



Construction of the Mathematical Meaning of the Function–Derivative Relationship Using Dynamic Digital Artifacts: a Case Study

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Abstract

This article describes construction processes of mathematical meaning of the function–derivative relationship, as it is studied graphically with a dynamic digital artifact. The discussion centres on a case study involving one student during his interaction with the artifact. He was asked to explain the connection between two linked dynamic graphs: the graph of a function and the graph of its derivative function. The study was guided by the semiotic mediation approach, which treats artifacts as fundamental to cognition and views learning as the evolution from meanings connected to the use of a certain artifact to those recognizable as mathematical, that is, connected directly to the mathematical object. In the course of three rounds of data analysis, the student was shown to progress from a point-specific view to an interval one, and to move toward a construction of the meaning of the derivative as a function. The actions of the student and his interactions with the artifact that enabled him to construct the mathematical meanings of the function–derivative relationship are identified and described.

Keywords Derivative · Digital tools · Function · Graph · Semiotic mediation

Given the central role of the derivative in calculus, and its presence in many educational curricula, researchers have sought to understand the learning processes associated with it. Special attention has been paid to students' understanding of the concept of derivative as a slope of a function, as the limit of the difference quotient expression, and as a rate of change (Zandieh 2000). Studies that investigated the learning of the derivative symbolically and graphically have found that students tend to give greater weight to the symbolic register than to graphical means of presenting the subject (Yoon and Thomas 2015). Yet, several scholars have argued that learning the derivative

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concept graphically and, more specifically, exploring the function–derivative graphs’ relationships, may lead to a better understanding of the concept of derivative (Haciomeroglu et al. 2010; Vrancken et al. 2008; Ariza et al. 2015).

The interest in learning the concept of derivative as a relationship between two graphs has led scholars to define the concept of *students’ graphical understanding*. Asiala et al. (1997) defined *students’ graphical understanding* of the function–derivative relationship as the ability to grasp the latter directly from graphical information rather than from explicit symbolic representations and calculations connected to differentiating the function. Researchers who have explored the graphical understanding of the derivative by calculus students reported a range of difficulties, including co-ordinating the first derivative, the second derivative and the value of the limit to interpret or construct the graph of the relevant function (Baker et al. 2000; Aspinwall et al. 1997).

With these difficulties in mind, mathematics educators have also explored students’ conceptual understanding in a technological environment (Berry and Nyman 2003; Ubuz 2001). Most of the studies, however, investigated students’ graphical understanding of the derivative concept in the context of specific tasks. Only a few studies have looked into how students *actually construct* the mathematical meanings of the function–derivative relationship, as they interpret the linked dynamic function and derivative graphs, and explored how dynamic digital artifacts can serve to prompt these constructions. This article does so by means of a case study involving a high school student (Manhal), working with a dynamic digital artifact.

The case study follows his evolving sense of the function–derivative relationship to reveal the potential of dynamic digital tools in learning fundamental calculus concepts, as well as to provide some hints regarding the theoretical principles behind these concepts. It is based on the semiotic mediation approach (Bartolini Bussi and Mariotti 2008), which considers learning to involve an alignment between the personal meanings arising from the use of a certain artifact for the accomplishment of a task completion and the mathematical meanings that are deployed in the artifact. By means of this approach, I was able to identify the evolution of Manhal’s personal meanings of the function–derivative relationship into a recognizable mathematical meaning.

The article is divided into five sections. I begin with a literature review, discussing studies concerning the learning of the concept of derivative. In the second section, I consider the features of the particular dynamic digital artifact used in this study. In the third section, I present the theoretical framework, after which I describe the methodology of the research and the results. Finally, I discuss the main conclusions of the study, in particular the role played by the dynamic digital artifact in constructing the function–derivative relationship.

Literature Review

Although the literature dealing with the students’ understanding of the derivative concept is substantial, this section focuses on the literature related to students’ understanding of the relationship between the function and its derivative. The studies by Tall and West (1986) and by Asiala and colleagues (Asiala et al. 1997) were amongst the first to examine the students’ graphical understanding of a function and its derivative.

The latter found that students maintained two conceptions of the graphical interpretation of the derivative as a function: (i) seeing the derivative as the function, $x \rightarrow \text{slope at } (x, f(x))$; (ii) identifying f' with the line that is tangent at one point. Building on this work, Habre and Abboud (2006) studied the understanding of the function and its derivative by undergraduate students of a reformed Calculus I course, who used dynamic technological tools. The authors found that about half the students were unsuccessful in giving a geometric definition of the derivative, 40% of the students mentioned the slope of a tangent to the curve at a given point and the remaining 10% mentioned the instantaneous rate of change. Oehrtman et al. (2008) examined the students' understanding of the derivative as a function, claiming that, "coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function" (p. 36) was a crucial ability for understanding the co-varying nature of a function, but students were often unable to achieve such a co-ordination.

Orhun (2012) investigated how students find connections between the graph of the derivative function and the original function. The specific graph of the derivative function involved a change in slope, decreasing, increasing, a local maximum, a local minimum, and the point of inflection. The results indicated that the students found it difficult to see d/dx as an operator from "function to function" and to make connections between the graph of the derivative function and that of the original function, tending to interpret the graph of the derivative function as the graph of the function.

Other researchers have supported the findings mentioned above, even when the relationship between the function and its derivative was studied as part of scientific concepts. For example, Nemirovsky and Rubin (1992) explored high school algebra students' understanding of rate in physical settings while working with graphs, finding that many students drew a graph for the derivative that was similar to the graph of the original function, without appreciating the relation between a function and the derivative over the interval. Ariza and colleagues (Ariza et al. 2015) characterized the students' understanding of the function–derivative relationship when learning economic concepts, and suggested that some students use the concept of the derivative only in the algebraic register, whereas others make better use of the derivative when it is presented in the graphical register.

Other studies have suggested that the fact that students find it non-trivial to transition from the point-specific view to the interval view of a function may explain their difficulties in constructing meaning for the function and its derivative (Monk 1994; Sfard 1992). Some have shown that it is difficult for students to understand the relation between the point-specific and interval views, and the co-varying nature of the derivative in relation to the original function (Zandieh 2000; Oehrtman et al. 2008; Park 2015). Park explored how three calculus instructors addressed the derivative as a point-specific value and as a function in the beginning lessons about the derivative. The results show that: (a) the instructors frequently used secant lines and the tangent line on the graph of a curve to illustrate the symbolic notation for the derivative at a point, without making explicit connections between the graphical illustration and the symbolic notations; (b) they made a transition from the point-specific view of the derivative to the interval view mainly by changing the literal symbol for a point to a variable, rather than addressing how the quantity that the derivative indicates changes over an interval; (c) graphical mediators for the derivative as a function were limited to

illustrations of zero, positive and negative values of the derivative, rather than numerical values that change over an interval; (d) they often justified the property of the derivative function with the slope of the tangent line at a point of the function as an indication of the universality of the property.

Dynamic digital artifacts have been used extensively in teaching and learning the concept of derivative (Arzarello et al. 2009; Tall 2009; Sánchez-Matamoros et al. 2015). Generally, these studies focused mainly on learners' use of graphical illustration and symbolic algebra (which involved graphical illustrations of a tangent line at a given point), and their attempt to find the algebraic notation to explain how a secant becomes a tangent line.

Recently, researchers have shown interest in understanding the processes involved in learning calculus concepts as a relationship between two graphs. For example, Yoon et al. (2011) examined the role of teachers' gestures in constructing a graphical anti-derivative while working on a professional development task in a calculus modeling activity. They found that mathematical gesture spaces can encourage mathematical experimentation and lighten the cognitive load for students, but can be limited by a person's physical constraints. Swidan and Yerushalmy (2014) examined the learning process of the indefinite integral with dynamic digital artifacts. We explored the ways in which the affordances of dynamic digital artifacts allow students to correlate between the function graph and its indefinite integral function graph, and identified six mathematical elements in the process of correlating the graph of the function with the graph of its indefinite integral function. We focused mainly on how the dynamic digital artifact affordances helped students endow the function's graph and the indefinite integral function's graph with mathematical meanings, but paid less attention to the bodily interaction of the learners with the dynamic digital artifacts and to how the students constructed the mathematical meaning of the function–derivative relationship through those interactions.

Current cognitive theories (Bartolini Bussi and Mariotti 2008; de Freitas and Sinclair 2014; Radford et al. 2017) regard learning as a multi-modality, in which a variety of semiotic means (speech, body, gestures, symbols and tools) are co-ordinated. These interactions play a central role in fostering mathematical thinking. Based on these theories, and on the research studies reviewed above, I assume that interacting with dynamic digital artifacts that display linked graphs of a function and its derivative can aid learning and, in particular, facilitate the construction of the mathematical meaning of the function–derivative relationship. To understand how dynamic digital artifacts aid learning and facilitate the construction of the mathematical meanings of the function–derivative relationship, this study explores how these constructions happen by attending to the full multi-modal picture.

Theoretical Framework

Having described the literature focused on teaching the concept of derivative, I now return to the goal of this study, namely to examine how Manhal constructed mathematical meanings of the function–derivative relationship when he learned about it with the help of a dynamic digital artifact. Because this study was designed to capture how students construct mathematical meanings of these types of function–derivative

relationship, I framed it using the semiotic mediation approach (Bartolini Bussi and Mariotti 2008), which includes that the relation between artifact and knowledge is expressed by culturally determined signs and that the relation between the artifact and the learners in the course of accomplishing a specific task is expressed by signs such as speech, gestures, symbols and tools. Framing this study using the semiotic mediation approach as a theoretical tool made possible a thorough analysis of the interaction between objects deployed in the artifact and the subjective meanings ascribed to them by the student.

Bartolini Bussi & Mariotti proposed this model for a learning process that takes advantage of the potential of artifacts. It aims to describe how meanings related to the use of a certain artifact can evolve into meanings recognizable as mathematical. The semiotic mediation approach assumes that social interaction and semiotic processes play a key role in learning, particularly in situations in which learners are encouraged to use the artifact in order to solve a given task. In the context of artifact use, semiotic mediation describes the relations between personal meanings and mathematical meanings as a *double semiotic relationship*. On one hand, one concentrates on the use of the artifact for accomplishing a task, recognizing the construction of knowledge within the solution of the task. On the other hand, one analyzes the use of the artifact, distinguishing between the constructed personal meanings derived by individuals from their use of the artifact in accomplishing the task (top part of Fig. 1) and meanings that an expert recognizes as mathematical (bottom part of Fig. 1) when observing the students' use of the artifact in the process of completing the task (left triangle in Fig. 1).

Various kinds of signs are produced in the practical activity with the artifact. The semiotic mediation approach distinguishes three kinds of signs: artifact, pivot and mathematical. *Artifact signs* refer to the artifact and its use. This type of sign is produced through the social use of the artifact (upper right vertex in Fig. 1). These shared signs, generated through the social use of the artifact, may evolve into mathematical signs that refer to the mathematics context. The mathematical signs are related to the mathematical meanings shared in the institution to which the classroom belongs (right side of Fig. 1). Through a complex process of evolution of the artifact sign into a mathematical sign, other types of signs, which Bartolini Bussi and Mariotti term *pivot*

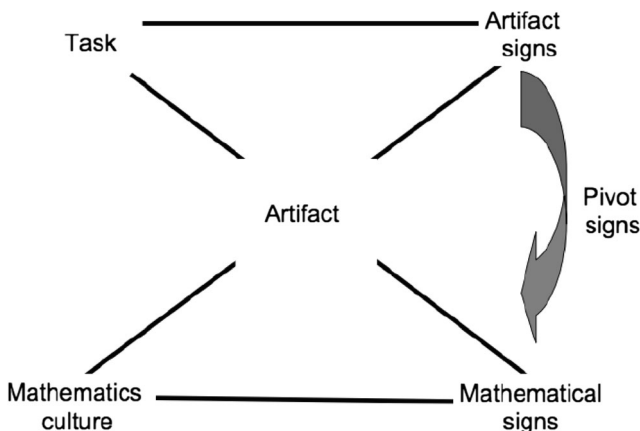


Fig. 1 Semiotic mediation model (Bartolini Bussi and Mariotti 2008)

signs, play a crucial role. These researchers suggest that the characteristic of these signs is their shared polysemy – that is, they may refer to the activity with the artifact as well as to natural language and to the mathematical domain.

The semiotic mediation approach assumes that any artifact can have valuable *semiotic potential* with respect to particular educational goals. They define the *semiotic potential* of an artifact as follows:

On the one hand, personal meanings are related to the use of the artifact, in particular in relation to the aim of accomplishing the task; on the other hand, mathematical meanings may be related to the artifact and its use. (p. 754)

The double semiotic relationship is referred to as the semiotic potential of an artifact. The potential is defined with respect to a particular design and set of pedagogical goals. Therefore, determining the semiotic potential constitutes a basic element in the design of any pedagogical plan based on the use of a given artifact.

Research Questions

To understand the ways in which Manhal constructed the mathematical meanings of the function–derivative relationship using a dynamic digital artifact, I set out to answer the following question: What are the signs through which Manhal’s personal meanings evolve into mathematical meanings when he associates the function graph with its derivative function graph?

Methodology

The dynamic digital artifact used in this study is the Calculus Derivative Sketcher (CDS) (Shternberg et al. 2004), in which two dynamically linked graphs are constructed according to the function–derivative relationship (see Fig. 2).

The basic features of the CDS comprise *construction* (which allows one to generate a graph of a function on the upper Cartesian system of the CDS using icon buttons) and *dragging* (which allows one to manipulate viz. move, stretch, etc. the graphs). The design of the CDS determines how the construction and dragging functions are actually performed. The graphic user interface of the CDS comprises seven icons  that are used to cover the different situations of a one-variable function graph (i.e. constant function, decreasing and increasing linear functions, decreasing and concave-up functions, decreasing and concave-down functions, etc.) (see Schwartz and Yerushalmy 1995). Users can sketch the graph of a function on the upper Cartesian system by using one or more of the icons. In response, the CDS displays a graph consisting of segments whose shapes resemble the respective icons that were chosen. In all cases, the CDS generates the graph of the derivative function in the lower Cartesian system (see Fig. 2).

When the upper graph is manipulated according to the features provided by the artifact design (see Fig. 3), the lower graph (anti-derivative function graph) changes to satisfy the mathematical relationship between the graph of the function and that

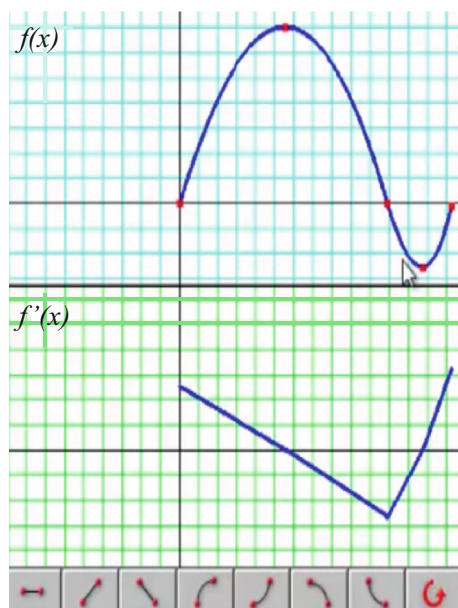


Fig. 2 Calculus Derivative Sketcher interface, which displays function graph in the upper Cartesian system and its derivative function graph in the lower system

of its anti-derivative. The educational intention of the software is to enable students, through exploration, observation and manipulation of the graphs, to notice principles linking the graphs: that would help inform them about the mathematical relationships between a function graph and its derivative function graph. In this context, the dragging tool plays a fundamental role in drawing students' attention to the relationship that links the two graphs.

Generally, in CDS, students can *drag the function graph by catching either the entire segment or its end points* (in the former case, the user can only drag the

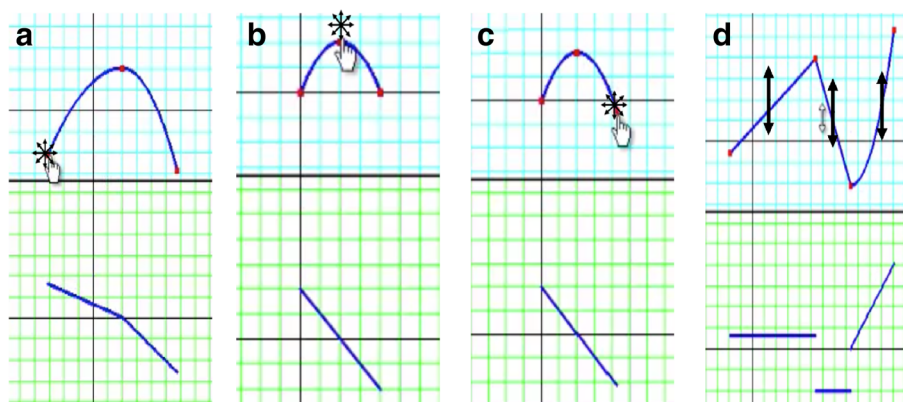


Fig. 3 Dragging options provided by the CDS design; **a**, **b**, **c** drag and stretch the function graph in all directions; **d** drag the segments vertically

segments vertically, whereas in the latter case the user can drag and stretch the segment in all directions).

Procedure

The case study reported here is part of a project aimed at examining the potential of digital dynamic artifacts (CDS is one of them) to support students in their construction of the mathematical meanings of calculus concepts. A case study methodology is an appropriate one because it enables the micro-analysis of the construction processes that were analyzed using the semiotic mediation approach. The procedure was conducted at a computer lab at the student's school. The interviewer (myself) briefly introduced the students to the artifact and showed him how to work with it. I explained, for example, how to create single and multi-segment graphs and how to drag and stretch the graphs. To identify the processes of evolution of personal meanings of the derivative function toward its mathematical meaning, I asked the students to explore graphically and explain out loud the possible connection between the two graphs on the screen. They were also given the instructions presented in Fig. 4.

Participant

This study focused on the work of Manhal, a 15-year-old student who volunteered to participate in after-school meetings. This article focuses on his learning processes both because he successfully completed the task and because he used several different mathematical signs (e.g. associating various kinds of points on the function graph with their corresponding points on the derivative function graph; associating segments on the function graph with corresponding segments on the derivative) to construct the mathematical meaning of the function–derivative relationship and clearly articulated his thoughts, both verbally and non-verbally. Manhal described himself as highly active in class, having good but not excellent grades in math (70–80), and being perceived by his teachers and classmates as a ‘smart’ student, but not as an industrious one. At the time of the case study, he had already learned the concepts of linear function, including the concept of slope, and of quadratic function, but not that of derivative. He was not familiar with any conventional function graphing software. Technological tools were not part of the students’ previous learning of functions. He learned about linear and quadratic functions with an emphasis on typical procedures and algorithms for solving exercises involving them, such as finding: the slope of a linear function, the vertex coordinates of a quadratic function, intersection points, the increasing/decreasing sub-domains of the function.

Data was collected in the course of a learning experiment that took place in a single meeting, lasting an hour and 20 minutes. It was recorded entirely with two video

The graphs in the upper and lower Cartesian systems are linked according to mathematical principles. Your task is to explore and explain possible connections between the two graphs. You should be able to construct such pairs of graphs yourself. While working on the task you may have to generate graphs using icon buttons and to change or drag the graphs as needed.

Fig. 4 Written instructions given to the student

cameras. One was located behind the student to capture his gestures when pointing to the computer screen. The second camera was set in front of the student to record him and the computer screen, which was done with the Camtasia studio software.

The videos were transcribed in Arabic, the student's mother tongue, and then translated into English. The findings presented below are the result of two rounds of data analysis. The first consisted of repeatedly reading the transcripts to search for different kind of signs involved in finding the possible connections between the graphs (Swidan and Yerushalmy 2014). The second identified how personal meanings evolved into mathematical meanings. As Manhal described the distinctive nature of a mathematical concept or its properties, and as he endowed the concept with mathematical meanings, these moments were coded. For example, the statement "the slope of the tangent line in this point is the lowest" was coded as the mathematical meaning of the inflection point.

During this round, I analyzed how the personal meanings evolved into mathematical meanings by distinguishing among artifact signs, pivot signs and mathematical signs. For example, terms such as: 'red points', 'blue segments', 'green grid' and 'icons' were coded as artifact signs. Expressions such as, "I move on" and "I go on" which are part of the natural language used to describe the movement of an object, but may also may refer the mathematical meaning of variation, co-variation, or variables, were coded as pivot signs. Mathematical terms such as 'zero', 'function', 'slope', 'positive', 'negative', 'increasing' and 'decreasing' were coded as mathematical signs. In addition, actions with the artifact and the student's gestures were also analyzed.

To establish the reliability of data analysis, two other mathematics education researchers with strong backgrounds in calculus, who were familiar with semiotic approaches, independently coded the data in both rounds. The other coders and I reached agreement after discussing the analysis process performed by each of us. The results of data analysis are presented in the following section.

Results

Manhal constructed the mathematical meaning of the function–derivative relationship through five processes while using the dynamic digital artifact: (a) recognizing the relationship between the two graphs; (b) recognizing the tangent slope of a curve; (c) recognizing the extremum point in the function graph; (d) recognizing the meaning of the inflection point in the function graph; (e) recognizing the meaning of the concavity of the function graph. Below I present an in-depth analysis of the evolution of mathematical meaning by Manhal of the function–derivative relationship.

Recognizing the Relationship Between the Two Graphs

The following analysis of the excerpt from the transcript of the interview describes how Manhal recognized the relationship between the functions graphs in the two systems. Initially, he created a straight segment, using the artifact sign () that overlapped with the x -axis in the upper Cartesian system. In the lower Cartesian system, there appeared a straight segment that overlapped with the x -axis. He endowed these two artifact signs with meanings using the mathematical sign 'zero'. This action suggests

that Manhal connected between the two Cartesian systems. Doing so, he used the artifact signs ‘upper system’ and ‘green system’.

1. Manhal [I] When I set it to **zero** in the upper system, in the green system also it is set to **zero**. [II] If I set this **function** to pass from **one**, that means each **square** is **one**, its **slope** is **one**. Here [*points to the lower system*] the **function** is **one**. [III] When I dragged the **function** [*the function graph*] and set it to **three**, that means when I go one **square** I will raise up **three squares** [*makes a staircase using the cursor; points to the horizontal lower side of a square, and points to the vertical sides of three squares*], in the lower it becomes **three**. [IV] Now I paid attention that this system [*points to the lower Cartesian system*] gives us the **slope** of this **function** [*the function graph*].

In the first utterance, Manhal connected the two systems using the mathematical sign ‘zero’ and the artifact signs ‘upper system’ and ‘lower system’. During the second utterance, he dragged the straight segment that overlaps with the x -axis, and created an inclined straight segment. He modified the inclined straight segment and set it to pass through the top-right corners of the first squares. To endow the artifact sign with meaning, he used four mathematical signs: ‘function’, ‘one’, ‘slope’ and ‘square’. He used the word ‘square’ not in its geometric context, but to quantify the slope of the function. His use of the word ‘function’ twice, suggesting that he was shifting from one Cartesian system to the other.

In the third utterance, Manhal continued to drag the inclined straight line, so that it passes through the top-right corner of the third square – its slope is 3. Now, he did not use the word “slope”, but instead pointed with the cursor at one square to the right, and then pointed three squares upward. This action suggests that he was gesturing the slope of the function graph. His words “in the lower it becomes three” and his tracing actions on the straight segment in the lower system suggested that he endowed this segment with mathematical meaning, a constant function, and was looking at its value. In the fourth utterance, by using the mathematical signs – slope and function – he explicitly recognized the relationship between the two Cartesian systems and endowed this connection with mathematical meaning: the function and its slope.

Recognizing the Tangent Slope of a Curve

The excerpt from the transcriptions of the interview that follows shows how Manhal recognized the tangent slope of a curved graph. As interviewer, I encouraged him to create a new function and to examine a variety of graphs.

2. Int: Can you determine a mathematical fact based on just one example?
3. Manhal: I made a conclusion just on that example.
4. Int: Okay!
5. Manhal: [I] Now I choose a **curved graph** [presses the fourth icon  (Fig. 5a)]. What is **slope**? [II] No, what is the characterization of the **slope** here? [III] I see that the **slope** was a **specific value**, and it begins **decreasing**. Why is it decreasing? If I

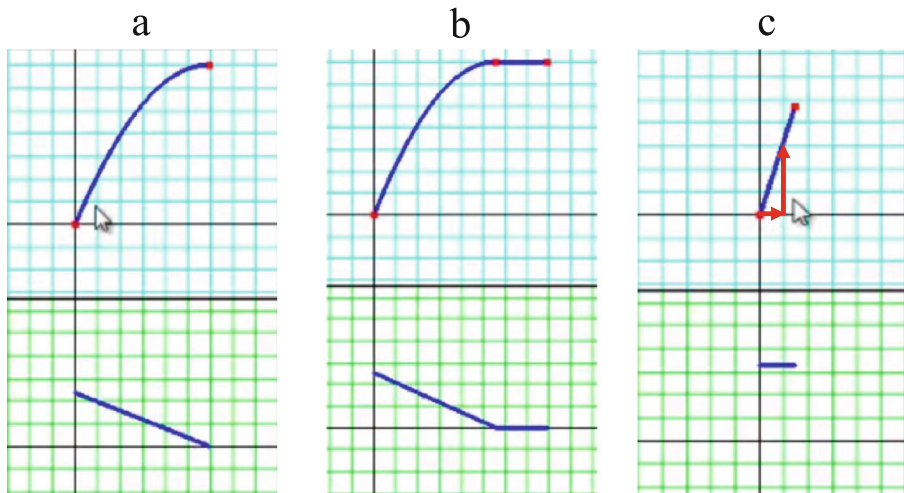


Fig. 5 a Concave-down function graph produced by the fourth icon; b a constant function graph added to the concave-down function graph; c Manhal draws a virtual staircase with the pointer

move on the graph like this [gesturing with his left hand (Fig. 6)], as long as I go, the **slope decreases**. If I add the first graph here (Fig. 5b).

6. Int: Okay!

7. Manhal: You see, **its slope is constant**. Here the **slope is varying** because the **function is curved**. Here, too, the **slope descends**, it starts at **two** and begins to **descend**.

In utterance [I], Manhal created an artifact sign using the fourth icon. His words “curved graph” suggested that he had endowed the artifact sign with mathematical meaning – a curved graph. His question “what is slope?” suggested that he was referring to the same mathematical sign ‘slope’ he had used previously. Changing his question from “what is slope?” to “what is the characterization of a slope?” suggested that he changed his strategy of making sense of the function graph. To come up with a conjecture regarding the characterization of the function slope, he used three mathematical signs: ‘slope’, ‘specific value’ and ‘decreasing’. He used the last two signs to determine the slope behavior. His words “I see that” suggest that he was perceiving the slope variation with his eyes. To answer the questions he raised, he performed a “*surfing gesture*” with his left hand, showing the way a tangent line of the function

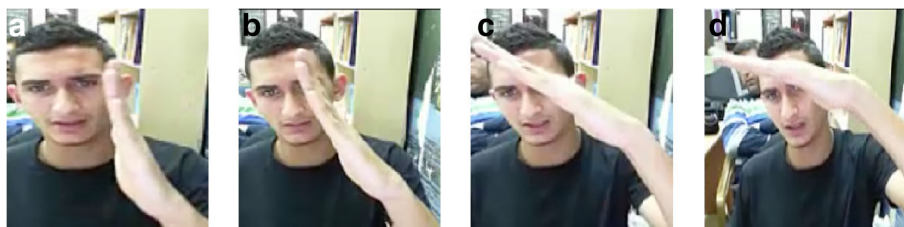


Fig. 6 Manhal illustrating the change of the tangent slope of a concave-down function graph by a surfing gesture

graph may behave (see Fig. 6). Doing so, he used the words “move on” as a pivot sign to exemplify the tangent movement on the graph.

The use of the pivot sign and the gestures enabled him to connect the movement on the function graph with various values of the slope of function (“as long as I go the slope decreases”). To verify his conclusions, he added the first segment to the curved graph, thus obtaining an artifact sign that consists of two segments. He endowed the artifact sign with mathematical meaning as a function in which the slope is constant and also made a comparison between the two segments based on their shapes. In other words, Manhal endowed the artifact signs with mathematical meaning – as a function – and characterized them according to the type of the function: one is constant whereas the other is curved. He shifted his attention from the upper system to consider the lower Cartesian system. His words “the slope descends, it starts at two and begins to descend” suggest that he was using the slope to name the Cartesian system.

Recognizing the Extremum Point in the Function Graph

To verify his conjectures, Manhal added a new segment [], so that a curved graph with an extremum point was created in the upper system (see Fig. 7).

8. Manhal: I think the **slope** of this **function** at this red point [*points to the maximum point of the function graph*] is **zero**. But if we conclude that **this function**, which is in the lower system, is the **slope**, I am now sure that the **slope** here is **zero**. Because in this **function** it is **zero**.
9. Int: Okay!
10. Manhal: So, the **slope** here is **negative**. Because here the **slope** was **positive** like this [*makes a surfing gesture with his left hand (see Fig. 8a–c)*], like this means it's going up, it means it's **positive**, then **zero** [*puts his left hand on the right hand (Fig. 8d)*], thereafter the **slope** will be **negative** [*makes a surfing gesture with his left hand (Figure 8e)*]. The **slope** begins to reduce, reduce, and reduce, until the **slope** becomes **zero**. After **zero**, the **slope** changes its direction, and starts to **descend**. This means the **slope** is **negative**. And here, I see that this point is **zero**. From this point, it starts to be **negative**.

In turn 8, Manhal conjectured that the slope at the extreme point is zero. Doing so, he used three mathematical signs (slope, function and zero) and an artifact sign (red point). To verify his conjecture, he referred to the derivative function graph in the lower Cartesian system as signifying the slope of the function in the upper Cartesian system. Signifying the derivative function graph as a slope of the upper graph, and linking the two graphs, helped him verify his conjecture that at the extreme point the slope is zero.

After attributing the value zero to the slope at the extremum point in the function graph, he endowed the value of the tangent slope of the decreasing concave-down segment with the mathematical meaning of a ‘negative slope’. To justify his claim, he referred to the right segment in the graph and endowed its tangent with mathematical meaning ‘positive slope’. Initially, he used two mathematical signs, slope and positive. After that he gestured with his left hand the tangent line and simultaneously articulated

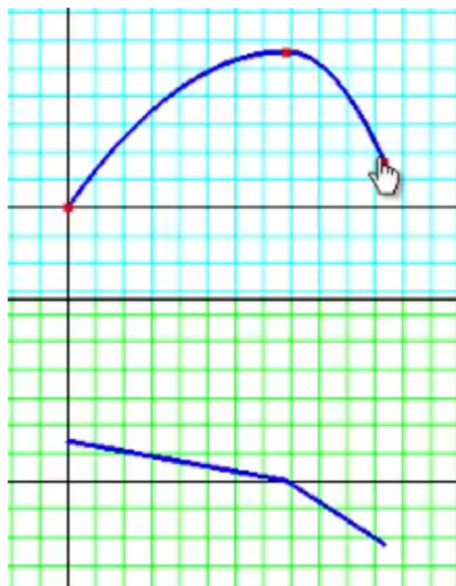


Fig. 7 Function graph with a maximum point

the words “like this”. In using the words “like this”, he referred to the pivotal sign ‘going up’, which evolved to the mathematical sign ‘positive’.

In a similar way, Manhal endowed the tangent slope of the left-side segment with its mathematical meaning – positive slope. He endowed the variation of the tangent slope with meaning by using gestures, speech rhythm and pivotal sign, while his use of the words ‘until’ and ‘after’ suggests that he was referring to the value of the extreme point as a ‘turning point’ in which the tangent slope changes its direction.

Recognizing the Meaning of the Inflection Point in the Function Graph

Manhal continued the exploration process of revealing the relationships between the two graphs. For this purpose, he added another segment to the function graph (see Fig. 9).

11. Int: Can you describe to which extent the negativity and positivity of the function is sharp?

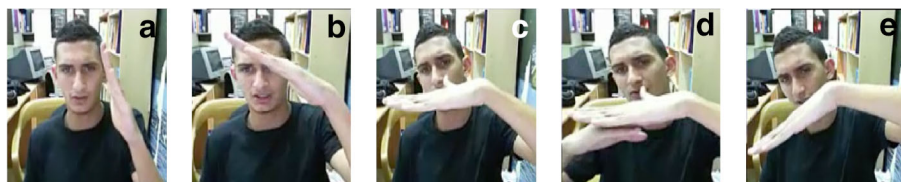


Fig. 8 **a, b, c** Manhal gestures the tangent line of the left segment of the graph; **d** Manhal gestures the maximum point; **e** Manhal gestures the tangent slope of the right segment of the graph

12. Manhal: At this point, [points to the leftmost point in the function graph] the **slope** is the biggest.
13. Int: How did you know?
14. Manhal: If I look at the lower system, which is the **slope**, this point is the highest point for the first segment. From this point the **slope** begins to reduce, reduce, reduce [traces the derivative function graph]. At first it reaches **zero**. When it reached **zero** it was in this point [points to the maximum point], because its **slope** at this point is **zero**. Here [points at the inflection point] it is the most **negative**.
15. Int: How do you know that?
16. Manhal: Here it is **zero**. I see here [traces the second segment] the **slope is negative**. Also, here in the **slope's system** I see that it is **negative** [traces the graph below x -axis]. When the **slope** reached this point [points at the inflection point in the upper Cartesian system] the **slope** changed. I also see it here in the **slope system** that it changed its value. This point is the lowest one in all the **function** and more specifically for this point.

To answer the question asked by the interviewer regarding the sharpness of the negativity and positivity of the function graph, Manhal referred to the leftmost point in the function graph (see Fig. 9). Using the mathematical sign 'slope', he endowed with meaning the value of the tangent slope ("the slope is the biggest"). Doing so, he focused on the lower Cartesian system and endowed it with the mathematical meaning

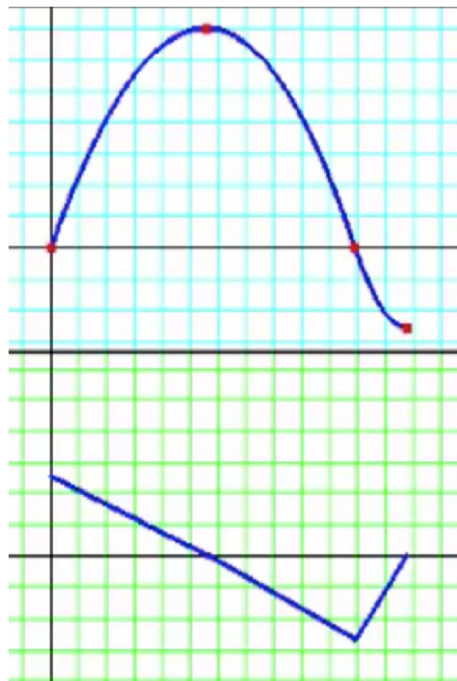


Fig. 9 Function graph with extreme points and an inflection point

of ‘slope’. Starting from the left-most point of the derivative function graph, he endowed with meaning the artifact sign (the first segment in the derivative graph) using various signs. He merged the mathematical sign (slope), pivot signs (begin to reduce), speech rhythm (reduce, reduce, reduce) and tracing the segment. Furthermore, he focused his attention on the intersection point of the derivative function graph, and ascribed it with mathematical meaning of ‘zero point’ (turn 14). He was aware that its value was zero because the intersection point of the derivative function graph corresponds to the extreme point in the function graph: “At first it reaches zero. When it reached zero, it this point [points to the maximum point], because its slope at this point [points to the maximum point] is zero” (turn 14).

After endowing the slope of the extreme point with zero, Manhal referred to the artifact sign (the third red point) and endowed its slope with mathematical meaning “most negative”. The extreme point in the function graph served as a departure point to him. He referred to its slope as “a zero”. His tracing the second segment in the function graph and articulating “the slope is negative” suggests that he endowed the artifact sign with mathematical meaning of “negative slope”. To verify his statement, he referred to the lower Cartesian system and endowed it with mathematical meaning “the system of the slope”. In a similar way, he endowed the artifact sign (the third red point) with the mathematical meaning of inflection point, i.e., the lowest value of the slope in its neighborhood: “This point is the lowest one in all the **function** and more specifically for this point”. By looking at the function graph, he conjectured that tangent slope changed its direction from decreasing to increasing. To verify his conjecture, he referred to the lower Cartesian system.

Recognizing the Meaning of Concave-Up and Concave-Down Graph Segments

Manhal added another segment to the function graph [see Fig. 10a]. The addition of the new segment did not draw his attention to new aspects of the graph. He dragged the function graph from the extremum points.

17. Manhal: [Adds the fourth segment] It is the same principle here. Let me drag this graph to see what would happen. [Drags the function graph from the extremum points (see Fig. 10b)]
18. Int: What do you observe?

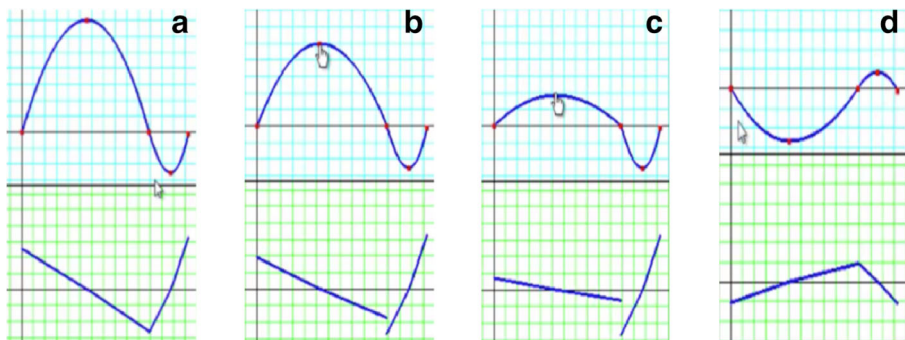


Fig. 10 Dragging the function graph to change the concavity domains of the function graph

19. Manhal: The **slope** of **all points** is lessening. I also see that the **slope** of this part [traces the concave-down segments] is always **decreasing**, until this point [the third red point]; the **slope** of this part [traces the concave-up segments] is **increasing**.
20. Int: What do you conclude?
21. Manhal: Just a moment! I want to check something [drags the function graph (see Fig. 10c)]. It is the same. When the graph is upward the **slope is decreasing**; when the graph is downward the **slope is increasing**. In this point (the third red point on the function graph), which separates the two graphs the slope, is the biggest or the smallest.

As Manhal dragged the function graph downward, he focused his attention on the change that occurred in the derivative function graph. Using the mathematical signs ‘slope’, ‘all points’ and the pivot sign ‘lessening’, he recognized how the tangent slope of the concave-down segment changed as the function graph changed. Additionally, he used the mathematical sign ‘slope’ and the artifact sign ‘this part’ to conjecture how the tangent slope of a concave-down and concave-up curve is behaving. To verify his conjecture, he reflected the function graph by changing the location of the extremum points. His statement “it is the same” suggests that he was referring to how the tangent slope is behaving in each curve of the function graph (“when the **graph** is upward the **slope is increasing**” and “when the **graph** is downward the **slope is decreasing**”). In both cases, he used the pivot signs ‘upward’ and ‘downward’ to endow with meaning the type of concavity (concave-up/concave-down) of the function graph. He used the artifact sign ‘the third red point’ and the word “separate” to endow the inflection point with meaning, i.e. a point that separates the concave-up and concave-down parts of the curve, at which the tangent slope reaches its extreme value.

Discussion

This study aims at characterizing the processes of constructing meaning of the function–derivative relationship learned with the help of dynamic digital artifacts. Its focus on Manhal, a 15-year-old student, as a case study, enabled the micro-analysis of the construction processes that were analyzed using the semiotic mediation approach. The results of the study show that his construction of meanings of the function–derivative relationship was performed through several actions and interactions: (a) transition between the two Cartesian systems (of the function and the derivative); (b) gesturing; (c) utilizing mathematical signs; (d) tracking graphs by the cursor. It seems that these actions were used by him to make the absent objects (e.g. the tangent line, numerical values) apparent, thus helping him to construct the mathematical meanings of the function–derivative graph.

For instance, in turn 1, Manhal made present what was absent using gestures with the pointer (making the staircase on the grid, as shown in Fig. 6c). This simple, highly accessible feature of the artifact helped him to quantify the squares in the grid and to compute the slope of the line. The important educational function of the virtual figures created by his gestures has been highlighted in other studies, which showed how this

feature contributed to exploration and argumentation processes both in geometry (Chen and Herbst 2013) and in calculus (Yoon et al. 2011). The implication of the case reported here is that the combination of the virtual figures with those visible on the screen forms a new landscape, in which students can dynamically explore the mathematical concepts embedded in the function graphs.

Furthermore, performing the gestures that simulate the motion of the tangent line along the function graph and tracing the function graph with the cursor, played a central role in the process of detecting the relationship between the function–derivative graphs. The common use of gestures was observed especially when he wanted to interpret the function–derivative relationship of a curved graph. In this case, the gestures are considered as an additional layer, which is not available in the artifact and which helps the student construct the meaning of the rate of change of the curved graph. This meaning represents a crucial ability to understand the concept of derivative function (Oehrtman et al. 2008).

In addition to gesturing, the transition between the Cartesian systems also played a central role in constructing the function–derivative relationship. The first transition occurred right at the beginning of the learning experiment, when the student identified the relationship between the two systems and named the lower Cartesian system the “slope”. It seems that this identification helped him use the word ‘slope’ to describe the function graph, and later, to explain the relationship between function and derivative graphs. Although Manhal focused mainly on describing the function graph, he examined the derivative function graph to verify or to refute conjectures he raised when describing the function graph (e.g. when he conjectured that the tangent slope at the extreme point in the function graph is zero, he transitioned to checking his conjecture in the derivative function graph) [line 8].

The student also transitioned from the lower to the upper system, when points that have special features (extreme, inflection and intersection points) in the derivative function graph drew his attention [e.g. turns 14–16]. It seems that this focus, as well as the transition between the Cartesian systems, encouraged the student to construct the meaning of the inflection point [turns 11–16] and of concavity domains in the function graph [turns 17–21]. The ability to construct the meaning of the last two concepts, seemingly, was promoted by the design of the artifact, which highlighted the points with special features in each graph. In any case, the success of constructing the meanings of the inflection point and of the concave domains testifies to the progress in the learning process (Orhun 2012).

This study also sheds light on the process of artifact signs evolving into mathematical signs by use of pivot signs. I do not intend to sum up here the in-depth analysis I presented in the Results section, but rather I will discuss the evolution of the word ‘slope’ and its role in constructing the meaning of the function–derivative relationship. This study stressed the importance of the word ‘slope’, which plays a role not only in connecting the graphical and symbolic registers in the definition of the limit as a process and as the result (Lobato et al. 2012; Park 2015; Zaslavsky et al. 2002), but also in graphically constructing the meaning of the function–derivative relationship. The word ‘slope’ assumed several meanings in the course of the learning experiment. At the beginning of the experiment, it was used as a numerical value for a straight-line inclination “the slope of the line is 1”. Later, the word ‘slope’ was used as a name of the lower Cartesian system (“this system is the slope”), as a dynamic object which has

certain properties (increase/decrease, positive/negative), by which the student described the shape of the graph and, toward the end of the learning experiment, to describe the shape of the graph segments. This evolution of the meanings of the word ‘slope’ indicates that Manhal transitioned from the point-specific view to the interval view (Monk 1994; Park 2015).

A semiotic analysis of the word ‘slope’ highlights the unfolding of the semiotic potential of the digital artifact. Becoming aware of the semiotic potential of the dynamic digital artifact may serve the teacher as a tool to help students in constructing mathematical meanings from their personal meanings. Our results show that, in some cases, the meanings that Manhal attributed to the artifact signs did not evolve into the mathematical signs or, at least, were not expressed in mathematical terms (for example, the inflection point and the concavity sub-domains). These moments are crucial in the learning process and invite the teacher to intervene, to mediate culturally the transformation of personal meanings into mathematical meanings.

Although this study did not examine the educator’s teaching of the concept of derivative, the results of this study may give teachers some ideas about possible ways to teach this concept. Previous studies have found that teachers made a transition from the point-specific view of the derivative to the interval view mainly by changing the literal symbol of a point to a variable, rather than addressing how the quantity that the derivative reflects changes over an interval. This study illustrates how Manhal transitioned from a point-specific view to an interval view. Additionally, paying attention to student use of “surfing gestures” as exemplified by him, and the use of the same gestures consciously by teachers, may help them not only to demonstrate the property of the derivative function, but also to become more aware of the derivative as a function. This awareness is especially important to enable students to use the graphic register for the derivative to illustrate the variation of the numerical values over an interval (Asiala et al. 1997).

The case of Manhal helps make clear the semiotic potential of artifacts that dynamically link function and derivative graphs. Because such artifacts force students to explore the function and derivative graphs using only iconic signs and dragging tools, they provide an opportunity to relate the graphs in ways that avoid over-reliance on procedural approaches, which might be suggested by the explicit presence of numerical or symbolic information. The relationships between the function and derivative graphs that Manhal investigated were not limited only to connecting zero, positive and negative values of the derivative and function, but extended also to values changing over an interval. Therefore, I surmise that the artifact can be exploited by educators in the design of learning environments, emphasizing the shift from a point-specific to an interval view in the teaching of the derivative concept.

The aim of the present study was to identify the processes of constructing the mathematical meanings of the function–derivative relationship. To this end, I focused on the learning processes of one student who succeeded in completing the given task. Therefore, potential obstacles stemming from using the dynamic digital artifact are not discussed here, although these potential obstacles exist and should be explored in depth in future research. In addition, further research is needed in the authentic educational settings, to identify how students construct mathematical meanings of function–derivative relationships and to explore the teacher’s role in fostering the evolution of

mathematical signs.

Compliance with Ethical Standards

Conflict of Interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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