fit' solutions and those accepting such solutions as legitimate in a DGE. Teacher A's first comment (in section 3.1) that this is not what 'a standard geometric construction should mean' strongly suggests that he is using the rules of the FAEG game in making his judgments. By contrast, Teacher B's first comment (also in section 3.1) suggests that she is implicitly accepting a different set of 'rules of the game' that are indigenous to the DGE environment. She refers to 'a way that we can't do by compasses' and her reference to dragging so that 'you can adjust the radius of the circle' uses, within the context of the problem, precisely the 'hidden' postulate that we described in section 5.

How are such different perspectives likely to affect the geometrical experience of students in the classroom? Does it matter whether or not different teachers have such different perspectives, as long as they are made clear to the students? Do teachers make the 'rules of the game' clear to students and if so, how? These are important questions that certainly cannot be answered here. Our paper provides a starting point for such questions that will clearly necessitate a great deal of further research into what is actually happening in the classroom and into the students' own perspectives on their experiences, as contrasted with the perspectives of the teachers. However, without teachers clearly identifying the 'rules of the game' one possible outcome could be the existence of a kind of 'mismatch' between a teacher's understanding and expectations and that of the students. We hope that this paper may act as a stimulus for further discussion and research into dragging solutions of geometric constructions and the impact that teachers' perceptions may have on students' learning through DGEs.

References

- [1] Hoyles, C., 1993, Microworlds/Schoolworlds: the transformation of an innovation. In: C. Keitel and K. Ruthven (Eds) *Learning from Computers: Mathematics Education and Technology* (Berlin: Springer-Verlag), pp. 1–17.
- [2] Hoyles, C. and Healy, L., 1999, Linking informal argumentation with formal proof through computer-integrated teaching experiments. *Proceedings of PME 23: Psychology of Mathematics Education 23rd International Conference*, 3, pp. 105–112, Haifa, Israel.
- [3] Hadas, N., Hershkowitz, R. and Schwarz, B. B., 2000, The role of contradiction and uncertainty in promoting the need to prove in dynamic geometry environments. *Educational Studies in Mathematics*, **44**, 127–150.
- [4] Hanna, G., 2000, Proof, explanation and exploration: an overview. *Educational Studies in Mathematics*, **44**, 5–23.
- [5] Furinghetti, F. and Paola, D., 2003, To produce conjectures and to prove them within a dynamic geometry environment: a case study. *Proceedings of PME 27: Psychology of Mathematics Education 27th International Conference*, 2, pp. 397–404, Honolulu, USA.
- [6] Leung, A., 2003, Dynamic geometry and the theory of variation. *Proceedings of PME27: Psychology of Mathematics Education 27th International Conference*, 3, pp. 197–204, Honolulu, USA.
- [7] Hölzl, R., 1996, How does 'dragging' affect the learning of geometry. *International Journal of Computers for Mathematical Learning*, **1**, 169–187.

- [8] Arzarello, F., Micheletti, C., Olivero, F. and Robutti, O., 1998, Dragging in Cabri and Modalities of Transition from Conjectures to Proofs in Geometry. *Proceedings of PME 22: Psychology of Mathematics Education 22nd International Conference*, 2, pp. 32–39, Stellenbosch, South Africa.
- [9] Straesser, R., 2001, Cabri-geometrie: does dynamic geometry software (DGS) change geometry and its teaching and learning. *International Journal of Computers for Mathematical Learning*, 6, 319–333.
- [10] Leung, A. and Lopez-Real, F., 2000, An analysis of students' explorations and constructions using Cabri geometry. In: M.A. Clements, H. Tairab and W.K. Yoong (Eds) *Science, Mathematics and Technical Education in the 20th and 21st Centuries* (Darussalam: Universiti Brunei), pp. 144–154.
- [11] Leung, A. and Lopez-Real, F., 2002, Theorem justification and acquisition in dynamic geometry: a case of proof by contradiction. *International Journal of Computers for Mathematical Learning*, 7, 145–165.
- [12] Arzarello, F., Olivero, F., Paola, D. and Robutti, O., 2002, A cognitive analysis of dragging practices in Cabri environments. *Zentralblatt für Didaktik der Mathematik*, 34(3), 66–72.
- [13] Balacheff, N., 1993, Artificial intelligence and real teaching. In: C. Keitel and K. Ruthven (Eds) *Learning from Computers: Mathematics Education and Technology* (Berlin: Springer-Verlag), pp. 131–158.
- [14] Winslow, C., 2004, Semiotic as an analytic tool for the didactics of mathematics. *Nordic Studies in Mathematics Education*, 9(2), 81–99.
- [15] Lopez-Real, F. and Leung, A., 2004, The conceptual tools of Euclidean and dynamic geometry environments. Paper presented at Topic Study Group 10: Research and development in the teaching and learning of geometry, ICME 10. Copenhagen, Denmark.
- [16] Polya, G., 1957, *How to Solve It*, 2nd edn (Princeton, NJ: Princeton University Press).
- [17] Eves, H., 1990, *An Introduction to the History of Mathematics*, 6th edn (Philadelphia: Saunders College Publishing).
- [18] Lopez-Real, F., 2004, Using the history of mathematics as a starting point for investigations: some examples on approximations. *Teaching Mathematics and its Applications*, 23(3), 133–147.
- [19] Mariotti, M.A., 2002, Influence of technologies advances on students' maths learning. In: L. English, M. Bartolini Bussi, G. Jones, R. Lesh and D. Tirosh (Eds) *Handbook of International Research in Mathematics Education* (Mahwah, NJ: Lawrence Erlbaum Associates), pp. 695–721.
- [20] Magnani, L., 2001, *Abduction, Reason and Science* (Kluwer Academic: NewYork).