

# The use of interactive visualizations to foster the understanding of concepts of calculus: design principles and empirical results

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**Abstract** Calculus and functional thinking are closely related. Functional thinking includes thinking in variations and functional dependencies with a strong emphasis on the aspect of change. Calculus is a climax within school mathematics and the education to functional thinking can be seen as propaedeutics to it. Many authors describe that functions at school are mainly treated in a static way, by regarding them as pointwise relations. This often leads to the underrepresentation of the aspect of change at school. Moreover, calculus at school is mainly procedure-oriented and structural understanding is lacking. In this work, two specially designed interactive activities for the teaching and learning of concepts of calculus based on dynamic geometry software are presented. They accentuate the aspect of change and the object aspect of functions using a double stage visualization. Moreover, the activities allow the discovery and exploration of some concepts of calculus in a qualitative-structural way without knowing or using curve-sketching routines. The activities were used in a qualitative study with 10th grade students of age 15–16 in secondary schools in Berlin, Germany. Some pairs of students were videotaped while working with the activities. After transcribing, the interactions of the students were interpreted and analyzed focusing to the use of the computer. The results show how the students mobilize their knowledge about functions working on the given tasks, and using the activities to formulate important concepts of

calculus in a qualitative way. Also, some important epistemological obstacles can be detected.

**Keywords** Interactive learning activity · Functional thinking · Calculus · Qualitative study · Epistemological obstacles

**Mathematics Subject Classification (2010)** 97C30 · 97C70 · 97D40 · 97D70 · 97I20 · 97I40 · 97I50 · 97U50

## 1 Introduction and rationale

Since the term of functional thinking and its connection to calculus is of special importance for this work, it will be discussed first. Calculus was introduced to the German curricula as a result of the so-called *Meraner Reform* of 1905. The reform was initiated by Felix Klein and was meant as a reform of the whole complex of mathematics and nature science education at school. The term *functional thinking* was used for the first time in Germany within this reform, and the *education to functional thinking* was a main task of this reform. Functional thinking was meant in a very broad sense: as thinking in variations and functional dependencies spanning the whole mathematics education—always emphasizing the *aspect of change*. Calculus was seen as a climax in mathematics education and the education to functional thinking as propaedeutics to calculus. In her outstanding dissertation, Krüger (2000) describes the background of this reform and addresses especially the education to functional thinking as a special task within the reform.

The connection between functional thinking and calculus is also similarly described by Tall (1996): “One purpose of the function is to represent how things change. With this meaning it is natural to move on to consider the

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calculus concepts of the rate of change (differentiation) and cumulative growth (integration) together with the remarkable fundamental theorem of calculus that tells us that differentiation and integration are essentially inverse processes. [...] It [calculus] is both a climax of school mathematics and a gateway to further theoretical developments” (p. 289).

Commonly in didactic literature, there are several aspects of functional thinking specified (Vollrath 1989; Dubinsky & Harel 1992b; Sfard 1991):

- *The mapping aspect*: Functions are regarded as point-wise relations (static view).
- *The aspect of change*: Dynamic view of functions including the *aspect of covariation* of values as well as the description of change in sense of concepts of calculus (e.g. rate of change).
- *The object aspect*: regarding global characteristics of functions, and the treatment of functions as algebraic objects.

The last two aspects are similar to the meaning of functional thinking during the *Meraner Reform* and are of importance when dealing with concepts of calculus in the sense of Tall (see above). Sfard (1991) points out that the transition to an *object view* is especially difficult and needs special attention.

Of course, the above aspects are only discriminable in theory and these distinctions are introduced for the sake of didactical research. In fact, all aspects are strongly connected with each other. For example, when trying to describe the *object property* monotonicity or symmetry, it is done in a *dynamic way* by saying: ‘For all  $x \geq y$ :  $f(x) \geq f(y)$ ’ or ‘For all  $x$ :  $f(x) = f(-x)$ ’.

The presented work is aligned with the term of functional thinking within the *Meraner Reform*: functional thinking with an emphasis on the aspect of change and the object aspect is seen as propaedeutics to calculus.

### 1.1 Students’ difficulties related to functional thinking and calculus

In didactic literature, many difficulties in connection with functions and functional thinking are described. Here, only some of the difficulties with regard to the focus of the study are pointed out. An overview of misconceptions and problems related to functions is given in the papers in Dubinsky & Harel (1992a), for example.

#### 1.1.1 Concept definition, concept image, and problems with the aspect of change

In German schools, the following definition for function is usually taught:

*A function is an assignment rule by which each element  $x$  of a set is assigned to exactly one element  $y$  of (another) set.*

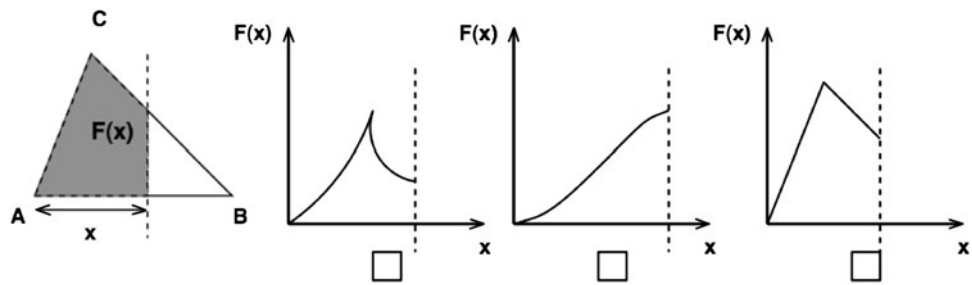
This definition normally appears in grade 8 (student age: 13 years) and is introduced by investigating examples and non-examples often on finite or countably infinite sets. The above definition is, however, abstract and very general. There are many aspects and associations that are not ‘covered’ by this definition. For example, the *aspect of change* is not evident in this definition. The neglect of the dynamic view in the definition, the primacy of numerical settings (Goldenberg et al. 1992) and the lack of qualitative examinations of functional dependencies and graphs (Janvier 1978) cause the students’ difficulties with the *aspect of change* and the *idea of covariation* (Falcade et al. 2007).

Being able to reproduce the definition does not mean that one understands the whole scope of the notion of function with all aspects and facets. Fischer & Malle (1985) also state that the true benefit of using functions becomes clear not until questions of calculus are addressed. Since the students only get to know a few classes of functions (starting with linear functions, quadratic functions, polynomials, etc.), the generality of the definition does not seem to be appropriate (Fischer & Malle 1985; Malik 1980). The ‘limited’ experiences of the students with functions and functional dependencies form their *concept image*. The *concept image* is the set of all mental images and characteristics connected with the concept of function. The *concept definition* is the definition for function one would give (Vinner & Dreyfus 1989). Vinner and Dreyfus pointed out that students make decisions about whether a given example is a function or not, based on their concept image and not on the concept definition. The concept image often contains images like: a function has to be *one* rule, has to ‘look reasonable’, is injective, does not have any jump discontinuities. The concept image and concept definition are often non-consistent.

#### 1.1.2 Epistemological obstacles and the work of Sierpinska

The work of Vinner & Dreyfus (1989) and their description of the concept image are related to the work of Sierpinska (1992) who describes typical epistemological obstacles when working with functions. Originally, the term *epistemological obstacle* was shaped by Brousseau (1983). It is assumed that the process of learning is not linear. In the course of learning, there are mental barriers or breaks that appear. Those barriers are natural and result from a genetic progression of the subject. The *illusion of linearity* is an example for such an obstacle. Besides in Sierpinska (1992), this obstacle is also described by Freudenthal (1983) in the

**Fig. 1** The dashed line moves rightwards.  $F(x)$  is the area of the gray part of the triangle dependent on the distance  $x$ . Which graph fits and why? (Schlöglhofer 2000)



following way: “Linearity is such a suggestive property of relations that one readily yields to the seduction to deal with each numerical relation as though it were linear.” (p. 267).

Epistemological obstacles depend on the mental images, ideas and interpretations one has of a notion or concept. In this sense, they are related to the concept image of functions. Obstacles are objects that need to be understood and therefore need to be identified in order to tackle them productively in the learning process. In this sense, they are valuable and essential elements in the process of conceptualization.

Some more epistemological obstacles related to functions are (Sierpiska 1992): functions are identified with tables and tables are not worth being studied as objects, laws in physics and functions have nothing in common, only relationships describable by an analytic formula are functions.

### 1.1.3 The teaching of calculus at school

Tall (1993, 1996) gives an overview of students’ difficulties in calculus. One main problem is the “consequent student preference for procedural methods rather than conceptual understanding” (Tall 1993, p. 17). Dreyfus and Eisenberg (1991) talk about the students’ reluctance to visual concepts of calculus and their preference of a numerical and symbolic approach. Also, many German authors point out that calculus at school is often procedure-oriented and consists of curve-sketching routines rather than structural understanding (e.g. Blum & Kirsch 1979; Bender 1991; Hahn & Prediger 2008). This has led to an ongoing request for a qualitative-structural approach to school calculus—a request which is in fact 100 years old and was already formulated at the time of the *Meraner Reform* (Krüger 2000).

### 1.1.4 An illustrative example

The following example illustrates and connects some of the difficulties with the concept of function described above. This example led to the idea of the first interactive activity and is therefore presented in more detail. The problem in

Fig. 1 was given to more than 100 students who started their bachelor in mathematics at Technische Universität Berlin.

The main mistake was to put a cross under the graph on the right side together with a reason like “The area of the graph is just like the area  $F(x)$ ”.

Here, the graph is interpreted as a photographic image of the situation resulting from an insufficient dynamic view of the functional dependency. We call this a *graph-as-image-misconception* (Clement 1989). One reason might be that usually in class the pointwise view of functions is predominant, whereas the dynamic view is underrepresented.

The student solution below shows how this example leads to fundamental concepts of calculus (Fig. 2).

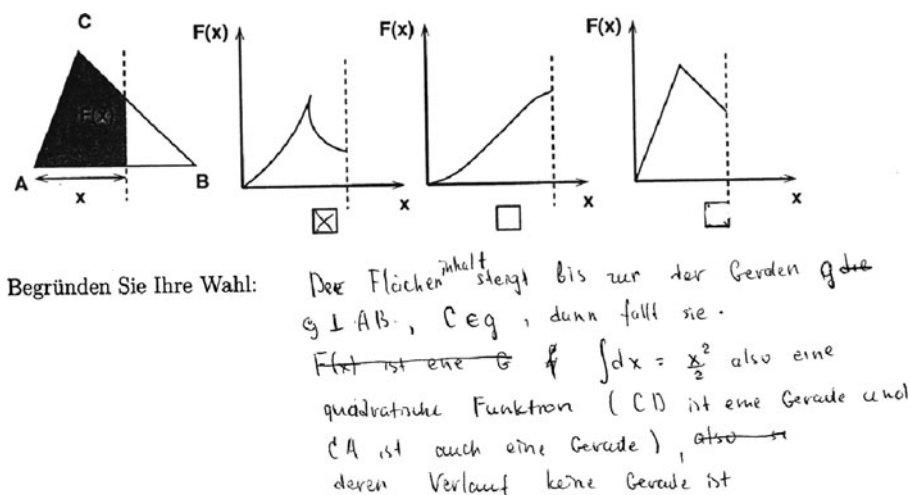
This student used the concept of integration by identifying the triangle with a piecewise linear function whose antiderivative must be quadratic. This answer is still combined with the *graph-as-image-misconception*, because the chosen graph decreases after C is passed. The student uses the concept of integration without activating a dynamic view of the function.

The mathematical background of this example is the *Fundamental Theorem of Calculus*: The triangle interpreted as piecewise linear function is the derivative of the area function  $F$ , and the area function is an antiderivative. Besides monotonicity of the area function, there is an inflection point where the quality of the growth changes: the growth of the area increases until this point and decreases afterwards. Note that the inflection point in this example cannot be found by applying the common routine using the second derivative, because the piecewise linear function is not differentiable in C.

## 1.2 Objectives

The presented work is meant as a contribution to a qualitative-structural approach to school calculus using the computer. Using the DGS (dynamic geometry software) Cinderella (Kortenkamp & Richter-Gebert 2006), learning activities including interactive visualizations were built, and design principles emphasizing the dynamic and object view of functions were developed. An emphasis of the aspect of change and the object aspect is easy to achieve

**Fig. 2** Student reasoning (translation): “The area increases until the line  $g$ ,  $g \perp AB$ ,  $C \in g$ , then it decreases.  $\int dx = \frac{x^2}{2}$  thus we have a quadratic function (CB is a straight line and CA is a straight line also), whose run is not a straight line”



when using a DGS. While computer algebra systems (CAS) are input/output based, and give back information and changes asynchronously, the use of DGS allows interactivity and immediate response (Kortenkamp 2007). This is used to emphasize the dynamic view of functional dependencies. The activities were used in a qualitative study on grade 10 students (age 15/16) in German ‘Gymnasiums’ (comparable to English ‘grammar schools’), before the students had learned about curve-sketching routines. Two of the activities, the design principles, the framework and some results of the study are presented below.

Since the activities allow only predefined forms of manipulation (mainly by moving points on graphs and in situations including a functional dependency), it is asked if the students are just moving around some geometric objects like dots or lines or if they really ‘see and do’ mathematics, respectively, analysis when manipulating the visualization.

More precisely, it is asked if and by which processes the students formulate certain concepts of calculus using the dynamic visualization. The role of the computer and the special design of the activities are of particular interest. Since the activities are meant to support a process of conceptualization, it is also examined if certain epistemological obstacles occur.

While the concept image and epistemological obstacles depend on experiences one has had with a notion or concept, it is assumed that the activities described below could enrich the experiences in a positive way fostering the process of conceptualization in calculus.

The exact formulations of the research questions will follow after the description of the activities.

### 1.3 The use of technology

Before the description of the activities, a few comments about the use of technology for conceptualization in

calculus are made. In calculus courses at school, CAS are often used. They allow the combination of different representations of functions, such as symbolic and graphic ones. In Tall (1996), some common approaches to concepts of calculus using the computer are described. Visual approaches include the use of computer graphics to magnify function graphs. Numerical representations occur when using spreadsheets, e.g.: “Placing the data (from an experiment, or from a function formula) into a spreadsheet gives opportunities for investigative exploration” (Tall 1996, p. 315).

As mentioned before, DGS could be used to emphasize the dynamic view of functional dependencies or as Ferrara, Pratt & Robutti (2006) write:

“But the big revolution in teaching mathematics with technologies was the introduction of dynamicity in software: A dynamic way to control and master the virtual objects on the computer let the student explore many situations and notice what changes and what does not. And the mathematics of change is the first step on the road to calculus.” (p. 257)

There are few studies about the use of DGS when dealing with the function concept or concepts of calculus. Two approaches shall be mentioned here.

One interesting study (Falcade et al. 2007) is about the design of a teaching experiment using a DGS (here: Cabri) environment to introduce the idea of function. The use of DGS, especially the dragging tool, allows the students to experience the idea of covariation. In Falcade et al.’s study, the Trace tool plays an important role and is used “to introduce the twofold meaning of trajectory, at the same time global and pointwise” (Falcade et al. 2007, p. 317). Falcade et al. regard these tools as semiotic mediators in the sense of Vygotsky (1978): “[...] a Vygotskian perspective may explain the contribution of tool mediated action to concept formation: signs generated in relation to

the use of a tool, through the complex process of internalization accomplished after social interchange, may shape new meanings.” Functions are introduced as geometric functions and variation is represented by motion. With the guidance of the teacher, such a tool can be internalized by students, and the concept of independent and dependent variables can be constructed using the tool as semiotic mediator.

Within the CalGeo-Project (<http://www.math.uoa.gr/calgeo>), learning environments based on dynamic geometry were developed to teach calculus concepts. The topics in the environments include the introduction to infinite processes, limit, continuity, derivative and integral. Besides using the DGS as function plotter with the special feature of magnification, DGS was used to visualize the concept of limit in a dynamic way. For example, the students explored the concept of derivative dynamically and at the same time used the magnification tool to ‘see’ the property ‘local straightness’. The material was tested in classroom accompanied by a special teacher training. The outcome of the project is that it seems to be a fruitful approach to achieve conceptualization, but there is also a need for further investigation and more systematic research (Zachariades et al. 2007; Biza & Zachariades 2007).

## 2 Interactive activities: basic ideas and design principles

Basic idea is the use of *interactive visualizations* to accentuate the dynamic component of functional thinking and to aim at the development of qualitative-structural concepts with regard to propaedeutics of calculus.

For the design of the activities, the following presuppositions were taken: The activities are designed to be applied at the end of grade 10 or the beginning of grade 11 at a ‘Gymnasium’. At that point, the students have already had certain experiences with different classes of functions (like linear functions, polynomials, and exponential functions) usually approaching them in a numerical setting. Concepts of calculus are not taught so far. Graphs are usually ‘drawn’ using tables of values and connecting points in the plane. This method does not emphasize the dynamic aspect of the functional dependencies. To approach concepts of calculus intuitively, dynamic visualizations of graphs are considered to be helpful as described in the preceding paragraph.

In the above context, three learning activities were developed in a joint work with Andreas Fest, PH Ludwigsbürg. The activities consist of Java applets embedded into a webpage and can be used with a standard Internet browser (*low technical overhead*). Next to each applet some work orders are given. These work orders are also

given on an accompanying worksheet on which the students can write their answers down. The activities and related teaching material (worksheets, etc.) are accessible at <http://www.math.tu-berlin.de/~hoffkamp> or as digital content on the website of the Intergeo Project <http://i2geo.net>.

In the next paragraph, the design principles of the activities are described in more detail and a mathematical analysis is given.

### 2.1 Design principles and the learning activity ‘Dreiecksfläche’ (‘area of a triangle’)

The design principles are also described in Hoffkamp (2009, 2010), but an overview will be given in the following. Figure 3 shows the learning activity ‘Dreiecksfläche’ (‘area of a triangle’).

The activity allows the students to explore the functional dependency between the distance AD and the dark area within the triangle. Moving D visualizes the dynamic aspect of the functional dependency simultaneously in the situation and the graph. Moving B and C changes the triangle and the graph as a whole. As already described in Sect. 1.1.4, the underlying mathematical background is the *Fundamental Theorem of Calculus*. What are the ‘mathematical aims’ of the activity for the students? The students are asked to describe and give reasons for the shape of the graph based on the dynamic visualization. In particular, the curvature and the change of curvature at the inflection point need to be described dynamically and related to the change in area within the triangle.

For example, the monotonicity property is now very easy to formulate: ‘There is more and more dark area added when moving D to the right.’ The inflection point is dynamically observable as a point where the quality of growth changes. This allows the students to find formulations like: ‘After the black point over C the added area decreases’. Note that the area function  $F$  is piecewise quadratic, and does not belong to the common classes of functions taught at school.

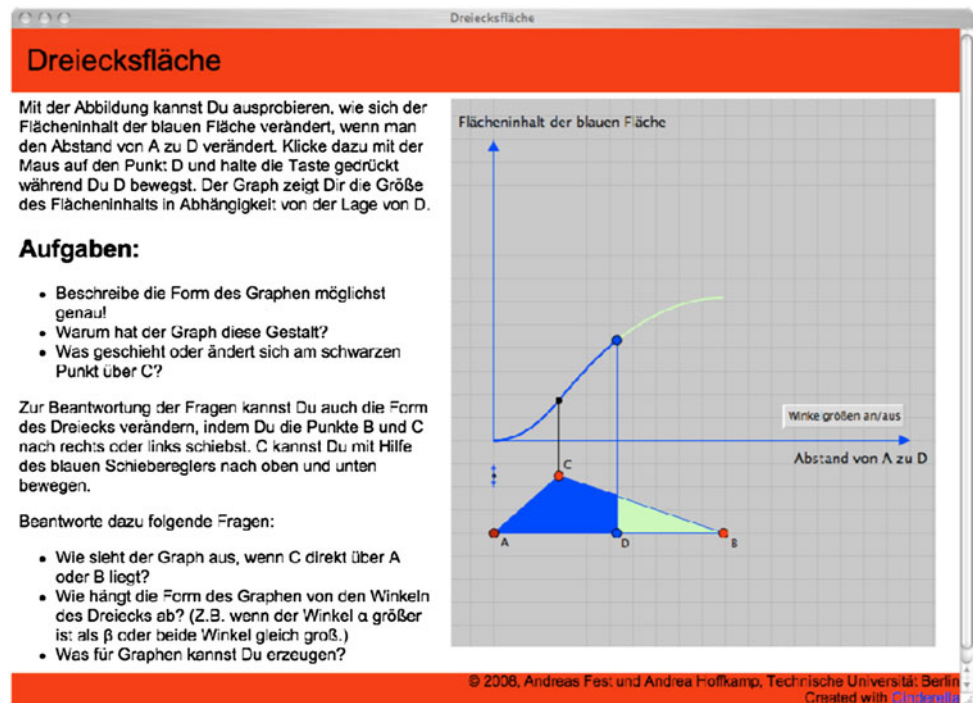
The underlying design principles are:

Starting point of the activities is always a dynamic *connection situation-graph*. The *graph* representation form was chosen, because it relates to the dynamic aspect of functional thinking in an eminent way and contains the information about the function ‘at a glance’. This is why graphs are important representations of functions in calculus.

Each activity contains *two levels of variation*:

First, moving D allows *variation within the situation*. The aspect of change becomes simultaneously visible in both representation forms—situation and graph. The underlying idea is the idea of supplantation (Salomon

**Fig. 3** Screenshot of the interactive learning activity ‘Dreiecksfläche’ (‘area of a triangle’)



1994). To understand a dynamic situation, one needs to construct an ‘executable’ mental model to achieve mental simulation. Supplantation means the visual support of mental simulation processes.

The second variation level allows—by moving points B and C—to change the situation itself and the function as a whole. This variation form will be named *metavariation* in the following. It relates to the object view of the function. Metavariation forces the detachment from concrete values and leads to a qualitative view of the functional dependency and its local and global characteristics. Characteristics like ‘monotonicity’ or ‘existence of an inflection point’ are emphasized by being invariant under metavariation (or in case of the inflection point ‘nearly invariant’).

When working with the applets, the students are always asked to verbalize their observations. The role of *language as a mediator* between the representations and the mental images of the student is of special importance in the conceptualization process. Janvier (1978) examined the interaction between a graphical representation and a given situation. He emphasized the importance of a constant interrelation between graph and situation for the transfer between the representations. In this context, Janvier points out the relevance of verbalization of the properties detected within the different representations. He considers language as a mediator between the representations and the students’ mental concepts.

The *principle of contiguity* (Mayer 2005) is essential. Representations referring to each other are close in space

and time. Especially, the fact that movement takes place where the mouse pointer operates is important to achieve an integrative visualization where dynamic can be experienced.

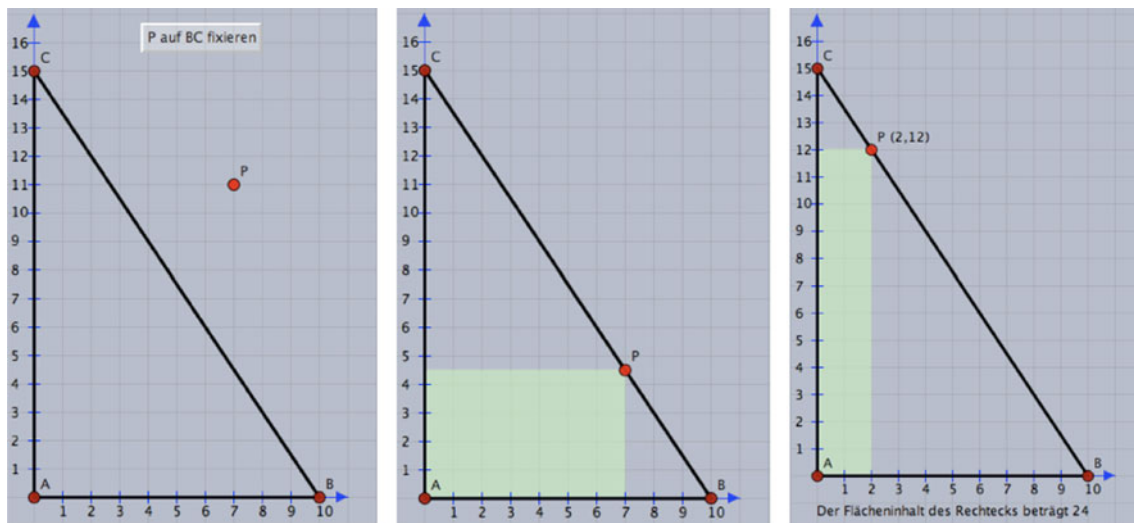
## 2.2 Learning activity ‘Einbeschriebene Rechtecke’ (‘inscribed rectangles’)

Figure 4 shows applet constellations within the first part of the activity ‘Einbeschriebene Rechtecke’ (‘inscribed rectangles’).

Students are asked to move point P on the line BC so that an inscribed rectangle appears. After fixing P on BC using a button, P can be moved on BC and the relation between the area of the rectangle and the x-coordinate of P can be explored by answering questions like: ‘Describe how the area of the rectangle changes, when moving P on BC.’, ‘Which values does the area take? Also 100?’, ‘Are there different rectangles with the same area? If yes, give examples for it.’ When using a ‘hint’-button the coordinates of P and the value of the area appear.

Figure 5 shows the applets within part two and three of the activity.

The left screenshot in Fig. 5 shows the *variation within the situation*. Moving P on BC leads to simultaneous movement of the point in the area graph leaving a trace. Students are asked to answer the questions: ‘When is the area at its maximum?’, ‘Why is the area graph shaped like this?’, ‘Explain which values for the area do not appear,



**Fig. 4** Applet constellations within the first part of the activity ‘Einbeschriebene Rechtecke’ (‘inscribed rectangles’)

appear once, twice using the graph.’ On the right side in Fig. 5 the applet referring to the *level of metavariation* is shown. Students are asked to move B and C on the axes and describe the influence on the graph. Two main questions are: ‘Give two examples for triangles so that the rectangle with maximum area is 15. Describe how you found these triangles.’ and ‘Give two examples of triangles so that the maximum is taken at  $x=4$ . Describe how you found these triangles.’ In particular, the situation in Fig. 5 shows an isosceles triangle where the rectangle with maximum area is a square.

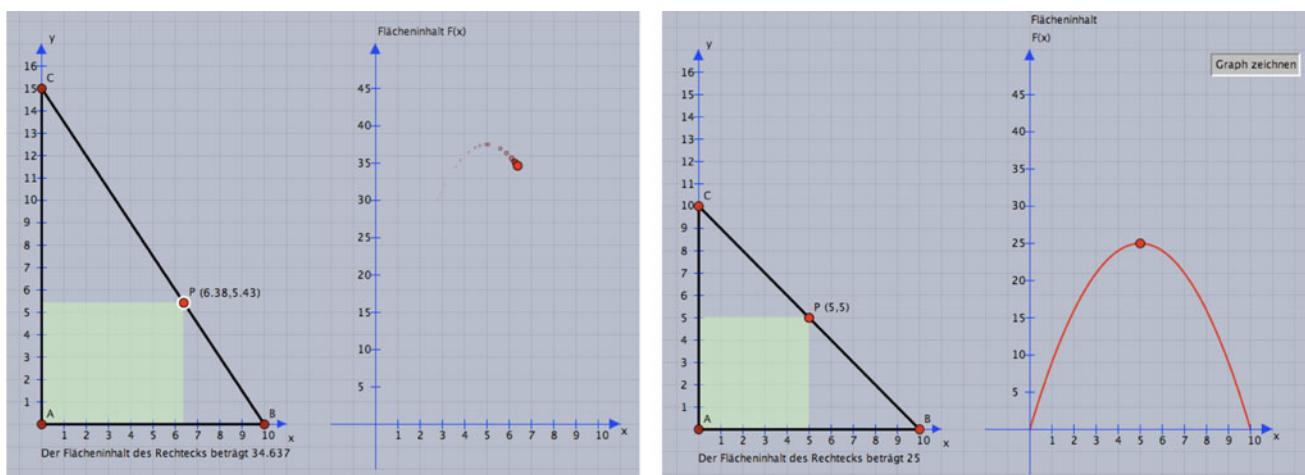
Mathematically, the students are asked to characterize certain properties of the functional dependency, such as symmetry, maximum, non-linearity, using a dynamic approach that connects a geometric situation with a function graph.

### 3 Qualitative study

#### 3.1 Research questions

The study was guided by the following research questions:

1. Which terms and mental concepts with regard to a dynamic view of functional dependencies do the students develop when working with the activities, especially when local and global characteristics of the functions are described qualitatively in connection to concepts of calculus?
2. What kinds of interaction processes can be described? Here, the interactions between the students and the student–computer interactions are of particular



**Fig. 5** The two levels of variation within the activity ‘Einbeschriebene Rechtecke’ (‘inscribed rectangles’)

interest. What is the role of the two levels of variations (variation *within* the situation, variation *of* the situation/metavariation) within the interactions?

3. Which epistemological obstacles occur when the students discuss the special properties of the visualized functions? Are there hints that the students could overcome certain obstacles using the activities? Can we find obstacles that may be tackled productively in class when teaching the concepts of calculus?

### 3.2 Framework of the study

After a pretest and a makeover, the activities were employed within the framework of a qualitative study in grade 10 of two different grammar school classes ('Gymnasien') in Berlin, Germany. Note that the concepts of calculus like differentiation, integration, and curve-sketching routines are taught in grade 11 and 12. At that time the students should have been familiar with certain classes of functions like linear, quadratic, polynomial, logarithmic, exponential, and some trigonometric functions.

Both classes consisted of about 25 students and were only taught by the researcher during the time of the study within the conventional lessons. For each activity, a block period of 90 min was used. An additional single lesson of 45 min was used in the end to take a short test and to answer a questionnaire. In the first 45 min of each block period, the students worked nearly autonomously in pairs at one computer. For each activity, a worksheet was provided. Usually, one of the students was in charge of handling the applet while the other wrote the answers down (in alternating roles). The second 45 min was used to discuss the questions and answers with the whole class.

Per class two pairs of students were videotaped while working with the activities using the software 'ScreenFlow'. This software allows to record and watch the students, their discussions, and the computer actions simultaneously. These recordings served as the main material for the analysis. The discussion with the whole class was also videotaped. Additional material for analysis consisted of the students' worksheets, the short test in the end, and the questionnaires.

### 3.3 Methodological aspects

It was expected that the diverse material for analysis would enable the researcher to get a sophisticated impression of the students' train of thoughts. The main objective was to understand the students' negotiations in order to constitute mathematical sense. Therefore, it was essential to analyze the recordings of the pairs of students working on the activity *sequentially* using a *microethnographical approach*. By such an analysis, it is possible to get insights

into the *microculture* of this special classroom situation. For the analysis, the researcher applied the principles of the tradition of the *German interpretative classroom studies* (Maier & Voigt 1991; Voigt 1995). The basic assumptions of this approach are that the teaching and learning of mathematics are instances of a social process. Mathematical meaning of symbols, means, and classroom activities are not naturally or automatically there, but need to be negotiated. Mathematical sense is interactively constituted. The aim is the *re-construction* of meaning by a deep sequential analysis of texts (mainly transcripts and texts written by students). The results of such analyses allow the recognition of the polymorphy of possibilities within individual cases and the detection of similarities or regularities. They help to sensitize for perception, and the own didactical attitude and didactical action.

The main material for analysis consists of the recordings of the pairs of students working on the activity. In the first step, raw documents for analysis were created by going through the recordings sequentially. These documents consist of large tables with the following columns: time, paraphrase, computer action, raw transcript, tentative interpretation. The information from the students' worksheets was also integrated.

These raw documents serve as overviews of a whole session of a pair of students working with the digital material. The interpretation of the actions and conversations and the re-construction of mathematical meaning are based on the chronology of their appearance.

Based on these raw documents, some episodes were chosen for detailed transcription. Episodes were especially chosen where the students discussed lively and controversially or where there was a lot of computer action. However, episodes in which wrong mathematical statements occurred were also chosen. These episodes were analyzed more deeply, also taking into account the worksheets, questionnaires and tests.

When interpreting the transcripts by going through them line by line, the raw documents allowed the researcher to embed the interpretation in the chronological course of events. This was especially important when statements of students were open to several interpretations, and gave hints for the most likely interpretation of a sequence. They also allowed the researcher at each point to look forward and backwards in the chronology of events.

## 4 Selected results

In the following, some selected results relating to the interactive activities 'Dreiecksfläche' ('area of a triangle') and 'Einbeschriebene Rechtecke' ('inscribed rectangles') are presented. We focus on how the students construct

mathematical sense using the dynamic visualizations, and which mathematical terms they find and use during the learning process. The impact of the level of metavariation on the learning process is of particular interest. Some occurring epistemological obstacles are also pointed out.

For the demonstration of the results, the author chose to depict some recurrent phenomena. The phenomena are described by presenting different episodes in the form of excerpts of transcripts, student remarks on worksheets, and their interpretation.

#### 4.1 Selected results: ‘Dreiecksfläche’ (‘area of a triangle’)

##### 4.1.1 Distinction between ‘stock and change’

The main difficulty of this activity was the distinction between ‘stock of area’ and ‘change of area’. Stock and change had to be described in the situation as well as in the graph representation form. It appeared that the description within the situation was more difficult. In the situation, one has to describe the aspect of change by analyzing the growth of area qualitatively, whereas in the representation form graph this can be described using terms like ‘slope’ or ‘gradient’.

We present an excerpt of a transcript showing the discussion of the question ‘Why is the graph shaped like this?’ Prior to the beginning of the transcript, S1 stated that the graph increases at first and decreases after point C is passed. S2 does not agree. The excerpt shows a very small part of a long discussion which got pretty loud in the end. Finally, a third student (S3) intervened.

- S2: But for me it does not decrease, it definitely increases.
- S1: But it, can’t you see this here (*points at the monitor*)
- S2: If it, if it would decrease, it would go down again like that (*points at monitor*)
- S1: It does not have to-
- S2: This is decreasing.
- S1: No, it really declines. It’s the ratio that declines.
- S2: No.
- S1: Look, if it goes down like this.
- S2: Ok, then let’s say the ratio declines, but here (*points to the monitor*) it still increases. It also increases here and also there.
- S1: Yes, but however that is always the ratio. However, the graph is the ratio.
- S2: No (.) Yes, ok, the graph could be the ratio but it definitely does not decrease here (*points at monitor*)
- S1: But proportionally it decreases.
- [...]
- S3: How about saying ‘the slope decreases’?

- S1: (*smiling*) That’s right. Good.
- S2: I am warm.

For S2, the graph represents the *stock of area*, whereas S1 tries to find information about *change in the graph*, but is not able to find appropriate terms for describing her observations in the situation. She speaks about some ‘ratio’ that decreases after point C is passed, but does not specify which ratio she means. In fact she realizes that the curvature of the graph changes. S3 helps with the term ‘slope’ to describe the change on the graphical side.

Why is this difficult from a mathematical point of view? In fact the applet shows the *stock graph* (area function) next to the *change graph* (triangle interpreted as piecewise linear function is the derivative). The stock graph is monotonic increasing, while the qualitative change of growth when passing point C is simultaneously visible in the decrease of the derivative. In other words: *Change in the stock graph* means *stock in the change graph*. *Stock and change* are dynamically connected by the learning activity. Therefore, both—stock and change—are mentally accessible in an intuitive and implicit way. This difficulty is also evident from the students’ answer on the worksheet:

“Because the area increases in the beginning dependent on AD, then the slope slightly decreases, because the line AD decreases in proportion to the area. The graph always has to increase, because the area also increases all the time.”

On the graphical side, they were able to describe the change using the term *slope*, whereas in the situation they were not able to give a description and explanation for the shape of the graph. Their attempt of describing a ratio is a good start when thinking about a *rate of change*, but they do not make clear which ratio they exactly mean. They always seem to consider the distance AD and not the segments of distance on the *x*-axis for their ratio. Even if they form the ratio  $\text{distance(AD)}/\text{area}(\text{distance(AD)})$ , the statement *the ratio declines when passing C* is wrong.

The description of monotonicity was not a problem at all: It was described in both—the situation and the graph representation form.

During the above discussion, the students did not use the possibility to drag points. Later, they used the level of metavariation in the following way:

- S2: No, you have to move C.
- S1 (*moving C to the left side, the first section of the curve from (0/0) to the inflection point is now shorter than the latter one*) This should be as small as possible. (*moves C to left again*) Can you see that now? Can you see that it is just the same here. That it is going like this?

- S1 Yes, of course. I can see that, but it is not decreasing here!
- S2 Yes, the slope decreases.
- S1 Yes.

S1 uses the level of metavariation to make the change of the curvature at the inflection point more obvious. She experiences the inflection point as being invariant under metavariation ('Can you see that it is just the same here?'). In this sense, the students are forced to view the functional dependency in a qualitative way. In this case, metavariation is used to affirm the students' statements by emphasizing the characteristic properties of the area function.

#### 4.1.2 Epistemological obstacle 'slope in one point'

The next excerpt of a transcript illustrates one notable epistemological obstacle occurring in the learning process with the activity. The students try to describe the shape of the graph, especially the shape at the inflection point corresponding to C:

- S1: I'd say the slope of the function changes.
- S2: No, no, the slope, the function does not have a slope, because, if it would have one, it would be a line.
- S1: Move it.
- S2: (*moving C horizontally back and forth*) It. Look, because he, it, aah, there is no slope, because the slope is different everywhere. There is no exact slope especially not here (*moves C above the middle of AB*)

So far, these students encountered the term *slope* only in connection with *slope of lines*. That is why S2 does not accept the term *slope* in the case of the area graph at first. Moving point C (using the level of *metavariation*) seems to lead to another visual impression—the impression that the slope is *different everywhere* and that it is *not exact*, which could mean that there is no exact mathematical description possible. Metavariation serves as visual accentuation, which could lead to further mathematical investigation: S2 moves C and says 'the slope is different everywhere' although she stated 'the function does not have a slope' before. The use of metavariation led to two contradictory statements.

In fact student S2 formulates the need for an extension of the *concept of slope* from lines to general differentiable graphs by herself. This need is initiated using the level of metavariation. There seems to be an epistemological obstacle here (which is maybe related to the 'illusion of linearity' mentioned in Sect. 1.1.2). The ideas and mental images the student connects with the concept of slope are limited and the contradictory statements of S2 lead to the need of extension of the concept of slope. In class this could be used productively as a starting point for a

discussion about the slope of a function like the given one, and about the possibilities to extend the concept of slope in a consistent way.

#### 4.2 Selected results: 'Einbeschriebene Rechtecke' ('inscribed rectangles')

##### 4.2.1 Epistemological obstacle 'continuity'

The *concept of continuity* is strongly related to characteristic properties of the real numbers (like completeness or uncountability) and the idea of approximation. In diverse situations while working with the applets, the students faced observations requiring mental concepts concerning continuity. In the following, some examples are given.

Working with applet one (Fig. 4), the students make conjectures about the change of the area of the inscribed rectangle. The following excerpt is part of a discussion of a pair of students working with applet one:

- S2: Ok, so the area (*moving P on B, P is not fixed on the line BC*) is zero at B (*moving P slowly on BC from B towards C*) then it gets larger.
- S1: Why does it get larger?
- S2: Oh well, because from here it (*moves P on B and then slowly towards C*) here it is zero and then it grows at first.
- S1: How do you know that it is larger? It is just not that long.
- S2: Yes, but anyway, it is larger than zero. Thus it is getting larger at first.
- [...]
- S1: No, it is (.) I think the area is always the same!

The fact that the area changes *continuously* does not seem to be intuitively obvious for all students. S2 implicitly assumes the continuity of the area function ("the area is zero at B then it gets larger", "it grows at first"), whereas S1 thinks that "the area is always the same", although S1 recognized that the area is zero when P coincides with B or C. But this would mean a jump discontinuity of the area function. In fact the continuity of the area function is not really visible in the situation. If P equals B or C there is no green rectangle visible, but in all other cases there is. The continuity is better represented by the graph, because the graph visualizes the continuous change in area and moving the point visualizes this change in a dynamic way. This could serve as a dynamic basis for the development of the concept of continuity.

Closely related to the concept of continuity is the *concept of approximation*. When the students started to search for the maximum area, they usually gathered some approximate (imprecise) values. They were not aware of the necessity to choose more and more precise values while

coming closer to the maximum. In the following excerpt, the students work with applet one (Fig. 4), gathering some sample values for the area and trying to find the maximum.

- S2: Now, we have (*moves P to the approx. coordinates P(6.68|4.98)*) five times six-and-a-half that makes (.)  
 S1: Thirty.  
 S2: More than thirty. (*moves P to the point P(6|6)*)  
 S1: Six times six, thirty-six. It gets larger. (*starts to write*)  
 S2: Yes, until six times, thirty-six it gets larger (*moves P to the approx. coordinates P(5.2|7.1)*) Now we have seven times five that's thirty. So, up to point six six  
 S1: Up to the middle, isn't it? [...] Perhaps we can say, that it that it up to the middle of the line?—  
 S2: No. Because now we have seven times five, that's thirty-five. It decreases again.

As above most of the students conjectured at first that the maximum is taken in point P(6|6) with a maximum area of 36. Note that the maximum area of 37.5 is taken in point P(5|7.5), thus at the midpoint of the line segment BC. In the discussion above, the students gather some values by changing the y-value for P in ones from five to seven and taking approximate values for the x-value to compute the area of the rectangle. Although the maximum could lie in between the chosen points, they do not refine the steps on the y-axis. This shows that they are not aware of an implicit concept of the intermediate value theorem—one of the important theorems about continuous functions. Connected to this is the idea of approximation. After moving to point P(6|6) and computing an area of 36, S2 moves P to the approximate coordinates P(5.2|7.1) and computes five times seven equals 35 (in the beginning S2 even miscalculates “seven times five that's thirty”), which is lower than 36. In fact taking the ‘exact’ values 5.2 times 7.1 leads to the value 36.92 which is larger than 36. More precise values are needed to get closer to the maximum. Approximation means adding decimal places and therefore real numbers need to be seen as decimal numbers. Of course, real numbers can also be seen as complete ordered field or as being geometrically represented as number line. But in this case (for the concept of approximation) the representation of real numbers as decimal numbers is the best choice. The ‘misapprehension’ of the students above is again a good chance for a productive continuation of the lesson when dealing with the idea of real numbers and continuity. To deal with real numbers and related concepts, one has to establish suitable ideas about real numbers. Moreover, in certain situations adequate representations and ideas need to be activated.

A third example in the context of continuity shows that working with the applets enabled some students to describe continuity in their own terms. The following excerpt is part

of a discussion of the problem ‘Explain by using the graph, which values appear once, twice or never.’ The students are working with applet two (Fig. 5, left). The discussion starts with the statement that all values below zero and above 37.5 are never reached.

- S2: Well, or below zero. Larger than 37.5 or below zero (.) smaller than zero. (*writing on the worksheet*) The values larger than ( )  
 S1: Mhm. (*moving P*) Well, the value 25 is never reached either.  
 S2: Of course it is.  
 S1: Mmh. Look. (*moves P*) Not here. (*the area value 25.1 appears as a decimal fraction on the monitor*) Yes, twenty-five one- hmm well, ok, I-  
 S2: It will be reached once eventually.  
 S1: (*trying to reach the exact decimal fraction 25.0 by moving P, reaches 25.085*) Here (*pointing to graph?*) the value 25 appears also.  
 [...]
 S2: Each value is actually reached. (*moving P*) It is always possible to move the point a tiny bit.

S1 states that the “value 25 is never reached” while moving P. Obviously, his statement is based on the observation that he was not able to obtain the decimal fraction 25.0 beneath the applet. When moving P the student S1 observes the change of the data and not the change of the corresponding point in the area graph. But the data are not appropriate to reflect continuity, because there are at most four digits visible after the decimal point. Finite decimal fractions only allow a discrete view of the situation, whereas a graph allows a geometric approach as continuous movement towards certain values. It is likely that S1 looks at the graph, when saying “Here the value 25 appears also”. In the end, S2 expresses his notion of continuity based on the experience with the dynamic visualization within the graph by “It is always possible to move the point a tiny bit”. Variation within the situation allowed the student to really experience continuity. Moreover, the student found his own terms to describe continuity based on the dynamic visualization (‘move the point a tiny bit’). This intuition could serve as an intuitive basis for an infinitesimal approach to calculus (see also Tall 1996, p. 308/309).

#### 4.2.2 Metavariation and predicative thinking

The level of *metavariation* is especially difficult to understand in a functional-dynamic way in this activity. The metafunction, i.e. the function assigning the triangle ABC to the area graph, is two-dimensional depending on the points B and C. Moreover, it is difficult to simulate the change of the area mentally within a given situation

because there are two underlying functional dependencies: the dependency between the coordinates of  $P$  (which is a linear function) and the dependency between the  $x$ -coordinate of  $P$  and the area  $F(x)$  (which is quadratic).

Metavariation is easier to understand when finding a predicative description of the situation. A predicative description considers the structure and invariants. It does not care so much on dynamics, but on static structures and the embedded complex relationships. It aims at the properties and their interrelations (Schwank 1999).

It seemed to be a specific obstacle of the applet that it strongly suggested a functional-dynamic view instead of a structural view, and in this case the dynamicity hinders the students from getting deeper insights.

In fact, the students formulated both views. For example, one student's solution to the question 'Give two examples for triangles so that the rectangle with maximum area is 15. Describe how you found these triangles.' was:

"On the  $x$ -axis you need a number that results in 60 when multiplied with  $n$  or you take an arbitrary point on the  $x$ -axis and move the point on the  $y$ -axis as long as the maximum is 15."

Their structural approach is only understandable when watching the recording. It is meant, that the  $x$ -coordinate of  $B$  multiplied with the  $y$ -coordinate of  $C$  must result in 60, because the maximum area is one-fourth of the rectangle formed by completing the triangle  $ABC$  to a rectangle. Their second description is a functional-dynamic one giving an instruction how to use the applet to find the triangle. It does not integrate an object view of the situation.

Once the students had found an appropriate predicative description, it enabled them to think reversible in the sense of Piaget (1972). The students were able to reverse their mental operations. One pair of students used the following predicative description when characterizing the location of the maximum: "The maximum is found by reducing the coordinates of  $P$  by half and multiplying these values."

To answer the question of how they found the triangles, so that the maximum of the inscribed rectangles is 15, they reversed this operation mentally by saying "For example one can [compute] 15 divided by 3 [...] then you get 5 and you take 5 times 2 and 3 times 2 and get these values [ $x$ -coordinate of  $B$  and  $y$ -coordinate of  $C$ ]"

Please note that operational thinking in the sense of Piaget (1972) is needed to answer questions like 'What happens to properties, functional dependencies or relations if ...?' Operational thinking includes composition, association and reversion of operations (mentally or supported by visualizations). Therefore, it is strongly related to functional thinking.

## 5 Concluding remarks

The diversity of phenomena described and analyzed in Sect. 4 shows that the interactive visualizations do not just lead to the manipulation of some points or lines, but really activate the formation of an intuitive access to concepts of calculus.

It emerged that each activity has its own mathematical focus besides the common design principles accentuating the dynamic and object view of functions.

Whereas the activity 'area of a triangle' focused on the distinction between stock and change, the focus of the activity 'inscribed rectangles' is on the description of continuous change or movement with certain characteristics like symmetry, maximum, etc.

The results show that the interactive environments together with the given tasks asking for verbalization lead to various approaches to conceptualization processes. Verbalization forced the students to negotiate the mathematical meaning. It is assumed that these processes contribute to an enrichment of the concept image of functions integrating a dynamic view. For activating the processes, the students used the interactivity and the two levels of variation. For example, variation within the situation was used to make guesses about the change in area in the learning activity 'inscribed rectangles'. The students were enabled to gather some values and to test some characteristics of the area function, e.g., when they were asked which values appear twice or if there is a maximum value. In this sense, they worked mathematically using the applet. Moreover, the variation within the situation in the applets of 'inscribed rectangles' allowed the students to find an intuitive formulation of the idea of continuity. Their idea is strongly related to an infinitesimal approach to calculus.

Metavariation was used to emphasize certain characteristics of the function which are invariant under meta-variation. It also forced the students to develop and verbalize a qualitative view of the functional dependency. This was especially obvious in the discussion about stock and change when the students started to drag point  $C$ .

The microethnographical approach of the German interpretative classroom studies also allowed the detection of some epistemological obstacles ('slope in one point', 'continuity'). These obstacles are valuable hints for further didactical considerations and help to sensitize for perception and own didactical action. Overcoming such obstacles is an essential part in the conceptualization process and can be used productively in lessons. For example, the obstacle 'slope in one point' was detected using metavariation. The need for the extension of the concept of slope was formulated by the student herself.

The level of metavariation needs special attention. It allows experiences with the object aspect of functional

dependencies, but is difficult to understand and explain in a functional-dynamic way.

Metavariation relates to the object view of a function and emphasizes global and local properties of the dependency. But metavariation also evokes some obstacles specific for the applets. Nevertheless, in case of the activity ‘inscribed rectangles’, metavariation allowed the students to formulate a predicative description of the situation. This led to reversible thinking which is strongly connected with functional thinking.

In the opinion of the author, the environments could be of high value when continuing the development of routines and further concepts in calculus. For example, the activity ‘area of a triangle’ could serve as basis of the fundamental theorem of calculus. The underlying idea is the idea of supplantation as described in Sect. 2.1. The interactive visualization leads to mental simulation and a teacher could refer to it. Later on, the situation ‘area of a triangle’ could be examined by looking for an analytic description of the function and its derivative. Mental simulation allows a dynamic interpretation of the underlying functions.

Also, the level of metavariation can be described analytically, which leads to an elaborated image of the fundamental theorem of calculus.

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