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USING DYNAMIC GEOMETRY SOFTWARE TO ADD CONTRAST TO GEOMETRIC SITUATIONS – A CASE STUDY

ABSTRACT. There is no lack of suggestions concerning how Dynamic Geometry Software (DGS) may support heuristic approaches to problem solving. However, uses of DGS are often limited purely to a verifying role, in the sense that students are expected to vary or confirm empirically at the computer geometric data which are more or less given. By contrast, it seems worthwhile to seek other uses of DGS which go beyond mere confirmation so that the geometric situation is recognised in its particularity. This paper provides a case study that emerged from a project in which DGS formed an integral part of the pedagogical arrangement. The study is intended to show how the contrasting power of DGS might be utilised in a guided discovery setting.

KEY WORDS: Dynamic Geometry Software (DGS), heuristic problem solving, Dynamic geometry

INTRODUCTION

My starting point is a teaching episode that occurred six years ago. At that time I was a visitor to the Institute of Education, London, UK. During a period of conducting teaching experiments together with Celia Hoyles, Lulu Healy and Richard Noss, in which we used Dynamic Geometry Software (Goldenberg and Cuoco, 1998) at an urban comprehensive school, one of the participating teachers reported that he had given his students a worksheet designed to help construct the altitudes of a triangle and subsequently encourage ‘dragging’ the vertices – ‘to see what happens’. Rather strikingly (for the teacher, and for us), none of the students mentioned that all three altitudes intersected at only one point, and even when their attention had been explicitly directed to this remarkable fact, it transpired that for most of the students such an observation was neither something special, nor very interesting.

As we reflected with the teacher on his experience, we concluded that we should not interpret the students’ responses purely as a result of lack of interest, but rather as a phenomenon that had to be viewed in the light of the curricular situation that existed in the UK at the time (and, to my knowledge, still does). Geometrical constructions in a classical, euclidean style had not been incorporated into the National Curriculum, neither had



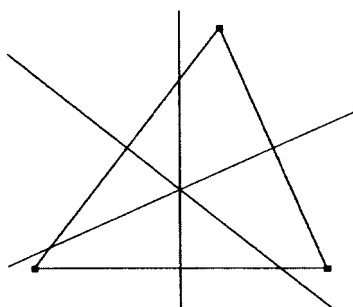


Figure 1. The perpendicular bisectors.

problems that touched on questions of incidence. Thus it was quite understandable that dynamic variations of geometric configurations failed to enable the students to appreciate the phenomena on the screen: dynamic geometry is based on the euclidean method, and therefore presupposes it to some extent within more advanced activities.

At some point in our discussion with the teacher, Celia Hoyles asked me whether I would expect something similar with students at a German *Gymnasium*, where a synthetic style of geometry is still intensively practised.¹ Or would I expect that the students' geometric perceptions – due to their previous experience with ruler and compass – had already been honed enough to recognise the common point of the altitudes (or perpendicular bisectors, angle bisectors, medians) as something special? My appraisal then was fairly optimistic; I thought the latter would be the case, but learned otherwise when I returned home and began dealing more systematically with my host's question.

For example, Hefendehl-Hebeker (1994) reports the following episode from a lesson in which the teacher introduces the perpendicular bisectors of a triangle. The set-up of the drawing (leading to Figure 1) comes to a close with the teacher asking if there is anything striking. A student responds: 'The bisectors all meet approximately in the middle.' As it turns out he does not emphasise the relation of the point to the bisectors, but its position inside the triangle.

The teacher picks this up but with a (presumably unintentional) change of meaning, because what he writes down on the blackboard as an 'assumption' reads: 'All three bisectors of triangle *ABC* intersect at one point.' Here, the student contradicts, insisting that this is *not* an assumption because it is correct anyway; the assumption is that the bisectors meet in the *middle*.

Clearly, there is a mismatch between the teacher's and the student's view of what is obvious and what is subject to proof. The chosen drawing

directs the student's attention to the phenomenon 'middle', which is of some importance in everyday life, but needs further conceptual refinement within a mathematical context.²

Another finding in the literature strengthens this point, this time in relation to dynamic geometry. Again in the case of perpendicular bisectors, Seyffert (1995) reports that none of the students she observed mentioned that the lines were concurrent. Instead, dragging the vertices of the triangle gives rise to observations other than those intended; for instance, that the point of intersection can be outside the triangle, or that the bisectors become parallel as the triangle collapses to a segment.³

So far, so good. I could have let the affair rest and not pursued it any further, however instructive the episodes may have been. However, there was something that concerned me, something that seemed to raise questions about the effective use of dynamic geometry, and this concern is addressed in the following section in more detail.

'VERIFYING' USES OF DYNAMIC GEOMETRY

I start with a critical view of the role that is attributed to DGS, e.g., *Cabri*, *Euklid*, *Geolog*, *Sketchpad*, *Thales*, . . . in the literature (e.g., Henn, 1994; Henn and Jock, 1993; Schumann and Green, 1994). In these texts, the reader is provided with numerous examples of how DGS can support the heuristic phase of problem solving, for example. However, a closer look at various examples may call into question the assumption that the software is used effectively to support a methodical *and* an active style of knowledge acquisition. Often, it seems to me, DGS is used only in a *verifying* manner: that is, learners are just supposed to vary geometric configurations and confirm empirically more or less explicitly stated facts. Examples frequently mentioned are:

- the intersection property of special lines in a triangle (see above);
- the sum of the distances between a point inside an equilateral triangle and its sides (see below);
- the Euler line (i.e., the line always carrying the circumcentre, the orthocentre, and the centroid).

Let me elaborate. Consider the difficulties involved in the following example by Elschenbroich (1997, p. 63):⁴ 'Draw an equilateral triangle. Choose an arbitrary point inside the triangle and measure the distance x , y , z to each side, respectively, cf. Figure (Figure 2). What can you say about the sum $x + y + z$? Is there any relation to other data in the triangle?'

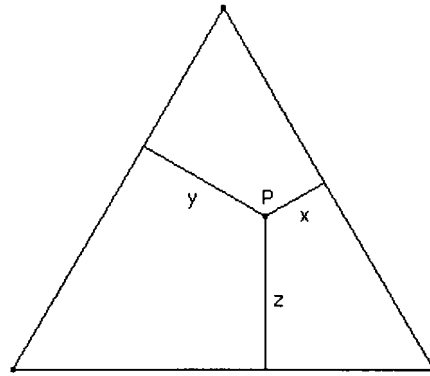


Figure 2. Sum of the distances.

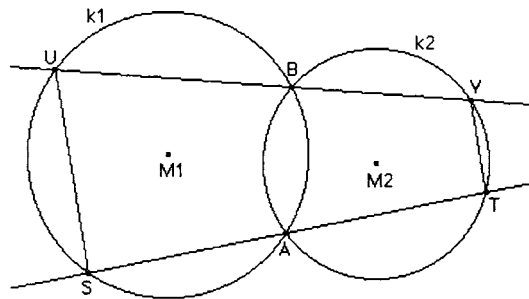


Figure 3. A trapezoid results.

Given such predetermined conditions, the question arises which learning processes are supposed to be initiated. Obviously, the task amounts to recognising that the sum of the distances is invariant. But such DGS activity can hardly be expected to yield more than empirical confirmation, nor is it likely to point to a promising solving strategy.⁵ In fact, such a strategy necessitates a mental restructuring of the triangle into appropriate equilateral triangles, thus requiring a purposeful view that already draws on heuristic skills but hardly comes to mind by just dragging P around.

Another example of a verifying use of DGS is provided by Jock (1994, p. 16):⁶ 'Draw two circles k_1 and k_2 , that intersect in two points A and B . On k_1 choose two points S and U . The line $g = SA$ intersects k_2 at T , the line $h = UB$ intersects k_2 at V . The points $STUV$ are vertices of a quadrilateral' (Figure 3).

'What is the quadrilateral like? Move S or U on k_1 to come to an assumption. Justify your assumption.'

The way the problem is posed is likely to initiate only a narrowly limited search process. Finding the assumption that the sides SU and TV are parallel comes with its empirical confirmation. Also, dragging the vertices S or U as proposed is unlikely to arouse any feeling of illumination, one that would help look for an explanation. The main challenge consists of finding an appropriate line which decomposes the given quadrilateral such that refined mathematical tools can be employed. For example, the segment AB gives us two quadrilaterals $SABU$ and $ATVB$, both of which are cyclic quadrilaterals having the property that opposite angles are supplementary. Thus, the interior angles at U and V , S and T , respectively, are also supplementary.

A verifying style of using DGS as described can be contrasted with a different style that seeks to offer more than just pure confirmation of geometric facts. DGS activities conducted in such a different style intend to create a heuristic context for a specific geometric question, for instance, by embedding the question into more general or more particular situations. Employing DGS in a learning situation, therefore, takes care not to isolate facts, but to put them in relation to each other. In so doing, it is hoped that students are helped to recognise the significance of a specific geometric property (see Introduction). This will be elaborated further in the following section.

‘CONTRASTING’ USES OF DYNAMIC GEOMETRY

As mentioned above, the altitudes of a triangle intersect in one point, the so-called orthocentre. Within a context of geometric loci one can, for instance, ask about the locus of the orthocentre under the restrictions that the vertices of the triangle are on a circle and one vertex is moving through the circle. The result is that the orthocentre, too, appears to be moving through a circle (Figure 4).

Following this observation, it seems reasonable to go about a proof. However, motivating it in the classroom can face difficulties. First, there is the plausible expectation on the part of the students that the outcome is not surprising given that the moving point itself was restricted to a circle. Second, the locus generated by DGS is sufficiently accurate, such that even an expert would hardly doubt the obvious shape of the locus, notwithstanding that s/he would look for an explanation: How is it that H moves on a circle that, apparently, is a reflection of the original circle in AB ? While DGS at first sight may decrease the need for proof in this situation, it can paradoxically do proof a good service if one is prepared to postpone it temporarily in order first to embed the result in a suitable context.

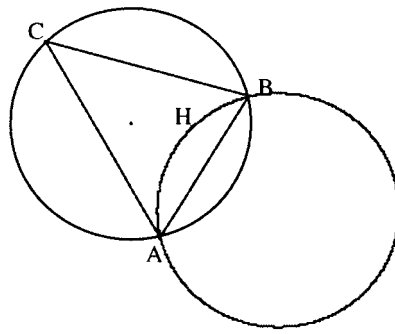
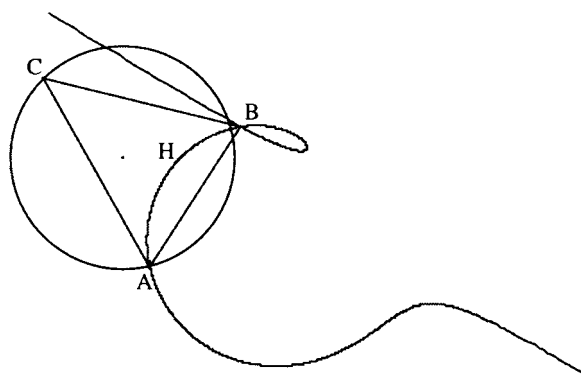


Figure 4. Locus of the orthocentre H .

What happens to the locus if not all vertices of the triangle are on a circle? For example, if B were outside – just a bit (Figure 5). Would we again expect H to move through a circle? Some of my own students at the time predicted this would be the case, suggesting that the locus would be a larger or smaller circle, possibly even ‘an oval’, but the result was baffling. Although the locus still has a distinctive section of a circular shape, as soon as C approaches B the orthocentre, in the words of a student, ‘turns crazy’.⁷ The earlier result – the locus as a circle – is now shown to be nothing that we can take for granted. Quite the reverse. Obtaining a circle almost amounts to being a stroke of luck, so to speak, once we continue to change the position of B or even additionally detach A from the given circle. It would seem that if students encounter situations during exploratory work that strike them as surprising, astonishing, or even intriguing, they are more likely to become seriously involved in subsequent analytical work. For it is not removal of doubt about the result that drives analytical work, but the quest for an explanation (de Villiers, 1998).

To summarise this and the preceding section I propose the following question as a means of preparation for the meaningful employment of DGS: *Is it possible to use DGS beyond the pure confirmation of geometric facts by creating an appropriate heuristic context in such a way that the significance of the geometric situation can be recognised more easily?*

Of course, no example can speak for itself or can guarantee that its significance will be revealed in the classroom. To this end, variable degrees of ‘mathematical maturity’ must be developed in the course of learning mathematics. However, I do believe that there are more purposeful and less purposeful examples for the use of DGS. At least, taking the specific approach into account, it would seem that the often mentioned fear that the computer hinders the development of an already problematic need for proof is too sweeping. It is the context in which the computer is a part of



the teaching and learning arrangement that strongly influences the ways in which the need for proof does – or does not – arise.

A CASE STUDY

Up to now examples have been presented in order to illustrate my concerns and to clarify what I understand by verifying and contrasting uses of DGS. I now turn to a case study that developed during a 2-year research project in which DGS formed an integral part of the pedagogical arrangement. The project was microethnographic in character within a theoretical and methodological framework of 'social constructivism' (Bauersfeld, Krummheuer and Voigt, 1988). It took place during regular geometry classes at secondary level (Hölzl, 1999).

This case study has been selected because of its relation to what has been set out in the preceding sections, namely the importance of finding opportunities in which the use of DGS is one that allows students to deal with situations that contrast with each other. It goes without saying that the study is not meant to have evidential value in terms of statistical representativity. Its prime purpose is to explore the approach described above and to point to those aspects that come to the fore once DGS is used in such a guided discovery setting.

Let us return to the perpendicular bisectors of a triangle. A plausible explanation of the fact that students find it difficult to recognise concurrent lines as something special is that they lack the necessary background against which incidence phenomena become meaningful. In fact, a closer study of triangles and their synthetic properties usually does not start until the introduction of systematic (i.e., deductive) geometry at the age of 13.

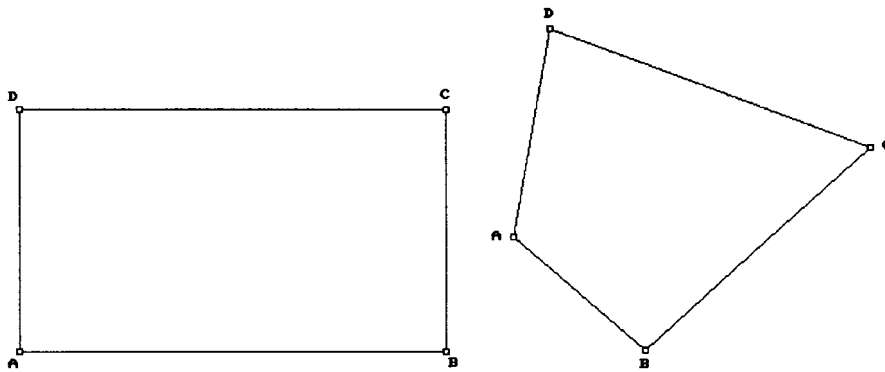


Figure 6. The predefined figures: a rectangle and a general quadrilateral.

So how, in the case of the bisectors, could opportunities be created that are more likely to lead to meaningful activities? Because we want the concurrent bisectors to be recognised as something special, it could be worthwhile looking for related examples in which such a property cannot be taken for granted. This was the starting point of a teaching unit on which the case study is based. The design of the unit was such that students were supposed to engage in computer work structured by a worksheet, before dealing with the perpendicular bisectors of a triangle.

The worksheet introduced *quadrilaterals* instead of *triangles*. In addition, it did not speak of bisectors and their intersection property, but of *circumcircles*. The underlying rationale was to transform the question of concurrency into a question of construction for which bisectors are a conceptual tool. The instructions on the worksheet intended a step-by-step process of generalisation: Students were first supposed to load a predefined square on the computer and work with it, followed by a rectangle, and eventually a quadrilateral that apparently had no particular properties (Figure 6). The task at each step was to investigate whether a circumcircle could be found. If students succeeded in constructing one, they were supposed to check it by dragging. In the case of a solution that had been found by eye the circumcircle would not have carried all vertices along with it.

My appraisal of the situation was that virtually all students would expect that a circumcircle existed in the case of the square and the rectangle due to symmetry. Additionally, the primary impression that the shapes convey would give a clear enough hint concerning the position of the circumcentre, so that constructing it by means of diagonals or bisectors would be straightforward. By contrast, the quadrilateral in the third step did not show any regularity that would give an immediate hint as to the

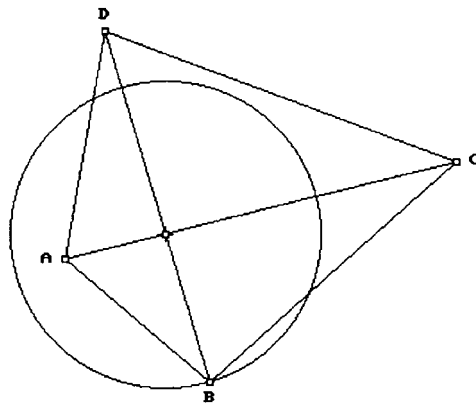


Figure 7. Using diagonals fails to identify the circumcircle.

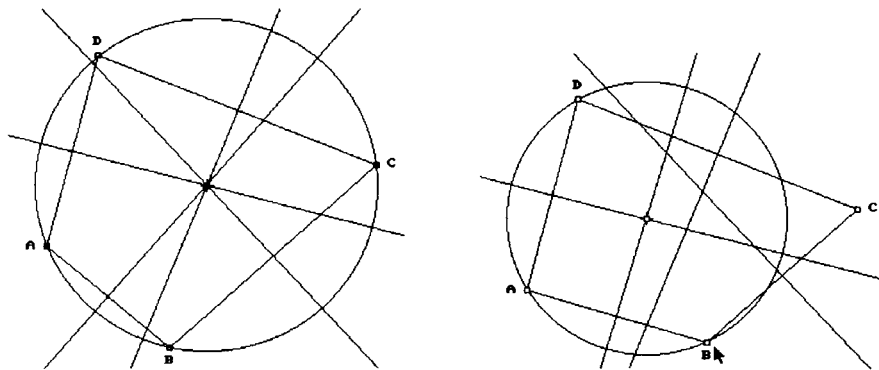


Figure 8. Using bisectors also fails as soon as a vertex is dragged.

position of the circumcentre; though it was reasonable to assume that students would try to use strategies that had worked before. At this point the worksheet, together with the DGS file QUADRIL.PIC,⁸ intended to provoke a conflict between the expectations of the students and the actual result, because;

- what worked with the square and the rectangle, namely, using diagonals to pinpoint the circumcentre, fails with the quadrilateral (Figure 7);
- whereas resorting to perpendicular bisectors leads to a circumcircle that is not preserved as soon as one of the basic points, say B , is dragged (Figure 8).

The conflict, deliberately incorporated into the activity, was supposed to switch to a more open mode of exploration in which subsequent steps

were no longer predetermined and active engagement with the problem situation induced (Becker and Shimada, 1997). With file QUADRIL.PIC, for example, it could be observed that the quadrilateral is only circumscribed by the constructed circle if all four bisectors meet in one point. If they fall apart, the circle is no longer the circumcircle of the quadrilateral.

The subsequent course of action was therefore hardly foreseeable for the students. It should show how the students would react to a concrete problem situation and how they would use the computer as a problem solving tool. Important for me, as a researcher *and* a teacher, were two things:

1. that the students would actively pursue the question of cyclic quadrilaterals, and thereby make the problem meaningful;
2. that they would eventually appreciate that there were some quadrilaterals with a circumcircle but also apparently some without.

The first point was relevant with respect to research methodology as the students' handling of the problem was supposed to serve as a *lens* on their geometric thinking and the influences of a computational environment.⁹ The second point was relevant with respect to our teaching objectives, as the understandings that the students would have developed could be exploited in the case of the triangle – having experienced that most quadrilaterals are *not* cyclic.

It should be noted that the prime criterion for evaluating success was *not* whether the students could completely uncover the underlying mathematics on their own. This could not be expected given the relative complexity of the task. Even if the question of the intersection property of perpendicular bisectors is reformulated as a problem of finding a circumcircle, the circumcentre must be interpreted in a metric sense, namely, as a point equidistant from the vertices. Such a point is not found immediately, but rather its position has to be delimited in a chain of conditions that require one to recognise each bisector as a locus of points equidistant from vertices of the corresponding segment.

Changes of perspectives and re-interpretations are meshed in a complete solution to the problem. It seemed therefore wise to assume that entangling all these threads was too difficult a task for students at the beginning of systematic geometry. The following is an account of how 13-year-old Cordula and Christina went about doing the tasks on the worksheet.

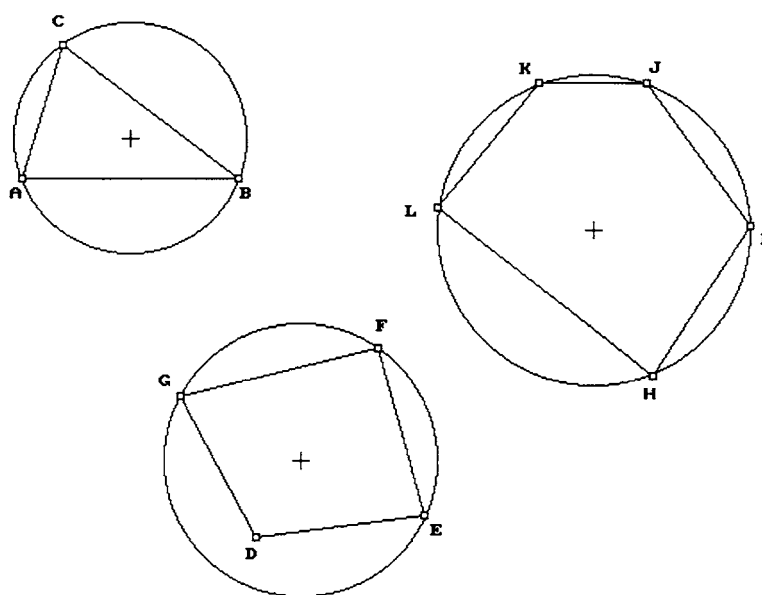


Figure 9. Introducing the concept of circumcircle.

Pedagogical Context

After distributing Notebooks for group work, the teacher gave some technical hints concerning *Thales*, then proceeded to open a file on his computer which was displayed on the wall (Figure 9).

The figures serve to illustrate the concept of circumcircle by example and counterexample. After an introductory discussion about some properties of the figures, the teacher handed out the worksheet summarising the students' task 'to find such a circumcircle – provided that there is one.'

Finding A Circumcircle For A Polygon . . .

Cordula and Christina were assessed as being about average attainment for the class by their official marks in mathematics. As far as technical aspects of DGS use are concerned, they had familiarised themselves with *Thales* well.¹⁰ The first two steps in the worksheet were processed fluently. I only mention that the students agreed on using diagonals to construct the circumcircle of both the square and the rectangle.

Eventually, they opened the file QUADRIL.PIC, commenting sceptically on the possibility of a circumcircle. Nevertheless they determined a circle by again choosing the intersection of both diagonals as centre and a vertex as radial point (Figure 7). The result on the screen confirmed their sceptical stance, but unlike before, the instructions on the worksheet did

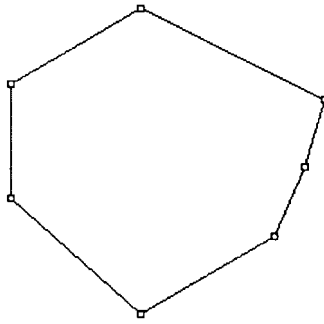


Figure 10. Testing the conjecture.

not simply suggest the next steps; they requested a justification,¹¹ thereby changing their mode of working.

In what follows, we can assume that the students were certain that the quadrilateral on their screen had no circumcircle. This assumption can be backed up by corresponding sections in the transcript where

- they realise that the preceding successful use of diagonals now fails;
- a perceptual impression is articulated that points to the missing symmetry: “Because somehow it’s not even”;
- an analogy to the quadrilateral on the screen projection is observed (Figure 9): ‘That one also hasn’t got any reflective symmetry.’

From their subjective point of view, the students were now obliged to justify their opinion and write it down. Cordula was thinking of a formulation – “No, because the distance to . . .” – when her partner interrupted with a surprising conjecture. Christina, looking at the projected figures shown in Figure 9, noticed that there was a circumcircle for the triangle and pentagon. All of a sudden she came up with a generalisation to the effect that

(*) a polygon with an odd number of vertices is cyclic.

The polygons square and rectangle were seen as exceptions because they show reflective symmetry.¹² Missing symmetry had already been mentioned to explain why both quadrilaterals projected on the wall, have no circumcircle. Cordula instantly took Christina’s conjecture on board and because she had saved her quadrilateral on disk, she, as she put it, could erase their dynamic sheet to try a “silly thing”. The “silly thing” turned out to be the construction of a crucial experiment (see Balachaff, 1988) a heptagon (Figure 10), drawn full of determination, and intended as a check for Christina’s conjecture.

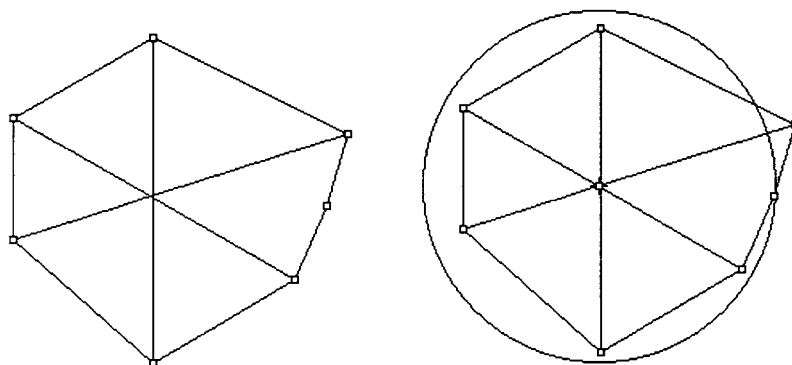


Figure 11. Trying to find the circumcircle.

However, the heptagon as a testing figure entailed a new difficulty: The circumcentre that was expected due to the conjecture (*), had to be determined, but “how do we best find the centre”? Christina proposed to draw in all diagonals but Cordula opposed, expecting an unnecessary effort: “Two diagonals will do”. And she succeeded in convincing her partner because a third diagonal, which she drew, went – accidentally for an expert, but as expected by Cordula – through the intersection of the others (Figure 11, left).¹³

Cordula was dominating the activity, but while she was preparing to intersect two of the diagonals, Christina felt uncomfortable, pointing her finger at the single vertex not yet part of a segment. As is evident from the video tape, she was worried that there was still a diagonal to this vertex that might *not* meet the point of intersection. But Cordula claimed there is ‘nothing to care about’ because they were already in a position to turn to the circumcircle. With Christina being reassured and the diagonals hidden, Cordula constructed a circle using the single point as a radial point (Figure 11, right).

The result did not meet their expectation and a reflective phase ensued. As a possible explanation that the circle just drawn did not turn out to be the circumcircle, the pair first mentioned the missing reflective symmetry, but eventually dropped the conjecture (*) when Christina brought in a new point of view.

Choosing The Polygon For The Circumcircle!

Christina resorted to her knowledge about sums of angles and attempted to sort the present results in a new way: Perhaps “all stuff” where the sum of the interior angles equals 180° and 360° , respectively, has a circumcircle?! Now, if that “funny pentagon” (Figure 9) with its circumcircle

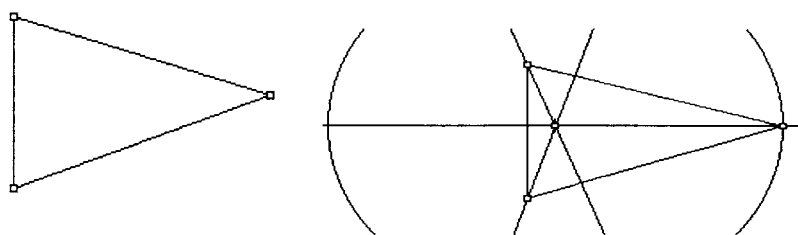


Figure 12. Investigating a triangle

had a total sum of 360° , then ... , but there she faltered because with the “funny quadrilateral” that was projected there was also a sum of 360° . Nevertheless Cordula too, after a short time of hesitation, fell in with her partner’s idea, except that she assumed a specific angle rather than a sum of angles to be a sufficient condition for a circumcircle. Both agreed anyway to have a go at “measuring angles”.

But which angles and in which figure? They first decided to try a pentagon, but then chose a triangle: “That’s easier”. After Christina drew a triangle, approximately isosceles (Figure 12, left) a new difficulty arose when she realised that a triangle has no diagonals to be used as before: “We can’t get a centre”. Nevertheless, pointing with their fingers to the screen they indicated a possible position for the centre and how it might be constructed: “Yeah, perpendicular line, exactly.” It seems that the position of the triangle directed their attention to a horizontal axis of symmetry, which they interpreted as a perpendicular line. But even before drawing a circle around the orthocentre with a vertex as radial point, they suspected that it would not work out well (Figure 12, right).

The pair ironically bemoaned the result of their attempt to construct the circumcentre. But, Cordula hit upon a new idea, which she does not communicate to her partner verbally but carried it out on the computer. The circumcircle to be looked for was drawn *first* while a triangle was inscribed *afterwards*. Using the appropriate commands yielded a triangle whose vertices could, in fact, be moved along the circle without leaving it. For the time being, however, the pair’s interest was in only a *single* triangle with its circumcircle and angles (Figure 13). Unfortunately, measuring all three interior angles did not reveal any striking regularity.

There is no doubt that Cordula had an important idea. If one assumes that interior angles play an important part in the quest for a circumcircle then it would be heuristically valuable to vary the data that are under observation. In what follows, the student switched to drag mode and first shifted the circle (by mistake) while afterwards she pointed with her mouse to the

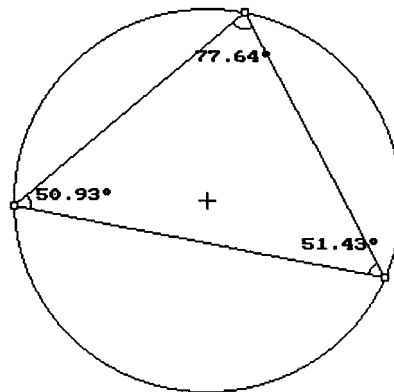


Figure 13. Measuring angles.

left vertex. By construction, the adjacent angles at the opposite side were changing while the angle at the varying point remained constant.¹⁴

Christina commented on what was happening on the screen, suggesting that although the vertices can be varied, it was of no use. Although the right angle theorem can be adjusted, it does not lead to anything worthwhile.¹⁵ The pair at a loss, Christina summarised what had been found so far: As can be seen from her remarks, all observations collected fall only into single cases that do not show any apparent relation:

- Only with a square and a rectangle is the existence of a circumcircle guaranteed by reflective symmetry,
- whereas the “Funny Pentagon” does not need any symmetry,
- but a quadrilateral that is not a square or a rectangle is not cyclic.

Cordula disagreed with the last point and her comments emphasise yet again her shift of perspective: Instead of starting with a quadrilateral and looking for a possible circumcircle, she imagined a circle on which to put a quadrilateral. Perhaps this new point of view clearly showed Cordula that there are in fact cyclic quadrilaterals *without* symmetry. Again, as in the earlier case of the triangle, Cordula set out to translate her idea into a DGS figure. She opened a new dynamic sheet, created a circle, and tried to inscribe a quadrilateral by means of the commands ‘point on object’ and ‘segment’. The attempt was successful, and: ‘This quadrilateral is quite different’. Subsequently, the pair again measured the interior angles (Figure 14), because “that must be something to do with the angles, aye, sums of angles”.

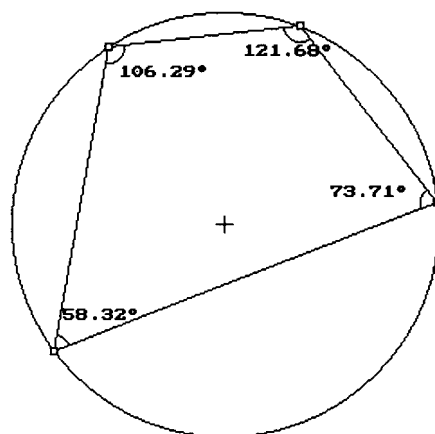


Figure 14. Inscribing the quadrilateral.

THE DISCOVERY – BY MEANS OF “THE HUMAN BRAIN”

In the following part of the work, the girls activated the integrated calculator in order to add up the results gained by measuring. Christina typed in the line up to “ $58.32 + 73.71 + 121$ ”, but then closed the calculator because of technical difficulties that arose. Replacing the calculator by “the human brain”, as Cordula put it, she and her partner started adding up mentally. While Christina concentrated on the digits to the right of the decimal point, Cordula attended to the integers of the opposite angles. The results of the calculations $58.32^\circ + 121.68^\circ$ and $106.29^\circ + 73.71^\circ$, respectively, were so striking that Cordula was immediately convinced of the significance of this observation: “Aye, hang on, could it be that the sum of the [opposite] angles always equals 180° ?! ”

After the first wave of joy subsided Cordula pressed on: ‘once again’, indicating that their remarkable result should be tested by a further example. From the point of view of an expert it is evident that the present situation with a quadrilateral on the screen should be varied by dragging its vertices. Interestingly, the pair did not come up with such a procedure. Instead Christina drew on a previous case to confirm Cordula: “Yeah, no doubt, because with a rectangle, 90° , add up to 180° ”. From the students’ point of view, a compelling argument had been found to explain why the quadrilateral they worked on had no circumcircle.

In the final phase of their work Cordula raised the question “if it also works with a pentagon”. Her partner even proposed trying a hexagon. In the end they decided to test a square, loading the predefined file SQUARE.FIG: “We do this, quick.” After a short quarrel about each part-

ner's share in the discovery, Cordula repeated their construction of the circumcircle. As in the case of the rectangle, the square fitted into the relation. Shortly before the end of the lesson the students started with a second worksheet that dealt with the circumcircle of a triangle.

DISCUSSION

If we look back at the students' work from a problem solving point of view then its high heuristic content is striking: its structure is formed by the dovetailing of subjective plausible conjectures and checks:

- (1) When – from the students' point of view – it was certain that the figure in QUADRIL.PIC was not cyclic, a *classification* of figures was made and *generalised* (p. 74).
- (2) To *check* this, the next favourable case, a heptagon, was considered (p. 74). When the result refuted the conjecture,
- (3) a *reflective phase* began where at the end the question was raised whether interior angles have any relation to a circumcircle (p. 76).
- (4) The students set out to follow this *new route* by investigating a triangle (p. 76). In doing so they *overcame* the *difficulty* of finding a circumcircle for a given triangle by a *shift of perspective* that put them on the right track for a significant discovery at the end of the work (p. 77).
- (5) But first, the pair themselves resorted to dragging as a means to *uncover* possible *properties* of the interior angles of a cyclic triangle (p. 77).
- (6) After a further reflective phase the *idea* of constructing the circle first was *transferred* to a quadrilateral. Inspecting measured interior angles, Cordula eventually became aware of a *regularity* that caused her to rejoice (p. 78).
- (7) The students finished their work by *subsuming* previous cases under that general rule, that was ultimately seen as its *confirmation* (p. 79). Judged by the teacher's intentions and goals laid out above (p. 72) the students' work was no doubt successful and illuminating, notwithstanding the fact that, from a normative point of view, closer scrutiny reveals weaknesses that are typical of learners at this level: First signs of systematic argumentation emerge as suddenly as they break down again due to their considerable susceptibility to interference.

Take, for example, the students' use of the diagonals. If their intersection did not provide the circumcentre, then the figure was seen as not being cyclic. The possibility that other lines may be appropriate to construct the circumcentre was not considered at any time; not even when, as in the case

of the triangle, altitudes were used as a substitute, although they obviously did not determine that central point inside the triangle whose position could be clearly indicated by pointing with fingers. Bearing in mind that it is common for students at this level not to distinguish between necessary and sufficient conditions, we should not over-emphasise such shortcomings. In fact, within the curricular context to which the study refers, the difference between a proposition and its converse was not explicitly dealt with until the following year.

Finally, we should consider whether the analysis of the students' work yields anything more than a nice story. Put another way: Are there certain points in the case study that can be stripped away from their individual and situation specific character so that it becomes interpretable within a broader context?

Of course, the way the students set to work and arrived at their discovery is entirely individual and unlikely to be repeated. In fact, the work of the other students showed different paths of thought. But the case study was not intended to prove that students will encounter a theorem concerning cyclic quadrilaterals once they work in an appropriate DGS setting. Instead it provides a concrete example of how the contrasting power of a DGS can be utilised to open up new ways of dealing with geometry, thereby initiating and supporting substantial learning processes.

Beyond that, the case study incorporates some issues of general interest because they emerged in the work of the other students, notwithstanding their different characters. These issues involve three aspects of the computational environment which are addressed in the following subsections.

A Pedagogical Aspect

There is the motivational fact that Cordula and Christina were actively engaged for some time in work that was by no means routine and which required a fair amount of endurance, because difficulties frequently arose and had to be overcome, with success only apparent at the end. It seems that the whole learning environment was appropriate for sustaining this process in a way in which the students knew that they could express their ideas on the dynamic sheet and get 'neutral responses' from the computer. There is no such kind of interaction with paper and pencil and often the teacher, unfortunately, is not seen as a neutral person but somebody who expects the right answer. Think of Cordula's "silly thing", i.e., the heptagon in phase (2). Would she have tried the same on paper or in the presence of a teacher?

A Cognitive Aspect

The decisive steps towards the pair's discovery were taken in phase (4) and (6), respectively, by a shift of perspective. One could argue that this is not particular to the computer because on paper it would have also been possible to draw the circle first and then to inscribe the triangle. Also, one could discount the subsequent dynamic measurement of angles because it was of no apparent help. But measuring angles is one thing, seeing the triangle move on the circle is another, and it seems that it is this that was particularly important for Cordula. Despite the apparent failure with the triangle, she held on to her shift of perspective by transferring it to the quadrilateral. Even *before* constructing a quadrilateral, Cordula clearly indicated that there *must* be non-symmetric, cyclic quadrilaterals (p. 77). A reasonable explanation for her insight is that she had a mental picture of an inscribed, possibly moving, quadrilateral.

This would provide an instance where experience and interaction with the software results in mental processes that are helpful even when there is no computer around. Cordula *knew* about the opportunity to create points on objects, knew about the opportunity to vary them. She had already done so with the inscribed triangle, and now – even without carrying it out – she probably thought in the same way with the quadrilateral: Its vertices could be created and moved freely on a circle thus resulting in non-symmetric, but still cyclic, quadrilaterals. This is the primary role of the computer in this episode: it became a cognitive tool (Dörfler, 1993). It helped establish a framework for mathematical actions that could eventually be carried out mentally.

Ideally, learners recognise the opportunities offered by the dynamic sheet in a specific geometric situation and act or reason accordingly. In such a case the computer is more than a mediating bridge, as its function cannot simply be reduced to a learning aid – to be discarded after concepts and procedures have been acquired – but the computer with its opportunities for action would be available possibly only in thought.

It is in this way that growth in abilities over the course of the project can be observed. For example, months after the students' work, the teacher and his class discussed a drawing exclusively at the black board but based on actions grounded in DGS. Again cyclic quadrilaterals played a role but this time with a focus on their variant parts (Figure 15): What changes if A is dragged around the circle?

This question cannot be answered unambiguously as long as it is not clear which *figure* belongs to the *drawing* (Laborde, 1993) i.e., as long as its underlying construction is not made explicit. The students became aware of this fact because they held divergent opinions about the behaviour

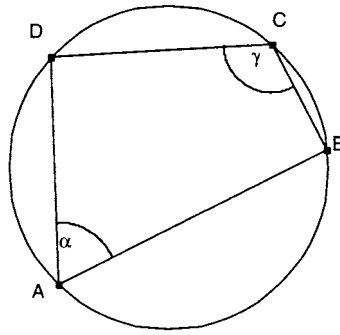


Figure 15. What changes if A moves around the circle?

of the drawing. After the teacher had specified that the points B, C, and D remained fixed while A was thought to be moving, the students recognised for themselves that α would not change because $\alpha + \gamma = 180^\circ$.

The principal feature of this observation is well-known in other software contexts. It points to one of the far-reaching expectations of the educational role of computer use as discussed in the community of math educators: That the computer is a medium particularly appropriate to find forms of representation for knowledge that the learning individual can use as cognitive models.

The mediating function of a computational learning environment, its role as a link between the individual experience of the learner on the one hand and the formal representation of a mathematical knowledge domain on the other have been investigated particularly by Hoyles and Noss for many years (Noss and Hoyles, 1992; Noss and Hoyles, 1996). In their studies they frequently observed the phenomenon that learners recognise mathematical relations on their own – not in a formal way but dependent on the means of expression appropriated by the microworld.

There is no doubt that this, too, points to a significant research problem in the case of dynamic geometry. Which kind of interaction with a DGS can be utilised for geometric thinking particularly when a computer is not at hand? What is the nature of geometric situations where learners are likely to resort to such dynamic thinking? How can we design learning environments for the use of DGS that foster such approaches?

An Epistemological Aspect

Another significant point in the case study has to do with the way the students use the dragging option. It is striking (for the observer) that Cordula and Christina mostly used their dynamic sheet as if it were static. This is particularly surprising at those points where an expert would

certainly use dynamic components, i.e., work in drag mode, for instance at phase (2) and, above all, at phase (7) to test the heptagon and the result respectively. However nothing of this kind happened; instead the original square was loaded again and subsumed under the general rule.

But such behaviour may be surprising only at first sight. It is less striking if one takes into account both the systematic, deductive style of geometry with which the students were still struggling and the requirements that come with an advanced use of drag mode. Taking up the latter point: What are the specific requirements that make for effective use of the drag mode? Certainly there is the paramount relation between drawing and figure (Laborde, 1993; Sträßer, 1991). A drawing is the materialised representation of an ideal geometric figure only characterised by its internal relationships. Dragging is a tool to find different representations of one and the same figure in continuous transition. Because dragging acts on a drawing with the effect being determined by the figure, a mediating function emerges. This mediating function can be used in at least two principal ways:

- A) drag mode as a test mode;
- B) drag mode as a search mode.

In the first case, a construction is checked as to whether it has the desired properties; in the second case, new properties are recognised. The resulting requirements for learners are different in each case. 'A' use presupposes an expectation as to the reaction of the drawing when it is being dragged. The movements on the screen can meet that expectation or not. However, in 'B' use, the changing appearance of the drawing must be evaluated under aspects which are still unknown. The task is to recognise the invariants of the figure and this means precisely to recognise what is *not* changing.

The analyses and supplementary observations from the overall research project indicate that such an advanced appreciation of the meaning of the drag mode is not a short term affair but develops in mutual dependence with the ability to grasp a geometric situation – a learning process that is characterised by different layers of conceptions.

As a first step, students viewed the drag mode as a graphics tool with which to change the appearance of a drawing once it had been set up. The drawing on the screen resembled a mechanical model with rigid or elastic joints (depending on the construction) that determine the behaviour of the drawing in drag mode. Such a conception could be observed above all in introductory phases for DGS during the project.¹⁶

With more experience, the students learned to use the drag mode even without explicit teacher intervention to check their own various conjectures and not only to make the final check of a construction. Drag mode

could already be seen as a test mode although this usage was still shaky.¹⁷ Even the above mentioned mechanical model still had its influence, for instance, when the students primarily used dragging to explore the movement properties of a drawing rather than the relational properties of a figure. In doing so they investigated whether and how parts of a drawing could be dragged. Most of their work could be attributed to this level of competence.

This leaves us with another significant but difficult research task. First, it is necessary to find long term arrangements for the integration of DGS in the learning of geometry that enable students to grasp a geometric situation where a drawing that is dragged is not only *one* drawing which moves and changes but a *sequence* of different drawings with certain common properties. Second, we need to find ways to help students focus on invariants rather than focus on details which suppress the overall impression of a drawing in its concentration on local relationships between parts of a figure.

NOTES

¹ Basically, Germany's school system is a tripartite system with the *Gymnasium* on top (providing qualification for the university). Next comes the *Realschule* (preparing pupils for business and trade) and finally the *Hauptschule* (nowadays realistically a school for the remainder). However, transfer across the various types is now easier than it used to be. In addition (and depending on political issues) a number of Germany's states favour comprehensive schools.

² As Hefendehl-Hebeker points out, a nice learning opportunity has, in fact, arisen: The question of what constitutes the middle of a triangle, or more general, of a polygon, allows different ways to be probed. Discussion may produce such diverse criteria as 'having the same distance from the sides (vertices)', or 'being the centre of rotational symmetry', etc.

³ Such observations may again be the source of fruitful exploration. Here, they only serve to substantiate the argument.

⁴ My own translation.

⁵ It should be added that Elschenbroich mentions in a footnote that one could argue whether one had not better pose the problem in a more open manner, for example, 'What can you say about x , y , z ?' (ibid., 63). In doing so, however, the heuristic role of DGS would not be significantly enhanced: Dragging P while watching *three* simultaneously changing variables does not suggest to consider their sum.

⁶ My own translation.

⁷ It should be noted that the dynamic flavour, unfortunately lost on paper, adds its own quality to the generation of the locus. The reader is invited to use a DGS on her/his own, taking care that s/he produces the locus by dragging, not automatically!

⁸ The DGS *Thales*, running under MS DOS, had been used throughout the project.

⁹ I prefer the term *lens* to *window* (see Hoyles and Noss, 1996) because a lens magnifies smaller parts of reality, may defamiliarize or even distort them. The look *through* a window, however, reminds us less of the medium's influence on what is examined.

- ¹⁰ It was the seventh time that students had worked in pairs with their PC-Notebooks in their usual classroom. Also, the teacher, i.e., the author, made intensive use of *Thales* during traditional lessons whenever it seemed appropriate.
- ¹¹ If students did not think there was a circumcircle, they were asked to give reasons why.
- ¹² Considered purely logically, (*) does not exclude polygons with an even number of vertices. The difference between necessary and sufficient conditions, however, is not something that students recognise for themselves at this stage.
- ¹³ This, obviously, backs up the argument presented above in relation to concurrent lines.
- ¹⁴ A phenomenon that goes unnoticed. In fact, as other case studies also indicate students at this level concentrate above all on parameters that *change* rather than parameters that are *invariant* (see also Goldenberg, 1995).
- ¹⁵ If one side is a diameter, then the opposite angle is equal to 90° .
- ¹⁶ Certainly this is not independent of the setting in which the introduction took place but further elaboration of this is beyond the scope of this paper.
- ¹⁷ In this regard Cordula's and Christina's work provides a case in point. On the one hand, the students appreciate the drag mode even as a heuristically effective tool; on the other, they check their various conjectures only in static drawings.

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