

Generating Conjectures in Dynamic Geometry: The Maintaining Dragging Model

Anna Baccaglioni-Frank · Maria Alessandra Mariotti

© Springer Science+Business Media B.V. 2010

Abstract Research has shown that the tools provided by dynamic geometry systems (DGSs) impact students' approach to investigating open problems in Euclidean geometry. We particularly focus on cognitive processes that might be induced by certain ways of dragging in Cabri. Building on the work of Arzarello, Olivero and other researchers, we have conceived a model describing some cognitive processes that can occur during the production of conjectures in dynamic geometry and that seem to be related to the use of specific dragging modalities. While describing such cognitive processes, our model introduces key elements and describes how these are developed during the exploratory phase and how they evolve into the basic components of the statement of the conjecture (premise, conclusion, and conditional link between them). In this paper we present our model and use it to analyze students' explorations of open problems. The description of the model and the data presented are part of a more general qualitative study aimed at investigating cognitive processes during conjecture-generation in a DGS, in relation to specific dragging modalities. During the study the participants were introduced to certain ways of dragging and then interviewed while working on open problem activities.

Keywords Conjecturing · Dragging schemes · Dynamic geometry · Invariant · Instrumented argument · Maintaining dragging · Path

Abbreviations

CL Conditional link

DGS Dynamic geometry system

A. Baccaglioni-Frank · M. A. Mariotti (✉)
Dipartimento di Scienze Informatiche e Matematiche, Università di Siena,
Piano dei Mantellini, 44, 53100 Siena, Italy
e-mail: mariotti.ale@unisi.it

A. Baccaglioni-Frank
e-mail: abaccaglinifrank@gmail.com

GDP	Geometric description of the path
III	Intentionally induced invariant
IOD	Invariant observed during dragging
MD	Maintaining dragging

1 Introduction

In this paper we introduce a model describing a process of conjecture-generation that was initially conceived as the outcome of theoretical reflections upon previous studies, and developed after our empirical investigations. In particular, it advances the line of research developed by Arzarello et al. (1998, 2002), by explicitly describing and focusing on certain possible steps of the cognitive processes that may occur when students engage in explorations within a dynamic geometry system (DGS) that involve the use of dragging. Our primary goal was to further investigate cognitive processes that occur during the phase of conjecture-generation in the solution of open problems. Such cognitive processes are associated with specific “uses” of the dragging tool that previous research has classified into “dragging schemes” (Arzarello et al. 2002).

The didactic significance of open problems can be found, for example, in their potential to foster proving processes, by creating curiosity in the solver and giving a clear sense to the argumentation process. In particular, dynamic explorations of open problem situations can foster the production of conditional statements, and the construction of proofs with strong functional relationships between these processes (Boero et al. 1996a, b; Mariotti et al. 1997; Bartolini Bussi 1998). The generation of a conditional statement is an issue of great interest with respect to the didactical goal of the construction of meaning of “a ‘mathematical theorem’ as a system consisting of a statement, a proof and a reference theory” (Mariotti 2000, p. 29). In particular, we wanted to study the contribution that specific uses of the dragging mode, conceived as an instrument (Rabardel and Samurçay 2001; Rabardel 2002), may give to the issue of conjecture generation.

The attempt to construct our cognitive model led to a first a priori conception, which we then refined during an empirical study (Baccaglini-Frank 2010) where we tried it out as a tool of analysis on the collected data. In this paper we will present the first results of our study: the cognitive model, examples of its use on some collected data, and two new notions that have arisen from our reflections and analyses.

2 Theoretical Background

This section describes the theoretical roots of our study (Baccaglini-Frank 2010). In particular we present three main concepts that can be found in the literature: the idea of what a dynamic geometry system (DGS) is, the notion of open problem in a DGS, and that of dragging in a DGS. We grounded our study, more than within a particular “theoretical framework”, upon these founding notions, developed over the years, and modified according to our needs. Thus, each notion will be first collocated in the literature, and then contextualized within our research.

2.1 Working in a Dynamic Geometry System

Generally speaking, we may consider two different worlds: the mathematical world of Euclidean geometry, and the phenomenological world of experience, which includes experience within a DGS. Between these two worlds is what has been referred to as “the experimental-theoretical gap problem” (Lopez-Real and Leung 2006, p. 667). Such an experimental-theoretical gap originates from the fundamentally different natures of the two environments students need to transition to and from. On one hand there is an environment embedded in real time, where dragging can allow a real-time physical experience, and dynamism. On the other hand there is the formal axiomatic Euclidean geometry, a crystallized world, without time, where logical dependency, that is conditionality and deductive reasoning, is the hierarchical organizer. Although processes of conjecture-generation may be facilitated by the dynamic component (whether they are given by a software or only by the mind of the conjecturer), proofs need to be constructed within the Theory, which in traditional mathematics is “static”. Passing from the phenomenological world to the mathematical world through the production of conditional statements is not trivial, as widely documented, for instance by Boero et al. (1996a).

However, a DGS may become a potential bridge between these two worlds, providing teachers with new insights and tools to overcome students’ difficulties.

2.2 Open Problems in a DGS

The terminology “open problem” (Arsac 1999; Silver 1995) refers to a problem or question stated in a form that does not reveal its solution or answer. More precisely, in geometry, open problems have been characterized in the following way.

The statement is short, and does not suggest any particular solution method or the solution itself. It usually consists of a simple description of a configuration and a generic request for a statement about relationships between elements of the configuration or properties of the configuration.

The questions are expressed in the form “which configuration does...assume when...?” “which relationship can you find between...?” “What kind of figure can...be transformed into?”. These requests are different from traditional closed expressions such as “prove that...”, which present students with an already established result. (Mogetta et al. 1999, pp. 91–92)

It may happen, as in some of the previous research, that the production of conjectures is an explicit request in the text of an open problem (for example, Boero 1996a; Boero et al. 2007; Arzarello et al. 2002; Olivero 2001, 2002). When this is the case, we will use the terminology *conjecturing open problem*, to distinguish it from other types of open problems. In this case, the solution consists in elaborating a conditional relationship between some premise and a certain fact, a conclusion. This relationship may be expressed by means of a conditional statement relating a premise and a conclusion. Such a conditional statement constitutes the formulation of the conjecture.

As research pointed out (Mariotti et al. 1997) the generation of conditionality seems to require the “crystallization” of a mental and/or physical exploration into a statement: from a “dynamic” exploration of a problem to a “static” conditional expression, through the focus on a “temporal section” (Boero et al. 1999, 2007).

“...the conditionality of the statement can be the product of a dynamic exploration of the problem situation during which the identification of a special regularity leads to a temporal section of the exploration process that will be subsequently detached from it and then “crystallized” from a logic point of view (“if...then...”)." (Boero et al. 1996a, p. 121, emphasis in the original).

This involves the identification, within a dynamic experience, of the two components of a (static) conditional statement: a “condition” that will become the premise and a “fact” that will become the conclusion. Searching for a “condition” is frequently referred to during the explorations as finding “when” (Arzarello 2000, 2001) something happens.

The dynamic nature of the exploration of open problem situations becomes particularly evident in the context of a DGS, where the figures can actually be explored dynamically through the dragging mode. This makes DGSs an ideal environment for posing conjecturing open problems and for observing and investigating processes of conjecture-generation. As far as the solution process of a conjecturing open problem goes, we will consider a characterization into two phases: a conjecturing phase, during which students engage in exploration of a dynamic figure and argumentation leading to the written formulation of a statement (which we consider the product of this first phase, and will refer to as “conjecture”); and a proving phase, during which students attempt to prove their conjecture (we consider the proof produced as the final product of this second phase). In this paper we will focus on the first phase. In particular, the problems we created are characterized by the description of a sequence of construction steps, which students are asked to follow, producing a dynamic figure, followed by an open question explicitly asking for a conjecture. From now on, we will refer to this type of conjecturing open problem as a *step-by-step construction problem*.

2.3 Dragging as an Instrument

The use of the drag mode has been widely treated in the previous literature, both through observation of students’ spontaneous behaviors (for example, Olivero 2002; Arzarello et al. 2002) and teaching experiments aimed at introducing dragging (Gousseau-Coutat 2006; Restrepo 2008; Baccaglini-Frank et al. 2009).

Dragging becomes particularly useful in the solution of open problems, because it allows the solver to be guided and supported by interacting with the software, as described by Laborde and Laborde:

... the changes in the solving process brought by the dynamic possibilities of Cabri come from an active and reasoning visualisation, from what we call an interactive process between inductive and deductive reasoning. (Laborde and Laborde 1991 p. 185)

Moreover, dragging can help the user interpret the exploration in terms of logical dependency, in the following way. In a DGS dragging can be done by the user, through the mouse, which can determine the motion of different objects in two ways: direct motion, and indirect motion. The *direct motion* of a base¹ element (for instance a point) represents the variation of this element on the plane. The user can select the base element and drag it on the screen. The *indirect motion* of an element occurs when this element appears to move

¹ We note that a base point is a free point (or a semi-free point if it is linked to an object) that therefore can be dragged anywhere on the screen (or along the object it is linked to).

on the screen in a manner that depends on the movement of a selected base point (or object) that is being dragged. Therefore, the use of dragging allows the user to feel “motion dependency,” which can be interpreted in terms of logical dependency within the mathematical context (Mariotti 2006). This becomes a key feature in the development of conjectures originating from the investigation of open problems in a DGS. In fact the use of Cabri in the generation of conjectures is based on the interpretation of the motion dependency through dragging in terms of logical dependency. In other words, the subject has to be capable of transforming perceptual data into a conditional relationship between what will become premise and conclusion of the statement of a conjecture (Mariotti 2006), a task which is not at all trivial. The consciousness of the fact that the dragging process may reveal a relationship between geometric properties embedded in the Cabri-figure directs the way of transforming and observing the screen image (Talmon and Yerushalmy 2004).

Arzarello and his colleagues were the first who gave a description and classification of expert solvers’ uses of the dragging mode (Arzarello et al. 2002), using the terminology of “dragging schemes.” Such terminology was not conceived within the instrumentation approach (Vérillon and Rabardel 1995; Rabardel and Samurçay 2001; Rabardel 2002), and for this reason it has to be interpreted in a generic way as “dragging modalities.” However, other researchers (for example, Leung et al. 2006; Leung 2008; Strässer 2009) have fruitfully analyzed the use of dragging under the lens of the instrumental approach: dragging becomes an artefact that can support the task of generating a conjecture. Such use of dragging may develop through a process of instrumental genesis (Rabardel 2002). If we consider dragging to be an artefact and place a user in the context of solving a problem (task), it is possible to identify specific utilization schemes associated to dragging (*dragging utilization schemes*, in the following “dragging schemes”), some of which have been identified by Leung and other researchers (for example, Lopez-Real and Leung 2006; Leung et al. 2006; Leung 2008).

Within our framework, and therefore in this paper, we make an explicit distinction between *dragging modalities* and *dragging utilization schemes*, in order to separate *what might be observed externally* as a particular way of dragging (dragging modality) from the description of *an internal mental construct of the solver* (dragging utilization scheme) associated to a particular way of dragging.

3 Methodology

The main goal of our general qualitative study was to “zoom into” and provide a description of certain cognitive aspects of a conjecture-generation process that previous literature seemed to indicate might be related to the use of particular dragging modalities. In order to accomplish this, we proceeded by re-elaborating and refining ideas from the previous literature, and by constructing a model which then we tried out as a tool of analysis. The first phase of our study consisted in the a priori construction of a model, starting from previous literature and from preliminary data obtained through a pilot study.

During the second phase, the empirical phase, we collected data from students from three classrooms of different high schools and tried out the model as a tool of analysis. This phase required the collection of data in which it would be possible to observe the phenomenon we were interested in: use of dragging modalities, specifically *dummy locus dragging*, in conjecture-generation. Since in the literature dummy locus dragging was described as a dragging modality used rarely in a spontaneous manner by students (Olivero 2001), we decided to first introduce participants to dragging modalities, in particular to

one, similar to dummy locus dragging, that we called *maintaining dragging* (MD). Our hypothesis was that by introducing students to these dragging modalities we would observe more cases of their use, and thus be able to better “zoom into” the cognitive processes we were interested in. We note that our general study was aimed at describing a cognitive process that might occur during conjecture-generation through dragging and not at investigating any process of instrumental genesis.

Before presenting the model we developed, we will describe the four dragging modalities we elaborated to introduce students to.

3.1 The Dragging Modalities

Drawing from the literature, we considered dragging modalities that seemed particularly significant for the investigation of step-by-step construction problems and that could be easily introduced. The four modalities we elaborated are described below:

- wandering/random dragging (Italian: “trascinamento libero”): randomly dragging a base point on the screen, looking for interesting configurations or regularities of the Cabri-figure;
- maintaining dragging (Italian: “trascinamento di mantenimento”): dragging a base point so that the Cabri-figure maintains a certain property;
- dragging with trace activated (Italian: “trascinamento con traccia”): dragging a base point with the trace activated;
- dragging test (Italian: “test di trascinamento”): dragging base points to see whether the constructed figure maintains the desired properties. In this mode it can be useful to make a new construction or redefine a point on an object to test a formulated conjecture.

Wandering dragging consists of randomly dragging a base point on the screen. It can be used to look for interesting configurations or regularities of the Cabri-figure. In this sense this modality is a sort of fusion between *wandering dragging* and *guided dragging*, described by Arzarello et al. (2002). *Maintaining dragging* (MD) consists of trying to drag a base point and maintaining some interesting property observed. For example, the solver may notice that a certain quadrilateral, part of the Cabri-figure, can “become” a square, and thus attempt to drag a base point trying to keep the quadrilateral a square. In other words, maintaining dragging (MD) involves the recognition of a particular configuration as interesting, and the user’s attempt to induce the particular property to become an invariant during dragging. Using Healy’s terminology such an invariant would be denoted as a *soft* invariant, as opposed to a *robust* invariant, which on the other hand derives directly from the construction steps (Healy 2000).

Our definition of *maintaining dragging* (MD) differs slightly from what in literature has been referred to as *dummy locus dragging*. In the literature dummy locus dragging is described as “wandering dragging that has found its path,” a dummy locus that is not yet visible to the subject (Arzarello et al. 2002, p. 68). Instead, we consider maintaining dragging (MD) the mode in which a base point is dragged, not necessarily along a pre-conceived path, with the specific intention of the user to maintain a particular property.

With *dragging with trace activated* we intend any form of dragging after the trace function has been activated on one or more points² of the figure. This tool arises from the combination of two Cabri functions, “dragging” plus “trace,” which together constitute a

² Although during the introductory lessons we did not explicitly specify particular points to activate the trace on, we only proposed to activate it on the base point selected to be dragged.

new global tool that can be used in the process of conjecture-generation. In particular, combining maintaining dragging with the trace activated on the selected base point can be particularly useful during processes of conjecture-generation.

The *dragging test* is a test that a figure can be put through in order to verify whether it has been properly constructed or not (Olivero 2002; Laborde 2005). After having conceived of a new property to add to the investigated figure, the *dragging test* can be useful to test whether the originally desired property is actually maintained in this case. In the current literature, the dragging test is conceived as a tool to test the validity of *robust* constructions. Our conception differs in that we consider *dragging test* the dragging mode in which a base point is dragged with the intention of *observing* two (soft or robust) invariant properties *occur simultaneously*. Thus we can distinguish the dragging test from maintaining dragging (MD), according to the intention of the user. This can be useful in the context of conjecture-generation tasks, because the dragging test, as we conceive it, can become a way of testing a (possibly still implicit) conjecture.

3.2 Our model for conjecture-generation

Our model describes some aspects of the cognitive process leading to the formulation of a conjecture, these aspects may be expressed through tasks that a solver can engage in. Since maintaining dragging (MD) plays an essential role in the particular conjecturing process we were studying, we named our model the *MD-conjecturing model*. In the description of the tasks the model introduces some new notions, corresponding to the main components that come into play during the conjecturing phase and that are elaborated into the final conjecture (considered as the product of this process). In this section we will introduce the MD-conjecturing model through a hypothetical exploration based on a step-by-step-construction, showing how these elements come into play.

Consider the following problem (Baccaglini-Frank 2010):

- Draw a point P
- and a line r through P.
- Construct the perpendicular line to r through P
- and choose a point C on it.
- Construct a symmetric point to C with respect to P and call it A.
- On the semi-plane identified by r containing A, draw a point D.
- Construct the line through D and P.
- Construct the circle with center in C and radius CP.
- Let B be the second intersection of the line through P and D with the circle.
- Consider the quadrilateral ABCD.

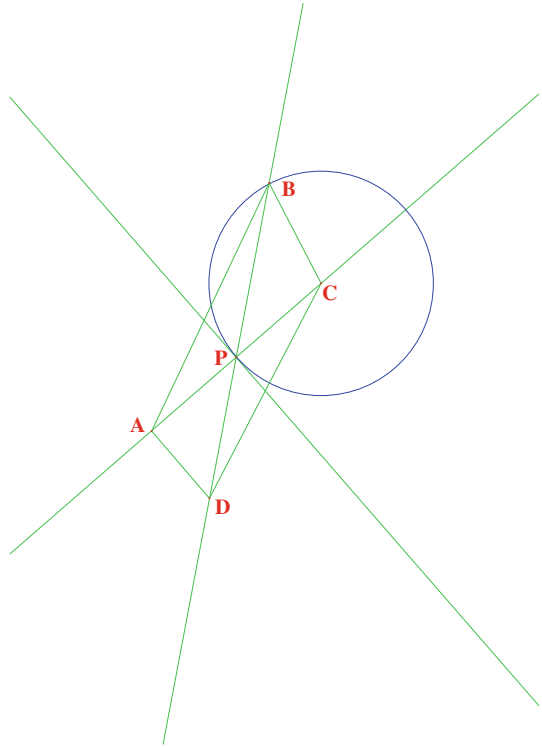
Make conjectures on the types of quadrilaterals that it can become, describing all the possible ways it can become a certain quadrilateral. Write your conjectures and then prove them.

3.2.1 Task 1: Determining an Intentionally Induced Invariant (III)

According to the MD-conjecturing model, initially the solver uses wandering dragging to move the various base points of the construction and notices *construction-invariants*,³ that

³ It can be useful to make a distinction between construction-invariants that are described in the steps of the step-by-step construction, as opposed to ones that are *consequences* of such steps.

Fig. 1 ABCD as a result of the step-by-step construction



is, geometrical properties of the figure which are true for any choice of the base points (Baccaglini-Frank et al. 2009).

In our example, let us assume the solver starts dragging the base point D (Fig. 1). While dragging, s/he sees that it is possible to make the quadrilateral ABCD into something that “looks like” a parallelogram. To continue exploring “when” the Cabri-figure becomes a particular type of geometrical figure (a “parallelogram” in our example) the solver may decide to apply maintaining dragging (MD). The invariant s/he will try to maintain, the intentionally induced invariant (III), can be the type of geometrical figure found in the previous part of the exploration. While dragging a base point of the Cabri-figure, the solver will concentrate on maintaining the III, visually. This means that at this point the III is a soft property of the Cabri-figure. Maintaining such a property while continuously dragging a base point may not be a simple task: it may depend heavily on the manual skills of the solver, and sometimes it may not be possible at all.

3.2.2 Task 2 (Part 1): Searching for a Condition Through Maintaining Dragging (MD)

While dragging the base point trying to maintain the III, the solver’s attention can shift to (and keep shifting back and forth to and from) the movement of the dragged-base point. The combination of sight and haptic perception may guide the solver’s interpretation of some regularity in the movement of the base point. Moreover, a solver can use MD with the explicit intention of looking for such a regularity. In order to do this the solver must be confident that dragging continuously the base point considered and maintaining the chosen

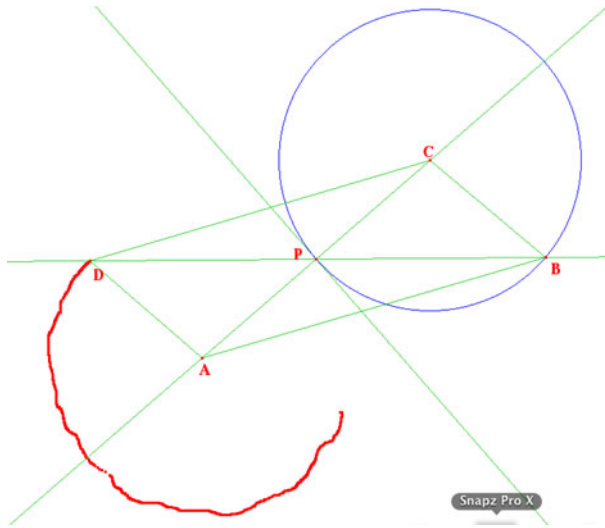


Fig. 2 Effect of maintaining dragging with the trace activated on the dragged-base point

III is possible. The solver may refer to the regularity explicitly as “something” to look for. We call this “something” a *path* and define it as *a continuous set of points on the plane with the following property: when the dragged-base point coincides with any point of the path, the III is visually verified*. We highlight how the path becomes a reification of the perceived regularity in the *direct movement* the solver induces on the dragged-base point. The possibility of explicitly dealing with the object we define as *path* seems to be fundamental in using MD for conjecture-generation. We have therefore further developed the notion of path, as a finding of our research, and will focus on this notion in a later section.

The belonging of the dragged-base point to a path determines the invariant observed during dragging (IOD), which yet needs to be expressed as a geometrical property. This can be accomplished by proposing a *geometric description of the path* (GDP), that is a description of the path in terms of a known geometrical object linked to the Cabri-figure. A tool that the user may choose to use in order to have additional guidance in determining an explicit path, and potentially providing a GDP, is *dragging with trace activated* on the dragged-base point. In our example the trace could be interpreted as something like a circle, and the solver could provide an argumentation like: “as we go down we have to also move over and move as if B moves on a circle” to reach a description of the path as a symmetric circle to the existing one. So the IOD would be: “D belongs to the circle with center in A and radius AP” (Fig. 2).

We note that the III is a property of the figure that can be induced only “indirectly” through the dragging of a base point. In this sense, the III is to be conceived as an *indirect invariant*. On the other hand, the movement induced on the dragged-base point is a form of “direct” movement, and therefore the IOD is to be conceived of as a *direct invariant*.

Through his/her perception, the solver can therefore interpret the induced invariant (III) as “caused by” the observed invariant (IOD). This interpretation guides the search for “the conditions” under which the III can be maintained (*indirectly*), and these conditions are controllable (*directly*) by the solver.

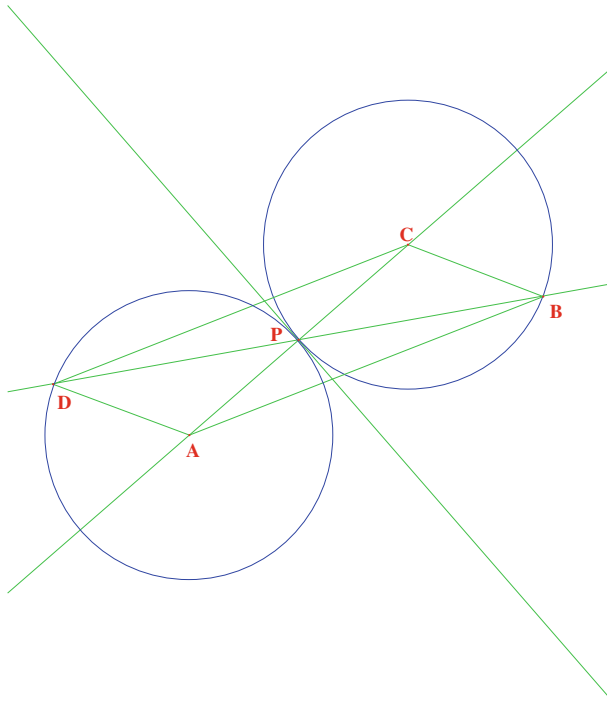


Fig. 3 A GDP has been constructed and a dragging test may now be performed

3.2.3 Task 2 (Part 2) and Task 3: Establishing and Checking a Conditional Link (CL) Between the Invariants

At this point the solver has identified two invariants, the intentionally induced invariant (III) and the invariant observed during dragging (IOD), which occur simultaneously. In order to become more convinced of the existence of a link between these properties, the solver may perform a dragging test in the soft or robust version. The solver may choose to add the IOD as a new robust property of the Cabri-figure (Fig. 3). The new property consists in the dragged-base point now being linked robustly to the object representing the GDP (in our example D can be linked to the circle with center in A and radius PA⁴): such a property is no longer a possibility, but something “true.” At this point the solver has changed both the epistemic and the logical value of the IOD. After the redefinition of the base point, if the III also visually becomes a robust invariant, the dragging test is passed, confirming the precise logical status of the invariants and of their relationship. The robust dragging test is extremely convincing not only of the fact that the IOD and III do occur simultaneously, but also that there is a precise conditional relationship between them. Constructing one property robustly led to the robustness of another. At this point we can formulate a conjecture taking our invariant observed during dragging (IOD) as the premise and our intentionally induced invariant (III) as the conclusion of this statement. Since the

⁴ Notice how this choice of the radius makes the constructed circle a D-invariant, (Baccaglini-Frank et al. 2009).

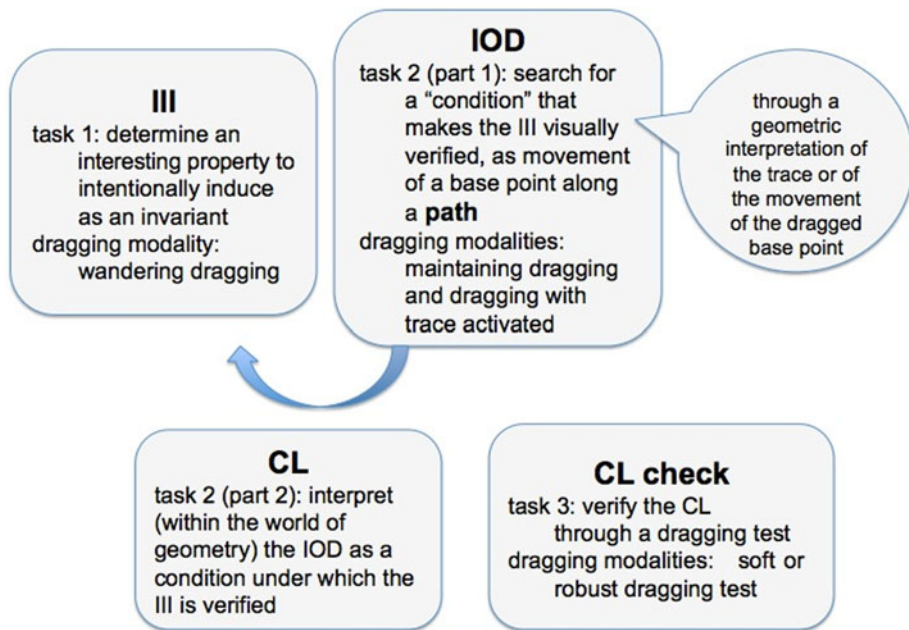


Fig. 4 Interplay of the elements of the MD-conjecturing model

IOD was “D belongs to the constructed circle⁵” and the III was “ABCD parallelogram” we obtain the following conjecture:

If D belongs to the circle centered in A with radius AP, then ABCD is a parallelogram.

We advance the hypothesis that a key requirement for conceiving a conditional link (CL) is the solver’s experience of simultaneity together with *direct control*, that is control over the direct movement induced by dragging a base point along the path. In fact, after discovering the IOD, the solver can directly act on the base point to maintain the IOD, and simultaneously feel and observe the maintaining of the III *indirectly*, as a consequence. Thus we suggest that a bridge between the experiential field (in dynamic geometry) and the formal world of Euclidean geometry is provided by the interpretation:

simultaneity + direct control → premise

simultaneity + indirect control → conclusion.

The three tasks in our model can be illustrated as in Fig. 4.

3.3 The Empirical Phase

The experimental phase involved high school students in Italian “licei” (high schools) between the ages of 16 and 17 and consisted of two parts. First, there were introductory lessons in the classroom during which selected dragging modalities were presented to the

⁵ Notice the transition from “D is dragged along the circle” to this crystallized form.

students; they were followed by interview sessions with the students who were asked to solve step-by-step-construction problems.

3.3.1 The Introductory Lessons

The introductory lessons were carried out over two class periods. The first preliminary lesson aimed at helping students overcome some difficulties, well-documented in the literature (for example, Talmon and Yerushalmy 2004; Restrepo 2008). These difficulties are related to awareness of the different status of elements of Cabri-figures, that is, awareness of the relationships of dependency between these elements determined by the steps of the construction.

Goals of Lesson 1:

- to distinguish between base points (in general, elements) and dependent points (in general, elements) of a Cabri-figure that originated from a step-by-step construction (given explicitly);
- to experience how different Cabri-figures that can represent “a parallelogram” can originate from different step-by-step constructions and thus have different base points (in general, elements) and dependent points (in general, elements);
- and to experience the different behaviors of such Cabri-figures when their base points are dragged.

The second lesson was focused on the introduction of the four dragging modalities we had developed. As students explored Cabri-figures they were asked to drag points in particular ways and describe their observations and perceptions (for example, how they moved their hand while dragging) with respect to a particular configuration and its movement. Students were asked to share their perceptions with the whole class, in a collective discussion during which the instructor gave names to specific “dragging modalities” while the students explained how they used them.

Goals of Lesson 2:

- to explore a Cabri-figure that originated from a given step-by-step construction by dragging its base points;
- to experience (physically) and describe different modalities of dragging base points of a Cabri-figure;
- to learn names for four “dragging modalities:” *wandering dragging*, *maintaining dragging*, *dragging with trace activated*, *dragging test*;
- to attempt to formulate conjectures on the Cabri-figure being explored through dragging, but with no guidance from the instructor.

Two examples of step-by-step constructions explored during Lesson 2 are:

Construction 1

- Construct a quadrilateral given its four vertices, A, B, C, D;
- construct the four perpendicular bisectors of its sides;
- label the intersection points of consecutive perpendicular bisectors E, F, G, H;
- and connect these points to form a second quadrilateral.

Construction 2

- Construct an isosceles triangle ACD with base AC;
- construct a point B on the plane so that ABCD is convex;

- construct the four external squares on each side of the quadrilateral;
- construct the centers of each square and let them be the vertices E, F, G, H of a second quadrilateral.

During the introductory lessons no explicit reference to how the dragging modalities contributed to the formulation of a conjecture was made by the instructor, nor were any other particular indications for using dragging modalities given. Students were told that these modalities of dragging “may be useful for exploring figures in dynamic geometry”. We underline how our study was *not a teaching experiment*: the teaching intervention had the limited aim of introducing students to different dragging modalities and to their names; there was no intervention aimed at introducing students to conjecture-generation.

3.3.2 Task-Based Interviews

The interviews in the general study were *semi-structured* and *non-directive*, in that during them the interviewer exhibited minimal direction and control as she observed students solving step-by-step-construction problems and sometimes asked them for some clarifications and explanations. These brief interventions during the activity-session were conceived and carried out to gain insight into the possible underlying cognitive processes.

Each interview was carried out during a 90-minute-long activity-session, in which solvers were presented with 2–4 problems (from a pool of 6 problems developed for the study), depending on how each exploration unfolded. The problems in the activity-sessions contained a step-by-step construction that was described in the text and that gave rise to a quadrilateral, say ABCD. The step-by-step construction was followed by a question of the type: “Consider the quadrilateral ABCD, make conjectures on the types of quadrilaterals it can become and how. Then prove your conjectures.” Before the interviews, the interviewer explained how she was interested in understanding a thought process and how solvers could help her achieve this goal by speaking out loud and explaining as much as they could to her. She also explained that any time she would ask “why?” it did not mean that the solver was wrong (Hunting 1997; diSessa 2007), but that she was seeking for an explanation with the aim of understanding the solvers’ thought process, thus valuing any clarification the interviewee might be able to provide and refraining from any type of judgment (Hunting 1997; Ginsburg 1981).

Since during the pilot study it was clear that students seemed to share their thoughts more openly when talking to a fellow student, which is coherent with what is described in literature (for example, Schoenfeld 2000), most of the task-based interviews of the final study had as subjects pairs of students. However some students were interviewed alone if they seemed able and willing to share their thoughts openly with the interviewer.

4 The MD-Conjecturing Model as a Tool of Analysis

In this section we will show how our model can be used to analyze students’ explorations. The transcripts we will use are part of the data collected during solving sessions of: two pairs of students, Fra and Gia, and Giu and Ste, of the 10th grade of a Liceo Scientifico in Italy, and a student, Isa, of the 11th grade of a different Liceo Scientifico in Italy. The students are working on the same step-by-step construction problem

analyzed in the previous section of this paper in which we introduced the model. During our analysis we identified “expert solvers” as those who, during the activities in the interviews, would engage in maintaining dragging (MD) and successfully use it to reach a conjecture. The table below summarizes the content of this section providing a brief description of what occurs during each excerpt with respect to each task of the model.

Task 1	Excerpt 1	Fra and Gia determine the III and application of MD with trace activated
Task 2	Excerpt 2	Fra and Gia search for an IOD, they propose a GDP through interpretation of the trace, then reject this GDP and re-launch the search for an IOD
Task 3	Excerpt 3	Giu and Ste find an IOD, and propose a GDP
	Excerpt 4	Fra and Gia propose a GDP, construct the GDP, and perform a dragging test (soft version)
	Excerpt 5	Isa proposes a CL between the invariants she has perceived
	Excerpt 6	Isa uses a robust dragging test to verify the CL, she has conceived

In the transcripts, the capital letters in bold refer to the solver who is holding the mouse. The letter “Int” refers to the interviewer

4.1 Task 1: Choice of an III and Use of the MD

In the episode below Fra and Gia decide to use MD to investigate “when ABCD is a parallelogram”. In a previous episode they have noticed that the property “ABCD parallelogram” can be substituted with the sufficient property “diagonals of ABCD congruent.”

Excerpt 1

Transcript	Analysis
(0:41) Fra : exactly. [he drags D a bit, in a way that looks like he is trying to maintain the property parallelogram]	Application of MD Identification of a possible III: “ABCD is a parallelogram”
(0:48) Gia : you see that if you do, like, maintaining dragging ... trying to keep them more or less the same...	(or III: “congruent diagonals”)
(0:57) Fra : exactly [murmuring]... well, okay. ...	
(2:41) Fra : For the parallelogram, uh, let’s try to use “trace” to see if we can see something.	Use of MD with trace activated
G: go, let’s try [speaking together with him]...uh, “trace” is over there. ...	The “something” searched for indicates the belief in the existence of a path to be described geometrically
[They have a little trouble activating the trace]	
(3:05) Gia : and now what are we doing? Oh yes, for the parallelogram?	
(3:07) Fra : yes, yes, we are trying to see when it remains a parallelogram	

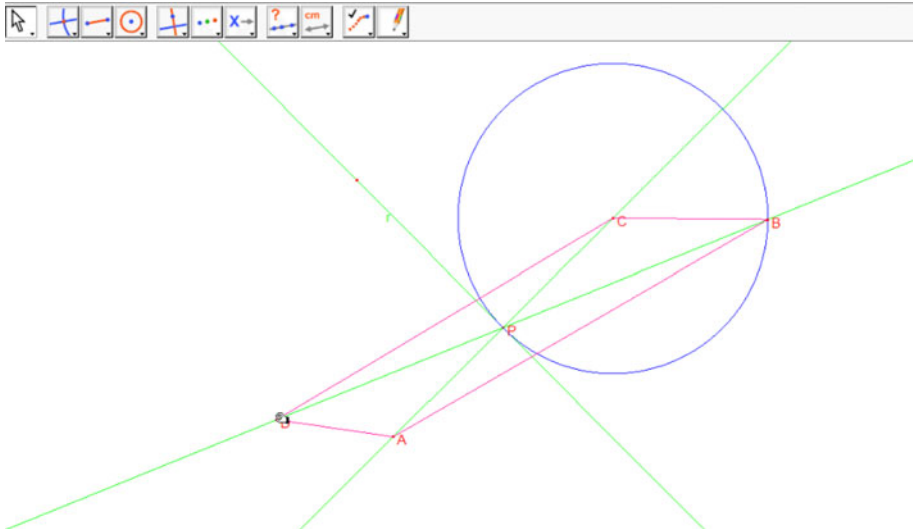


Fig. 5 Fra and Gia decide to use maintaining dragging with the trace activated

The solvers decide to use MD (Fig. 5) and they start to move point D looking at the properties “diagonals more or less the same” or “ABCD parallelogram” to maintain. We highlight Gia’s use of the terminology, “maintaining dragging” (0:48), introduced during the

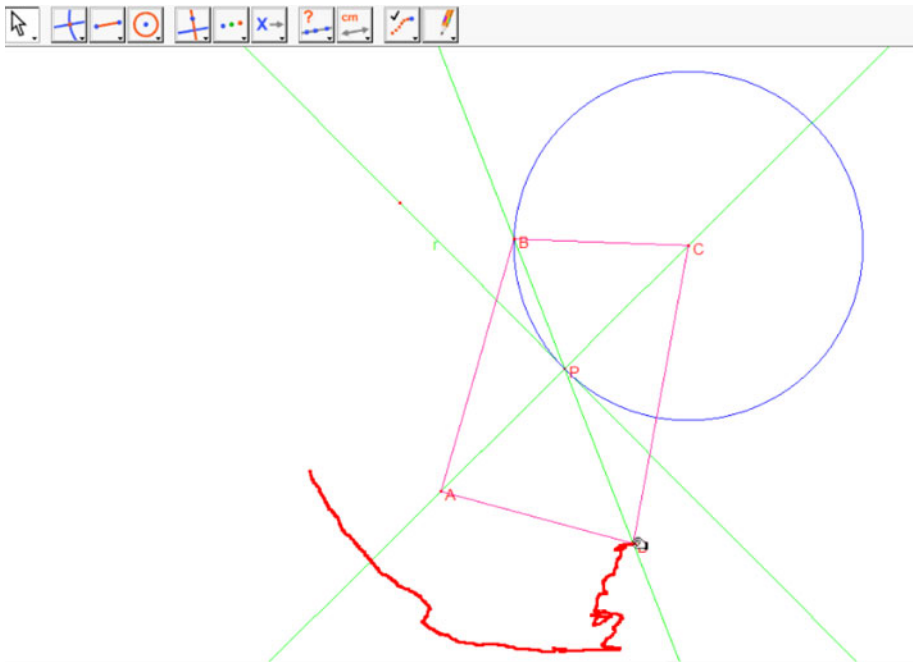


Fig. 6 Fra rejects the first GDP as he performs MD with the trace activated

in-class activities. After a while Fra proposes to activate the trace in order to “see something” (2:41). All this, and specifically the use of appropriate terminology can be interpreted as indicative of expert use of MD. It is interesting that during this application of MD the students express their intention either in a generic way (“to see something”) or in a more specific way (to see “when” ABCD is a parallelogram). In this second case the solvers seem to explicitly express the intention of searching for a *cause* that might make the parallelogram “happen.” It seems clear that the solvers believe in the existence of “something” that will make the “parallelogram happen.” The MD-conjecturing model refers to this “something” as *path*.

4.2 Task 2: Perception of the IOD and Establishment of the CL

According to the model, the next step consists in identifying an IOD. In the excerpt below we can observe how the solvers search for an IOD through the interpretation of the trace to reach a GDP (Fig. 6).

Excerpt 2

Transcript	Analysis
[silence while the MD with trace activated is carried out by Fra]	
(3:18) Gia: yes, okay the usual circle comes out.	First GDP: usual circle
(3:23) Fra : wait, because here...oh dear! where is it going?	
(3:35) Int: What are you looking at as you drag?	
(3:38) Fra : I am looking at when ABCD is a parallelogram. You try [handing the mouse to Gia]	
...	Continuation of the search for an IOD
[Gia tries dragging some other points looking for the “draggable” ones, and there is a short diversion on “the case of the rectangle”. Then Gia starts dragging point D.]	
(4:17) Fra: ...turn it. No, it’s not necessarily the same circle, because, I don’t know at some point I don’t know, keep going... by tomorrow... keep going... careful you are making it too long ...	
(5:01) Fra: but you see? This one is always longer than that one... it’s too long, if you go, let’s say, along the circle here, this one is too long. So, maybe it’s not necessarily the case that D is on a circle so that [Italian: “in modo che”] ABCD is a parallelogram.	
...	Indirect expression of a CL between generic IOD (D is on a circle) and III (ABCD is a parallelogram)
(6:36) Fra : exactly. Now there is this problem of the parallelogram in which we can’t exactly find when it is.	
(6:44) Gia: eh, uh, we discovered when...	
...	

Table c continued**Excerpt 2**

Transcript	Analysis
(7:02) Fra : because you see, if we then do a kind of circle starting from here, like this, it's good it's good it's good it's good, and then here... see, if I go more or less along a circle that seemed good, instead it's no good... Because, you see, in a certain sense B, at this point the circle	
(7:24) Gia : eh, it's linked to the circle	Rejection of the GDP
(7:25) Fra : exactly, and so in a certain sense it goes ... down along a slope and so... it's no... no good. So, when is it any good?	

At first Fra drags D trying to maintain ABCD a parallelogram, Gia seems to be observing the trace and (too?) quickly proposes a GDP (3:18). It is not clear what “usual” refers to: maybe to the interpretation of other trace marks⁷ that had led to a circle as a GDP. However what Fra sees does not seem to be the circle he had in mind (maybe the circle centered in P with radius AC) and he appears unhappy and confused when he does not understand “where it is going” (3:23). We note that the quick transition that the solvers seem to make from the trace mark to a GDP, though unsuccessful, is another indication of expert use of MD. As highlighted in the analysis of Excerpt 1, the solvers are “looking for something” and they already expect an object that they will be able to describe geometrically and even construct, based on the other elements of the figure (Excerpt 4). Of course this behavior was developed over time: this excerpt is taken from the last activity explored by this pair of students.⁸

After repeating his intention of investigating “when ABCD is a parallelogram” (3:38) Fra hands the mouse to G, asking him to try. Fra uses what he sees to argue why his initial idea (involving some circle he never describes explicitly) does not seem correct. Fra’s “same circle” probably refers to the one he has in mind, which did not coincide with what he saw in his trace (4:17 and 5:01). In spite of the incorrectness of the specific GDP, we can observe Fra’s consciousness of a conditional link (CL) between the III and the IOD he had hypothesized at time 5:01. Through his words—“*so that* [Italian: in modo che]”—we can see the CL appear between “D being on a circle” and “ABCD being a parallelogram”.

With his argumentation Fra rejects the proposed GDP, and re-launches the search for an IOD (6:36). Back to his argumentation, Fra tries to explain why a circle seems to be “no good” (he probably still has in mind his circle described in the analysis above). Although such a circle is never described geometrically, Fra and Gia seem to have a similar object in mind. However, most importantly, the solvers seem to have in mind a CL between the III and a hypothetical IOD that they are still searching for (6:36 and 6:44).

⁷ Before the problem analyzed in the excerpts, Fra and Gia had (during the introductory lessons and the preceding part of their interview) explored constructions in which maintaining dragging had led to lines (or segments), circles (or arcs of circles), and a parabola.

⁸ It is beyond the scope of this paper (and of the general study) to describe the processes of instrumental genesis of the maintaining dragging. However the analysis of solvers’ behaviors during successive activities, highlighted some elements that may characterize a process of instrumental genesis. This will be the object of future studies we plan to carry out.

Similarly, in the following example, we observe the solvers searching for an IOD. In this excerpt the solvers successfully reach an IOD and a GDP that satisfies them, by interpreting the mark left by the trace activated on the dragged-base point during MD (Fig. 7).

Excerpt 3

Transcript	Analysis
[1] Giu: Try to maintain these things here.	Ste is trying to simultaneously maintain the concurrence of the two circles and of the line through PD, which the solvers have identified as a sufficient property for the III “ABCD parallelogram” to occur.
[2] Ste : It’ll be hard.	
[3] Giu: You try!	
[4] Ste : eh, what am I doing?	
[5] Giu: There, more or less...yes, yes, yes, not too much, there.	
[6] Int: In the meantime you, Giu, tell me what you are looking at.	Giu proposes a first GDP.
[7] Giu: Come on, come on...	
[8] Giu: Uhm...it seems to be a curve. ...	
[9] Ste : It’s really hard! It moves!! [laughing]	Ste experiences manual difficulties in performing MD.
[10] Giu: I know.	
[11] Giu: I can only imagine...but I think that is it also, uh, that it is a circle...with center in A.	Giu refines the GDP.
[12] Ste : and maybe with radius P.	
[13] Giu: AP!	

Controlling the simultaneous concurrence of the two circles and the line through PD during dragging is a manually-difficult task that seems to require all of Ste’s (the solver who is dragging) attention. Spontaneously Giu offers oral guidance during the dragging task, as he can concentrate on the figure as a whole, not having to exercise manual control over the figure. Moreover Giu takes on the task of interpreting the regularity in the movement of the dragged-base-point using the hint of the trace mark left on the screen. Through this collaboration the two solvers are able to overcome difficulties involved in performing MD and perceiving an IOD. As in Excerpt 2, the solvers’ quick transition from the observation of the trace mark to a GDP, is indicative of expert behavior.

4.3 Task 3: Verification of the CL

At the end of Excerpt 2, as Fra discusses why the circle he had in mind is no good, his attention seems to shift to the movement of point B (7:02) and then to the figure as a whole. At this point Gia has handed the mouse back to Fra who starts using MD without the trace. In Excerpt 4 that follows, Gia proposes a new GDP and Fra proceeds to construct this geometrical object. Then, before they explicitly express a conjecture, the solvers feel the need to check their idea, in their words “try it” (8:34). Therefore they use a soft dragging test to become convinced that dragging D on the constructed object guarantees the simultaneity of the two invariants, the III and the IOD (Fig. 8).

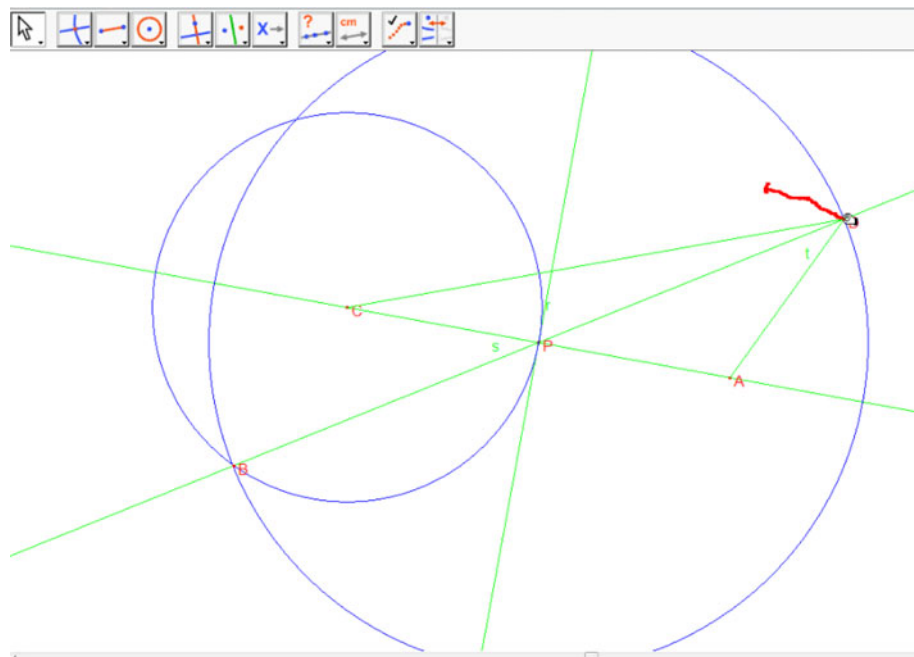


Fig. 7 Giu and Ste reach a GDP as they perform MD with the trace activated on the dragged-base point

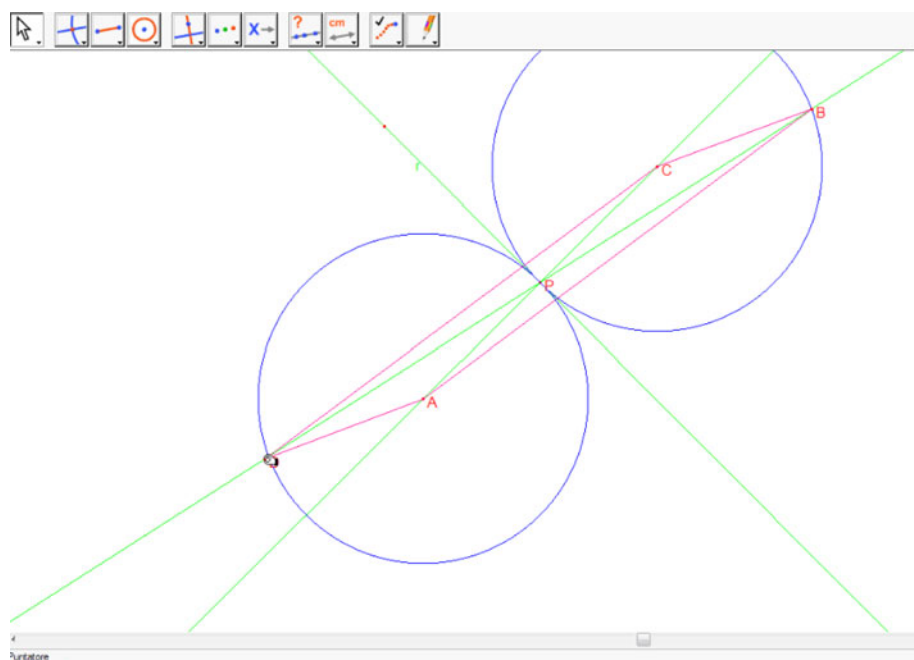


Fig. 8 A (soft) dragging test convinces Fra and Gia that their GDP is correct

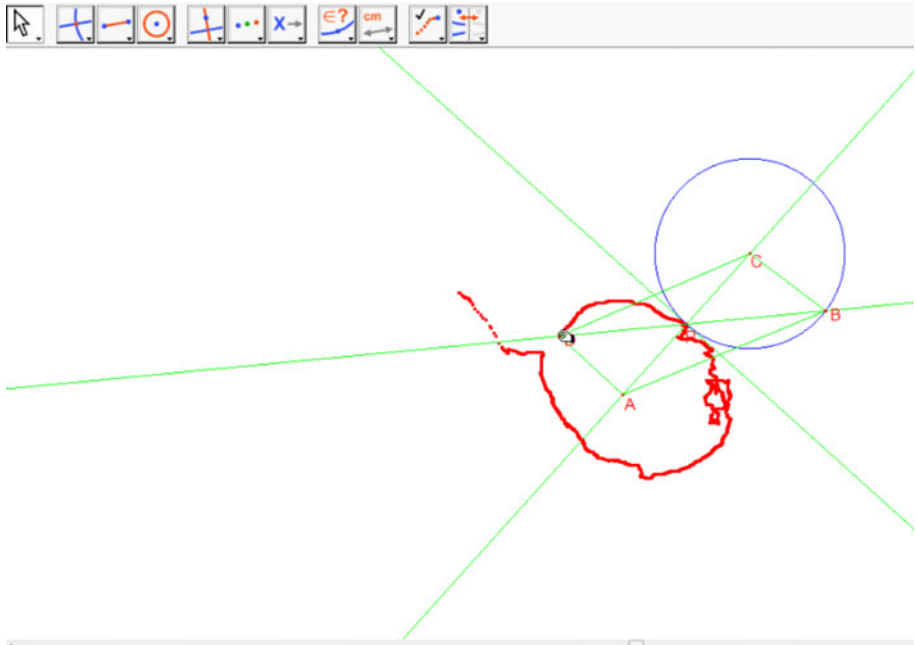


Fig. 9 Isa uses maintaining dragging with trace activated to determine an IOD

Excerpt 4

Transcript	Analysis
(8:17) Gia: circle... with center A and radius AP. I, I think...	Final GDP construction of the GDP
(8:24) Fra : let's move D. more or less...	
(8:26) Gia: it looks right doesn't it?	soft dragging test
(8:27) Fra : yes.	
(8:30) Gia: Maybe we found it!	
(8:31) Fra : but maybe... maybe only along this	
(8:34) Gia: Let's try to	
(8:35) Fra : let's try to, right, go the whole way around	
(8:37) Gia: like this yes, like this yes, like this yes	uncertainty is overcome through the soft dragging test
(8:38) Fra : yes, yes, yes	
(8:41) Gia: over here too, I think	
(8:43) Fra : yes	
(8:44) Gia: yes.	The soft dragging test seems to be quite convincing.
(8:45) Fra : I would definitely say so.	
(8:47) Gia: okay we found it.	

Gia and Fra give a GDP as a circle with center in A and radius AP (8:17), which F constructs. Fra proposes to drag D “more or less” along such a circle, thus performing a soft dragging test. Initially they seem to have some doubts about whether the whole circle is the path (Fra seems particularly doubtful), but the (soft) dragging test convinces them that it works “the whole way around” (8:35).

When Fra and Gia refer to it looking right (8:26), to “try to” (8:34, 8:35), and to having found it (8:30, 8:47) it is reasonable to assume that they are verifying a CL (like the one Fra had hypothesized in Excerpt 2, 5:01). This actually happens soon after when they express their conjecture: “if D moves on a circle with center A and radius AP, ABCD is a parallelogram.”

Here we provide a second example of checking the conditional link (CL) through a soft dragging test. In spite of some differences with respect to the previous solvers’ explorations, the interplay of the same elements of the MD-conjecturing model can be recognized. This Excerpt is from an exploration carried out by Isa, and it occurs after she has chosen “ABCD parallelogram” as her III. She is trying to interpret the trace mark as she drags D trying to maintain ABCD a parallelogram (Fig. 9). She proceeds using MD with the trace activated, and she interprets the trace mark to reach a GDP. She uses such a GDP to express the IOD she has perceived, inserting it immediately in a conditional statement.

Excerpt 5

Transcript	Analysis
Isa: parallel...here, ok, it collapses...it becomes a line, I mean all the points of all the lines coincide. Isa: and over here...ok...no, no, no, no Isa: There it collapses...so... Int: What is it that you are looking at here to do it? Isa: I am trying to make a parallelogram, uh, to put two sides parallel. Isa: and so AD and BC. Int: uhm. Isa: So now I need to go back a second... no, no, no, no... Int: eh, it's hard when you go close... Isa: alright, anyway, here it should...how nice!...be here. Isa: There... Int: Let's continue over here... Isa: So, like this...uhm...here it becomes easier...There, more or less Int: uhm. Isa: So, let's see to try it. So, if I move D on a circle with center in A, and, theoretically, radius AP...	Isa focuses on a particular case that she is not sure represents the possible III, that is ABCD is a parallelogram. She describes the III and the property she is using to obtain it. Isa has perceived the regularity to the extent that she can predict how to continue dragging the selected base point. In this conditional statement Isa provides a GDP and the CL she has conceived between the IOD and the III.
Int:...hmmm Isa:...I find that the quadrilateral is a parallelogram, except when, uh, D comes to lie on the line CA.	

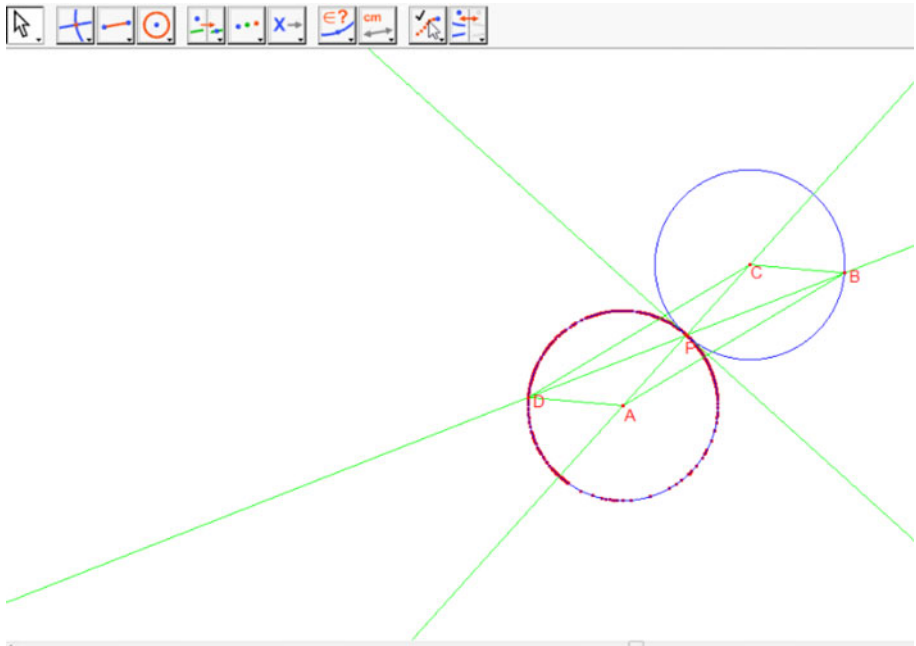


Fig. 10 Isa uses a (robust) dragging test to try out her conjecture

As Isa drags D she seems to be looking at the sides of the quadrilateral being parallel. She then confirms that she is looking at “AD parallel BC” (so this is her III). She hesitates when the quadrilateral seems to collapse, and expresses increasing and decreasing levels of difficulty in using MD. However she seems to exhibit expert behavior with respect to MD: she has an idea in mind, expressing what “should” happen. This indicates that she has conceived a path, and, moreover, that she is interpreting “dragging along the path” as a cause for the III to be visually verified. Isa seems to be controlling her dragging of the base point D along the path (described as the circle centered in P with radius AP), that is, she is performing a soft dragging test inducing the IOD. She interprets this invariant as a premise in her conditional statement.

The last excerpt is the conclusion of Isa’s exploration from which Excerpt 5 was also taken. Excerpt 6 provides an example of use of the robust dragging test as a final check of the solver’s conjecture.

Isa has written her conjecture, but does not seem confident enough to start proving yet. In fact, when Int asks her whether she wants to try to prove the conjecture, or work on it a bit more, she interrupts the question and immediately answers as shown at the beginning of this episode. She feels the need to construct the object described in her GDP and perform a robust dragging test (Fig. 10).

Excerpt 6

Transcript	Analysis
Isa: eh, for now I'll try to construct it... Int: Ok. Isa: So...this, now I need to construct a circle [she constructs a circle centered in A with radius AP]...where is it? to link D to the circle? Int: Under the perpendiculars. Isa: Ok, now let's try to move...[she starts dragging D, now linked to the circle] Yes...here it becomes a single point...and here again...Now we can also turn the trace off. Int: Now you seem to be pretty convinced. Isa: Yes.	Construction of the Cabri object representing the GDP robust dragging test, accomplished by dragging point D linked on the circle.

5 Discussion

The analysis of the episodes shown above and of various additional ones from other students' explorations, leads us to conclude that the model seems to appropriately describe expert use of the presented dragging modalities, in particular of MD. Taking an instrumental perspective (Rabardel 2002), our model can be viewed as a description of a *utilization scheme* constructed by the solver with respect to the task of conjecture-generation. We can refer to this utilization scheme as the *maintaining dragging scheme*. Now, "experts" can be described as solvers for whom MD has become an appropriated tool supporting the solution of a conjecturing open problem. In this case acquisition of the utilization scheme can be viewed as the outcome of a process of instrumental genesis.

Our analyses of explorations through the model, have led to an elaboration of the notion of *path*, and to the conception of the notion of *instrumented argument*. Briefly, the notion of path is a description of a mental construct that can arise during a dynamic exploration when the MD scheme is used, and it seems to be useful for describing certain students' behaviors during the exploration.

The second notion, that of instrumented argument, seems to provide a description of a particular way in which a DGS can contribute to the conjecturing phase, giving the solver a new means for developing arguments. In the sections below we will introduce and elaborate on these two notions, showing how they emerge from our examples.

5.1 Path and Its Evolution

Consider the following remark made by Fra in Excerpt 1, time 2:41: "For the parallelogram, uh, let's try to use "trace" to see if we can see something." The student is referring to "something" that seems to have to *happen for the parallelogram to occur*. This idea of "something" is consistent with the "it" Fra and Gia are trying to uncover using the trace in Excerpt 2, time 4:17, and with the object that Fra and Gia claim to "have found" in

Excerpt 4, times 8:30 and 8:47. These examples and analyses of other transcripts showed that expert solvers seem to first have to decide that it *is* possible to continuously drag the base point they are interested in in order to maintain the III they have identified. This can be interpreted as the fact that they need to have conceived “something” to look for, something that when the dragged-base point is on “it”, “the III happens.” We refer to this “something” as *path*. It is interesting to notice that the path is not associated to a particular geometric shape (or curve), nor does it (necessarily) coincide with the mathematical notion of locus⁹: the path may, in fact, be a proper subset of such a mathematical locus. In this respect, our notion of path seems consistent with the notion of *locus of validity* introduced by Leung and Lopez-Real (2002).

Once an expert solver has decided that MD can be applied, s/he engages in the search of an IOD, by looking for a geometrical description of such path (GDP). During this stage they may choose to activate the *trace* on the dragged-base point in order to visualize the movement in a different way and “discover” what the path looks like. Depending on the solver’s manual ability, the movement (and possibly the associated trace mark) will appear more or less “regular.” Thus, at this point, the path exists both at a conceptual level and it is represented at a figure-specific level, its representation includes figural and haptic components (the trace mark on the screen and the feeling of the movement). We would like to stress that though the trace mark obtained on the screen is *not* the path, this new object may provide hints towards its geometrical description (GDP).

We believe that analogical thinking is used in associating some perceived regularity to other regularities previously discovered during other experiences. Moreover, reasoning through “selection and generalization” (Boero et al. 1999) seems to be used by solvers who select a subset of positions from the movement (or points from the trace mark if activated) that have in common some property and from which a “general rule” can be inferred.

We think this process could also be described as a “continuous” case of Boero et al. (1999) description of the third type of “process of generation of conditionality” (PGC 3):

A ‘synthesis and generalization’ process starting with an exploration process of a meaningful sample of conveniently generated examples (p. 141).

Once the solver has reached a GDP, in order to visually (and manually) test in a more convincing way its CL with the III, s/he constructs it as a Cabri-object within the figure. Now the idea of path has been fully exploited and there is a “concrete” Cabri object potentially representing the path and that can be used to test the conjecture either dragging the base point along it “by eye” (soft dragging test) or linking the base point to it and verifying (robust dragging test) visually (and manually) the GDP and the CL between the IOD and the III. In this sense we can speak of evolution of path within a specific exploration process.

5.2 The Notion of Instrumented Argument

In the Excerpts we analyzed, in particular in Excerpts 2 and 4, the solvers seem to need to convince themselves or each other of a certain idea. For example, in Excerpt 2 Fra continues his argumentation leading to the rejection of the previous GDP (7:02). In such

⁹ The extent to which the path is associated to a curve or a locus seems to depend on the mathematical expertise of the solver. For a mathematician it will unconsciously be considered a locus (or a subset of a locus), but this will not necessarily be the case for a student.

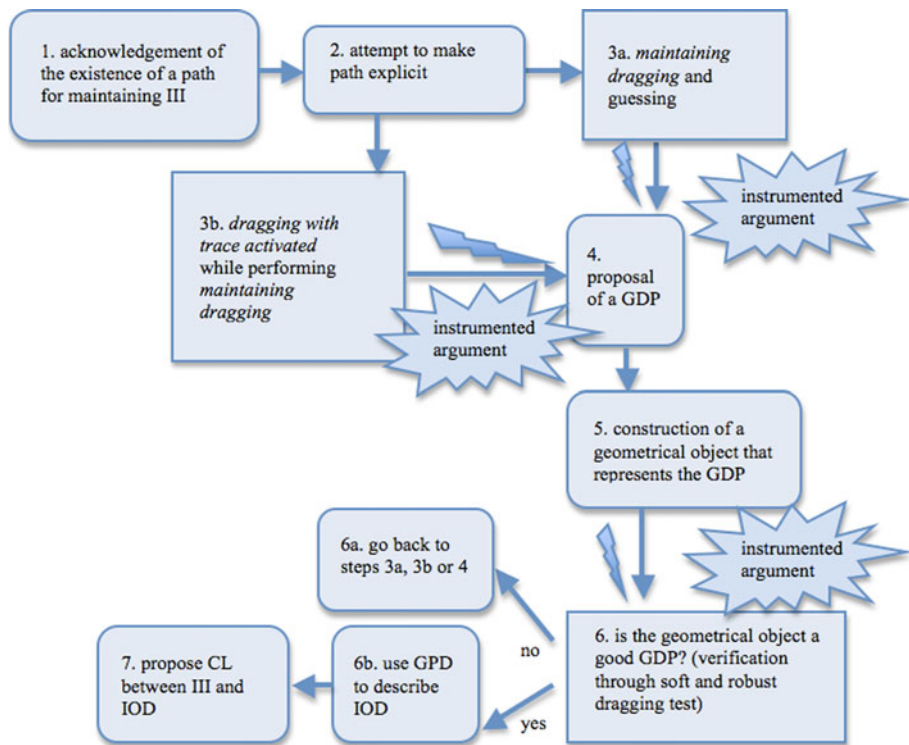


Fig. 11 Path and its evolution and relationship with instrumented arguments

argumentation he uses arguments with visual and haptic characteristics: “kind of circle starting from here [as he drags point D showing Gia what he means],” “you see,” “in a certain sense it goes...down along a slope [mimicking the movement with his hand].”

Moreover, in Excerpt 4 the solvers provide a GDP that they do not seem sure of. In particular Fra does not seem to be convinced that ABCD remains a parallelogram when D is dragged along the whole hypothesized circle. He therefore performs a soft dragging test which definitively convinces him and Gia of the GDP. Frequently we have observed that students use the words “try it” with respect to an idea (or possibly yet unexpressed conjecture) when they intend to perform a robust dragging test. From the transcripts we have so far analyzed within our study, this seems to be an even more convincing argument for solvers.

The analysis of our protocols highlighted a particular form of argument used by the solvers and intrinsically tied to the tools available in the DGS environment. We called it *instrumented argument*. With instrumented argument we intend an argument (thus part of an argumentation supporting a logical inference) in which the warrant is produced through the use of some of the DGS’s tools, in this case the dragging tool. As for any argument the goal of an instrumented argument is to convince oneself or someone else of a specific claim, thus changing its epistemic status, but in this case persuasion comes from the use of the dragging tool and its intrinsic logic.

Instrumented arguments in DGSs seem to be frequently used in conjunction with different versions of the dragging test, as in the episodes analyzed in this paper. Below are a

couple examples of instrumented arguments that we have observed in solvers' explorations.

An example of instrumented argument can be seen in the transition from a soft to a robust construction before a final (robust) dragging test is performed. In redefining the dragged-base point on a constructed object that represents the GDP (or reconstructing the Cabri-figure in order to add a property, with respect to the ones that already originate from the steps, to its base points)—an act that makes sense within a DGS—the IOD becomes robust, and the solver may subsequently refer to this property, in the argument, as being “true.” The (implicit) claim to defend is that a CL holds between the IOD and the III, and the instrumented argument consists in showing that when the IOD becomes “true”, the III in the new construction also becomes a construction-invariant (at a visual and physical perceptual level), thus robust, and therefore “true.”

Another example can be found in arguments in favor of a certain GDP. Before constructing the geometrical object that hypothetically represents the path, the solver may try to argue that his/her idea is right, speaking about the movement of the dragged-base point in physical terms and showing what s/he means by physically enacting the dragging movement on the screen. Thus, dragging and the feedback provided by the software are used as warrants supporting the solver's ideas about the GDP. This type of instrumented argument is also used to reject a given GDP. This can be seen in Excerpt 2 when the visual feedback seems to provide Fra with confirmation that what he had thought of as the GDP was “no good” (7:25). Frequently the instrumented arguments used to reject a GDP (in the most convincing way) make use of the dragging test after the GDP has been constructed. In this case the solver argues that while dragging the base point along (or even having linked it to) the hypothetical object that represents the GDP the III is not maintained (Fig. 11). Figure 11 illustrates possible appearances of instrumented arguments in the evolution of the path during a dynamic exploration.

We conclude with some remarks on our findings and on a few interesting inputs for further research.

6 Concluding Remarks

With the intent of describing certain cognitive processes related to the use of specific dragging schemes, we presented a model that we conceived and refined through theoretical and empirical reflections. The key elements we described are: the intentionally induced invariant (III), the invariant observed during dragging (IOD), the geometrical description of the path (GDP), and the conditional link (CL) between the invariants. Our concluding remarks deal with some issues that are related to the model and that point to interesting directions for future research.

The MD-conjecturing model does seem to appropriately describe and predict students' behavior, in cases in which the dragging schemes have been appropriated. The model can also be used to discover and describe cognitive difficulties that arise from episodes in which the model does not seem to apply (Baccaglini-Frank 2010). Moreover, if the dragging schemes were to be taught in classroom settings, the MD-conjecturing model can provide the teacher with insight into whether (and how) the schemes have been appropriated. In particular, an analysis of where the student fails to behave according to the model can guide the study of potential cognitive obstacles. Such analysis through the lens of the model can provide the teacher with new tools to help students overcome such

cognitive obstacles, by providing explicit descriptions of important cognitive steps involved in the acquisition of the schemes.

Furthermore, the MD-conjecturing model led to the new notions of instrumented argument and path, the scope of which goes beyond our model. For example, we noticed that the notion of instrumented argument seems quite appropriate for describing arguments used by students in episodes reported by Olivero and Robutti (2007). A first analysis of these researchers' results leads us to recognize instrumented arguments within the modalities described in the second category related to the shift from the theoretical field to the spatio-graphical field presented in the paper: the validation measuring modality, and the proof measuring modality (Olivero and Robutti 2007, p. 141).

Finally, we believe that the notion of instrumented argument, and potentially those of different types of instrumented arguments (Baccaglini-Frank *in press*), may become explicit tools to use during argumentations in a DGS, and these can be discussed in the classroom at a meta-level. Instrumented arguments may then become an explicit type of arguments that students may choose to consciously make use of.

The notion of path has rich potential, both for the study of proof in a DGS in general and as a concept that could help teaching and learning in the classroom. A preliminary analysis of proofs of some conjectures provided by solvers during this study shows that elements of the path, and of its geometric description especially, can be picked up in the proof and used as key ideas (Raman 2003). The path, and especially properties of the GDP, could therefore play a significant role in the generation of a proof, providing elements that help deductively link the premise to the conclusion of the conjecture. Since the path and the GDP arise during the argumentation phase in which the conjecture is developed, and they are again present in the proving phase, we advance the hypothesis that the path could, in some cases, constitute a sort of bridge between the argumentation leading to the conjecture and its proof. In this case, it could be considered an aid to gain *cognitive unity* (Boero et al. 1996a). Finally, if this were to be the case, then introducing the dragging modalities to students might foster and enrich the entire process of linking the making of a conjecture to its proof.

Acknowledgments The study this article reports on was carried out within the research project PRIN 2007B2M4EK (Instruments and representations in the teaching and learning of mathematics: theory and practice)—Università di Siena & Università di Modena e Reggio Emilia (Italy), with additional support from a dissertation fellowship from the University of New Hampshire (USA). We wish to thank the anonymous reviewers for their very helpful comments, questions, and suggestions on a previously submitted draft.

References

- Arsac, G. (1999). Variations et variables de la démonstration géométriques. *Recherches en Didactique de Mathématiques*, 19(3), 357–390.
- Arzarello, F. (2000). Inside and outside: Spaces, times and language in proof production. In *Proceedings of PME XXIV*, (Vol. 1, pp. 23–38), Hiroshima, Japan.
- Arzarello, F. (2001). Dragging, perceiving and measuring: Physical practices and theoretical exactness in Cabri-environments. In *Proceedings of CabriWorld 2*, Montreal.
- Arzarello, F., Micheletti, C., Olivero, F., Paola, D., & Robutti, O. (1998). A model for analysing the transition to formal proofs in geometry. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd PME*, (Vol. 2, pp. 24–31), Stellenbosch, South Africa.
- Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2002). A cognitive analysis of dragging practices in cabri environments. *ZDM*, 34(3), 66–72.

- Baccaglini-Frank, A. (2010). *Conjecturing in dynamic geometry: A model for conjecture-generation through maintaining dragging*. Doctoral dissertation, University of New Hampshire, Durham, NH.
- Baccaglini-Frank, A. The maintaining dragging scheme and the notion of instrumented abduction. In *Proceedings of the 10th conference of the PME-NA*, Columbus, OH (in press).
- Baccaglini-Frank, A., Mariotti, M. A., & Antonini, S. (2009). Different perceptions of invariants and generality of proof in dynamic geometry. In Tzekaki, M., & Sakonidis, H. (Eds.), *Proceedings of the 33rd conference of the IGPME*, (Vol. 2, pp. 89–96), Thessaloniki, Greece.
- Bartolini Bussi, M. G. (1998). Joint activity in mathematics classrooms: a Vygotskian analysis. In F. Seeger, J. Voigt, & U. Waschescio (Eds.), *The culture of the mathematics classroom* (pp. 13–49). Cambridge: Cambridge University Press.
- Boero, P., Garuti, R., & Lemut, E. (1999). About the generation of conditionality. In *Proceedings of the 23rd meeting of the IGPME*, (Vol. 2, pp. 137–144). Haifa.
- Boero, P., Garuti, R., & Lemut, E. (2007). Approaching theorems in grade VIII. In P. Boero (Ed.), *Theorems in school: From history epistemology and cognition to classroom practice* (pp. 249–264). Rotterdam: Sense Publishers.
- Boero, P., Garuti, R., Lemut, E., & Mariotti, M. A. (1996b). Challenging the traditional school approach to theorems: A hypothesis about the cognitive unity of theorems. In P. Luis, & A. Gutierrez (Eds.), *Proceedings of 20th conference of the IGPME*, (Vol. 1, pp. 113–120), Valencia, Spain.
- Boero, P., Garuti, R., & Mariotti, M. A. (1996a). Some dynamic mental processes underlying producing and proving conjectures. In P. Luis, & A. Gutierrez (Eds.), *Proceedings of 20th conference of the IGPME*, (Vol. 2, pp. 121–128), Valencia, Spain.
- diSessa, A. (2007). An interactional analysis of clinical interviewing. *Cognition and Instruction*, 25(4), 523–565.
- Ginsburg, H. (1981). The clinical interview in psychological research on mathematical thinking: Aims, rationales, techniques. *For the Learning of Mathematics*, 1(3), 4–11.
- Gousseau-Coutat, S. (2006). *Intégration de la géométrie dynamique dans l'enseignement de la géométrie pour favoriser la liaison école primaire collège : une ingénierie didactique au collège sur la notion de propriété*. Ph.D Thesis, Ecole doctorale des Mathématiques, Sciences et Technologies de l'Information, Informatique, Université Joseph Fourier, Grenoble, France.
- Healy, L. (2000). Identifying and explaining geometric relationship: interactions with robust and soft Cabri constructions. In *Proceedings of the 24th conference of the IGPME*, (Vol. 1, pp. 103–117) Hiroshima, Japan.
- Hunting, R. P. (1997). Clinical interview methods in mathematics education research and practice. *Journal of Mathematical Behavior*, 16(2), 145–165.
- Laborde, C. (2001). Integration of technology in the design of geometry tasks with cabri-geometry. *International Journal of Computers for Mathematical Learning*, 6, 283–317.
- Laborde, C. (2005). Robust and soft constructions. In *Proceedings of the 10th Asian technology conference in mathematics*, (pp. 22–35), Korea, National University of Education.
- Laborde, C., & Laborde, J. M. (1991). Problem solving in geometry: From microworlds to intelligent computer environments. In Ponte et al. (Eds.), *Mathematical problem solving and new information technologies*, (pp. 177–192). NATO ASI Series F, 89.
- Leung, A. (2008). Dragging in a dynamic geometry environment through the lens of variation. *International Journal of Computers for Mathematical Learning*, 13(2), 135–157.
- Leung, A., & Lopez-Real, F. (2002). Theorem justification and acquisition in dynamic geometry: A case of proof by contradiction. *International Journal of Computers for Mathematical Learning*, 7, 145–165.
- Leung, A., Chan, Y. C., & Lopez-Real, F. (2006). Instrumental genesis in dynamic geometry environments. In L. H. Son, N. Sinclair, J. B. Lagrange, & C. Hoyles (Eds.), *Proceedings of the ICMI 17 study conference: part 2* (pp. 346–353). Vietnam: Hanoi University of Technology.
- Lopez-Real, F., & Leung, A. (2006). Dragging as a conceptual tool in dynamic geometry. *International Journal of Mathematical Education in Science and Technology*, 37(6), 665–679.
- Mariotti, M. A. (2000). Introduction to proof: The mediation of a dynamic software environment. *Educational Studies in Mathematics. Special issue*, 44, 25–53.
- Mariotti, M. A. (2006). Proof and proving in mathematics education. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education* (pp. 173–204). Rotterdam, The Netherlands: Sense Publishers. ISBN 9077874194.
- Mariotti, M. A., Bartolini Bussi, M. G., Boero, P., Ferri, F., & Garuti, R. (1997). Approaching geometry theorems in contexts: from history and epistemology to cognition. In Pehkonen, E. (Ed.), *Proceedings of 21st conference of the IGPME*, (Vol. 1, pp. 180–195). Lahti, Finland.
- Mogetta, C., Olivero, F., & Jones, K. (1999). Providing the motivation to prove in a dynamic geometry environment. In *Proceedings of the British society for research into learning mathematics*, St Martin's University College, Lancaster, pp. 91–96.

- Olivero, F. (2001). Conjecturing in open geometric situations in a dynamic geometry environment: An exploratory classroom experiment. In C. Morgan & K. Jones (Eds.), *Research in Mathematics Education*, (Vol. 3, pp. 229–246), London.
- Olivero, F. (2002). *Proving within dynamic geometry environments*, Ph. D. Thesis, Graduate School of Education, Bristol.
- Olivero, F., & Robutti, O. (2007). Measuring in dynamic geometry environments as a tool for conjecturing and proving. *International Journal of Computers for Mathematical Learning*, 12, 135–156.
- Rabardel, P. (2002). *People and technology: A cognitive approach to contemporary instruments*. Retrieved online June, 16th 2009 from: <http://ergoserv.psy.univ-paris8.fr>.
- Rabardel, P., & Samurçay, R., (2001). Artifact mediation in learning. New challenges to research on learning. In *International symposium organized by the center for activity theory and developmental work research*, University of Helsinki, March 21–23.
- Raman, M. (2003). Key ideas: What are they and how can they help us understand how people view proof? *Educational Studies in Mathematics*, 52, 319–325.
- Restrepo, A. M. (2008). *Génèse instrumentale de déplacement en géométrie dynamique chez des élèves de 6ème*. Ph.D Thesis, Ecole doctorale des Mathématiques, Sciences et Technologies de l'Information, Informatique, Université Joseph Fourier, Grenoble, France.
- Schoenfeld, A. (2000). Purposes and methods of research in mathematics education. *Notices of the AMS*, 641–649.
- Silver, E. A. (1995). The nature and use of open problems in mathematics education: Mathematical and pedagogical perspectives. *ZDM*, 27(2), 67–72.
- Strässer, R. (2009). Instruments for learning and teaching mathematics an attempt to theorize about the role of textbooks, computers and other artefacts to teach and learn mathematics. In Tzekaki, M., & Sakonidis, H. (Eds.), *Proceedings of the 33rd PME* (Vol. 1, pp. 67–81). Thessaloniki, Greece.
- Talmon, V., & Yerushalmy, M. (2004). Understanding dynamic behavior: Parent–child relations in dynamic geometry environments. *Educational Studies in Mathematics*, 57, 91–119.
- Vérillon, P., & Rabardel, P. (1995). Cognition and artifacts: A contribution to the study of thought in relation to instrumented activity. *European Journal of Psychology of Education*, 10(1), 77–101.