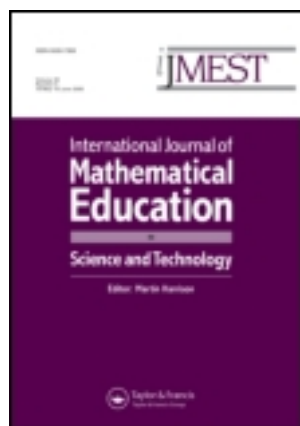


This article was downloaded by: [University of East Anglia Library]

On: 28 October 2013, At: 11:08

Publisher: Taylor & Francis

Informa Ltd Registered in England and Wales Registered Number: 1072954 Registered office: Mortimer House, 37-41 Mortimer Street, London W1T 3JH, UK



International Journal of Mathematical Education in Science and Technology

Publication details, including instructions for authors and subscription information:

<http://www.tandfonline.com/loi/tmes20>

Is the derivative a function? If so, how do students talk about it?

Jungeun Park^a

^a Mathematical Sciences, University of Delaware, 501 Ewing, Newark, 19716, USA

Published online: 05 Jul 2013.

To cite this article: Jungeun Park (2013) Is the derivative a function? If so, how do students talk about it?, International Journal of Mathematical Education in Science and Technology, 44:5, 624-640, DOI: [10.1080/0020739X.2013.795248](https://doi.org/10.1080/0020739X.2013.795248)

To link to this article: <http://dx.doi.org/10.1080/0020739X.2013.795248>

PLEASE SCROLL DOWN FOR ARTICLE

Taylor & Francis makes every effort to ensure the accuracy of all the information (the "Content") contained in the publications on our platform. However, Taylor & Francis, our agents, and our licensors make no representations or warranties whatsoever as to the accuracy, completeness, or suitability for any purpose of the Content. Any opinions and views expressed in this publication are the opinions and views of the authors, and are not the views of or endorsed by Taylor & Francis. The accuracy of the Content should not be relied upon and should be independently verified with primary sources of information. Taylor and Francis shall not be liable for any losses, actions, claims, proceedings, demands, costs, expenses, damages, and other liabilities whatsoever or howsoever caused arising directly or indirectly in connection with, in relation to or arising out of the use of the Content.

This article may be used for research, teaching, and private study purposes. Any substantial or systematic reproduction, redistribution, reselling, loan, sub-licensing, systematic supply, or distribution in any form to anyone is expressly forbidden. Terms & Conditions of access and use can be found at <http://www.tandfonline.com/page/terms-and-conditions>

Is the derivative a function? If so, how do students talk about it?

Jungeun Park*

Mathematical Sciences, University of Delaware, 501 Ewing, Newark, 19716, USA

(Received 17 August 2012)

This study explores how calculus students talk about the derivative as a function based on the concepts of function at a point and function on an interval. Twelve Calculus I students took a survey and explained their solution process during interviews. The analysis of students' discourse with the commognitive lens focusing on their use of words and visuals showed that most students did not appreciate the derivative at a point as a number and the derivative function as a function. Moreover, one of the common descriptions of the derivative as a tangent line implies that they considered the 'derivative' as a point-specific object but also a (linear) function defined on an interval. This description was closely related to their use of the word, 'derivative' for both 'the derivative function' and 'the derivative at a point.'

Keywords: derivative; function; calculus

1. Introduction

Research on students' thinking about calculus has grown over the past few years e.g., [1,2]. Among other concepts, the derivative is considered as a difficult concept due to the complexity in the definition, e.g., [3]. Related to previous research, this study explores how students describe and use *the derivative as a function* while they solve problems involving the derivative. This aspect, the derivative as a function, is crucial to explore because it becomes a basis for advanced courses (e.g., Differential Equations).

The motivation of this study comes from the observation on differences in language-related terms. In English, the relation between 'function' and 'function at a point' is equivalent to the relation between 'the derivative of a function' and 'the derivative at a point.' This equivalency often allows 'derivative' without 'of a function' to be used as 'the derivative of a function' (e.g., 'Is the derivative positive?'). However, in some other languages such as Korean and Japanese, the terms for these two concepts do not share a common word, and thus there is no confusion between the terms. Based on this observation, this study addresses, 'How do English-speaking calculus students describe and use the derivative as a function in task-based interview settings?' To this end, the students' use of words and visuals for the derivative was closely examined based on the developmental stages of thinking about function.

Investigating students' thinking through their discourse can contribute to the current literature about the role that mathematical language plays in students' learning. There has been research on how word use is related to children's thinking about early mathematical concepts, e.g., [4,5], but few studies have been done on advanced concepts. An explanation

*Email: jungeun@udel.edu

about students' use of keywords and visuals may extend our understanding of the role that language plays in the learning of the derivative, and guide instructors' classroom discourse. Existing studies have shown that children and grownups speak about mathematical topics differently, e.g., [6]. This study explores whether there are such discrepancies regarding the derivative and what the possible explanations might be. Thus, in the discussion, this study suggests some guides for instructors to address the derivative as a function.

2. Theoretical background

To explore students' descriptions and uses of the derivative as a function, this study used the developmental stages of thinking about function found in the existing literature. Here, 'developmental stages' is taken to mean that thinking about function is developmental, not that the concept itself is developmental. This section reviews existing studies on students' thinking about function and the derivative, and the features of mathematical discourse focusing on the use of words and visuals.

Developmental stages of thinking about function

There has been a rich body of research on how students understand functions. These studies provided several ways to conceptualize the function and developmental stages of thinking about function as (a) a pointwise object and (b) an object defined on an interval, e.g., [7]. Here, as mentioned before, 'thinking about function' was used in a broad sense to include teaching and learning the concept of function. Most studies describe the first stage as being able to generate an output value when an input value is given. Monk's [8] 'pointwise understanding,' Sfard's [9] 'interiorization,' and Dubinsky and McDonald's [10] 'action' describe this stage. The next stage is described as being able to see the dynamics of a function, i.e., all values at once. This stage is called 'across-time understanding', [8] 'condensation', [9] and 'process'. [10] These two stages are also found in Common Core State Standards for Mathematics. [11] In Grade 6, they address the ratio with tables of two quantities with whole number measurement, and expressions and equations with variables representing whole numbers. In Grade 7, they use variables to represent continuous quantities, and finally define function in Grade 8. These two stages for function can be applied to the derivative: *the derivative at a point as pointwise concept* (Stage 1) and *the derivative of a function as a function on an interval* (Stage 2).

Derivative at a point and derivative of a function

Existing studies about students' thinking about the derivative can be divided in terms of the two stages described above. The studies about the derivative as a point-specific value focus on students' thinking about the difference quotient in algebraic and graphical mediators, the limit on the quotient, and the tangent lines to a curve. These studies have shown that students' misconceptions about the limit (e.g., 0.99999... never reaches 1) [12] are closely related to their thinking of local linearity and a tangent line (e.g., the secant lines never reach the tangent line), which are the basis of the graphical mediator of the derivative. [13] The studies about the derivative as a function have addressed co-variation – what varies in a function, e.g., [14]. For example, Oehrtman et al. [1] discussed that 'the derivative of the volume of a sphere in terms of its radius' is its surface area, but the derivative of the volume of a cube in terms of its side is not the surface area (p. 154).

However, few studies have addressed the relation between those two types of understanding of the derivative. Monk [8] discussed these two types using students' written answers on four problems, including one about the derivative but did not give much detail about whether and how the students connect these two concepts. This current study expands on existing studies by exploring how students describe and explain the derivative as a function in terms of these two developmental stages.

Use of words and visuals in mathematical discourse

To explore students' discourse on the derivative, this study analyzed their use of words and visuals. This method is inspired by Sfard's [5] *Commognition*. Among other discursive approaches, e.g., [15,16], the commognitive approach characterizes discourse through speakers' use of words and visuals, metarules that govern such use, and the statements created as a result of repeatedly applying metarules. This approach views cognition and communication as two facets of the same phenomenon, and thus thinking as an individualized version of interpersonal communication.[5] Such a view allows researchers to explore both individual students' thinking and communication with others with the same theoretical lens and compare the nature of the discourses in these two different settings. This approach defines 'discourse' to indicate any verbal or nonverbal communication with others or oneself. Sfard [5] defines mathematics as discourse characterized by the following four features: word use, visual mediators, endorsed narratives and routines. Therefore, one or more persons' thinking can be explored by their use of words and visuals, discursive patterns, and the statements that they believe as true. In mathematical discourse, a *word* is a verbal or written expression that signifies mathematical objects. A word can be used differently in different contexts and by different speakers.[5] For example, the word *derivative* is used as the derivative at a point with a graph of a tangent line and as the derivative of a function in relation to differentiation rules. The related words –'slope' and 'rate of change' – and quantifiers –'any' and 'every' are also important to explore. *Visual mediators* refer to nonverbal, visual means of communication such as writing, drawing, and gesturing. For example, $f'(a)$ is usually used as the derivative at a point, and $f'(x)$ as the derivative function. Here, ' a ' and ' x ' are used as a number and a variable, respectively. The derivative at a point can be visually mediated by the graph of a tangent line, and the derivative function by its graph. *Narratives* are utterances that speakers can endorse as true or reject as false. *Endorsed narratives* refer to the narratives believed as true by speakers.[5] Students' endorsed narratives are often different from what the professional mathematics community endorses as true. *Routines* refer to metarules that determine discursive patterns, including courses of action, and when to start or end specific courses of action. For example, assume that there are several problems asking for the derivative at a point given the equation of a function. Suppose that for each problem, an instructor graphs the function, computes the derivative using its definition $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$, and compares the limit with the slope of the tangent line to the graph at $x = a$. This pattern can be seen as his or her course of action. These particular problems can be considered as the prompt, and the comparisons between the value of the limit and the slope can be considered as the closure of the performance of this course of action.

Based on Sfard's [5] framework, *endorsed narratives* are the results of repeatedly performed *routines*. However, in this study, the relations between routines and endorsed narratives in students' discourse were hard to explore because of the limited nature of the interview data. For this reason, instead of making connections between students' routines

and endorsed narratives, this paper reports students' endorsed narratives and routines that were identified at least three times in one student's discourse or at least three different students' discourse. This inference based on finding routines across multiple students' discourse is also supported by Sfard's [5] approach of using one theoretical lens for both one's thinking and communicating with others.

Developmental stages of thinking about the derivative

Based on the two stages for thinking about function, four stages for thinking about the derivative as a function were developed: the derivative as (a) a point-specific value (Stage 1), (b) a collection of values at multiple points (Stage 2), (c) a function (Stage 3), and (d) an operator (Stage 4). Stage 2 was distinguished from Stages 1 and 3 because Stage 2 can be used as an extension of the derivative as a point-specific object and thus may lead to the derivative on an interval. Stage 3 focuses on patterns of change in the derivative of a function on its own domain as the input variable changes. Stage 4 focuses on the derivative as an operator mapping a function to its derivative function.

3. Design of study

This study is part of a larger study consisting of classroom observation, student survey, and interviews with instructors and students. Three calculus classes at a large public university in the United States were chosen based on the instructors' background (mathematician vs. math educator) and teaching experience (e.g., years of teaching). The classes were observed for the derivative unit. Then, a survey was administered to the students in the classrooms, and interviews about their answers on the survey were conducted. Students were selected for interviews based on their written responses on the survey. This paper reports students' explanations on their solution processes.

The survey consisted of mathematical items. Most items came from the Calculus Concept Inventory.[17] They were chosen to give students flexibility to use all stages of the derivative without restricting to a specific stage (Appendix 1). Item 1 addresses Stage 1 by asking students to compute and interpret the derivative at a point. Items 2, 3, and 9 address Stages 1 and 3 algebraically and graphically. Items 4 and 5 address Stage 3 with graphs, and Stage 1 or 2 because students may use the derivative at a point or multiple points to find the graphs of the derivative function or the original function. Items 6 and 7 address Stage 3 with a possibility of using Stage 1 or 2 but did not give a specific context. Item 8 addresses Stage 1 with a tangent line at a point.

Of 99 enrolled students, 88 took the 20-minute survey in exchange for extra credit points. Two types of scores, raw and frequency, were computed and used to choose students for interviews. Raw scores (maximum 23) were based on correctness, and frequency scores (maximum 32) were based on all students' responses. The raw scores were used to find a heterogeneous group based on their performance. Frequency scores were used to find students whose answers were similar to the answers commonly chosen by other students. The 12 students selected for interviews showed a frequency score of 18 or greater and a wide range of raw score (from 8 to 21). These students had majored in engineering or natural science. Ten students had studied the derivative in high school.

Task-based semi-structured interviews were conducted individually lasting for about an hour. During the interview, students were asked to answer warm-up questions (Figure 1), and then the rest of the interview focused on their solution processes.

- | |
|--|
| Q1. What is the derivative?
Q2. Is your answer closer to the derivative of a function or the derivative at a point?
Q3. What is the other one?
Q4. Is there any relationship between the last two terms?
Q5. Is a function related to the derivative of a function or derivative at a point? |
|--|

Figure 1. Warm-up questions.

Analytical framework

The students' thinking about the derivative was explored by their uses of words and visuals. An analysis of the words focused on how nouns (e.g., derivative, rate of change, slope) were used with adverbs (e.g., at a point, on an interval) verbs (e.g., increase, decrease) or adjectives (e.g., positive/negative, general/specific). Visual mediators included written notations, drawings, and gestures. In written notations, the two concepts of the derivative as a point-specific value and as a function were examined with quantifiers such as 'a point,' 'any point,' and 'on an interval,' and numbers or letters for a point and variable (e.g., $f'(2)$, $f'(a)$, & $f'(x)$). Drawings and gestures can also be used for the derivative as a point-specific object and a function. The derivative at several points can be shown with multiple tangent lines, or arm gestures imitating these lines. A coding table (Table 1) was developed from an analysis of the textbooks and through an open coding of interview data.[18] In Table 1 'general' is used in Stages 3 and 4. In Stage 3, 'general' means the definition of the derivative of a function can be used at any point on the domain. 'The derivative' becomes specific when a specific point is given. In Stage 4, 'general' means the definition of the derivative at a point and the derivative of a function can be applied to any differentiable function, and 'the derivative' becomes specific when a specific function is given. In addition, the last stage could be further divided into three subcategories, the derivative as an operator at a point, at several points, and on a domain. However, because cases for Stage 4 were rarely identified, these categories were not separated in this paper. The students' endorsed narratives that used the words and notations in Table 1, and the routines in their use of words and visuals were also identified. This paper reports the endorsed narratives and routines that are identified in three students' discourses. Based on the categorizations in Table 1, transcripts were coded twice: once with Excel and once with Transana for intra-coder reliability.[19]

4. Findings

This section reports the students' responses to the warm-up questions and explanations of their solution processes on survey items. All names used here are pseudonyms.

Warm-up questions

During the warm-up session, students were explaining what the derivative is without any given context. To the first warm-up question (Figure 1), most students used the word, 'slope' (9 out of 12). Other answers were 'velocity,' and 'the way of deriving equations.' When they were asked the second warm-up question, seven students identified the 'slope' or 'velocity' as 'the derivative function,' with the phrases for Stage 3 'at all the points,' 'throughout the graph,' or 'over time.' Three students identified 'the derivative' as 'the derivative at a point' with the phrases for Stage 1 'at a point,' or 'at that single point.' Two students stated that the same explanation could be used for both Stages 1 and 3 (e.g., 'slope of the curve,' & 'slope

Table 1. Possible uses of words and visual mediators.

Stage	Word use		Visual mediator	
	Key terms	Attached phrases	Written notations	Graphs
1. Derivative at a point	Derivative, function, slope, tangent, rate of change, velocity	At a (specific, particular) point, time	$f'(1), f'(a), f'(x_0)$	A tangent line
2. Derivative at several points	Derivative, function, slope, tangent, rate of change, velocity	At multiple points	$f'(1) \text{ \& } f'(2), f'(a) \text{ \& } f'(b), f'(x_1), f'(x_2)$	Multiple tangent lines
3. Derivative on a domain	Derivative, function, slope, tangent, rate of change, velocity, how fast or slow	$f'(x)$	Graph of derivative of a function	Arm, hand gesture for tangent line Gestures for tangent lines at discrete points
Formula, function, equation	At any (every) point, everywhere, slowing down, increasing decreasing more and more positive/negative, becoming flat, depends on x [d], general			
Dynamic software showing the continuous change in slope of the tangent line	Hands straight following the graph of a function imitating the tangent line			
4. Derivative as operator	Derivative formula	Continuous general	$f'(1), f'(x)$	Map from a set of functions to a set of derivative functions

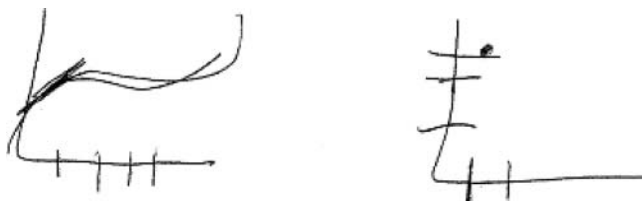


Figure 2. Joe's matching graphs of a function and the derivative at a point.

of the tangent line at a point'). Five out of 12 students also used visual mediators. Three students drew a tangent line to a curve at a point (Stage 1), one graphed the derivative of a function (Stage 3), and one graphed a function, and the tangent line at $x = 1$, estimated its slope as 3, and plotted (1, 3) on another plane (Figure 2).

Survey Item 1

Item 1 asked students to calculate the values of a function and the derivative at a point, and interpret the values in the problem context (Figure 3).

All students correctly answered parts (a) through (e), but answered (f), (g), and (h) differently. Most students did not use their answers to the warm-up questions – 'slope,' 'rate of change' – to explain their answers in (f), (g), and (h). For (f), 11 students answered 'miles,' and one answered, 'change in miles.' For (g), four students answered dollars, 5 dollars/mile, and three gave inconsistent answers. For (h), 5 of 12 students, interpreted $C'(2)$ as the change in $C(q)$ between $q = 2$ and $q = 3$, or between $q = 1$ and $q = 2$ with the units 'dollars,' and only one student interpreted it as the rate of change. Some of the correct units were not based on the rate of change. For example, a student who gave dollars/miles for $C'(2)$, interpreted it as $C(2)/2$. One student used 'change in miles' as the units for 2 in $C'(2)$, which is 'consistent with' the unit for $C'(2)$, change in dollars.

Survey Items 2, 3, and 9

Items 2 and 3 asked students to find $f'(2)$ when the equation $f'(x)$ is given algebraically or graphically (Figure 4).

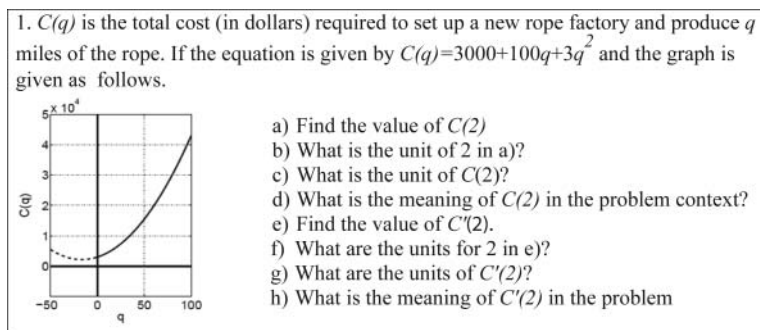


Figure 3. Item 1.

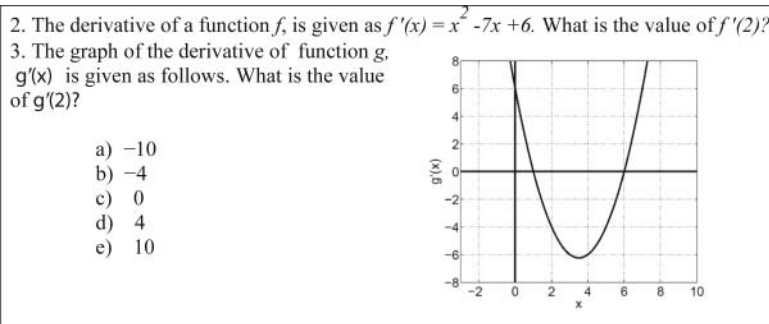


Figure 4. Items 2 & 3.

All 12 students gave the correct answer by substituting 2 in $f'(x)$ or reading the y value at $x = 2$ from the graph of $g'(x)$. Additionally, when they were asked, 'What does the number, -4 tell about $f(x)$ [or $g(x)$]?' nine students described it as 'slope,' 'velocity' or 'rate of change' at $x = 2$. Of the other three students, two said that it says that $f(x)$ [$g(x)$] decreases, but the slope is not necessarily -4 , just negative. One student found $f(2)$ by integrating $f'(2) = -4$ 'to get back to the original function,' obtained $-4x$, and 'plugged 2 in.' This last example shows that her 'derivative at a point' was not just a number.

Item 9, which asked for the slope of the tangent line at a point given the equation of the derivative of a function, made a good comparison for Items 2 and 3 (Figure 5).

Whereas all students answered Items 2 and 3 correctly, only seven students originally gave the correct answer in Item 9. After the students were reminded that they previously described the derivative as slope, and that the problem statement contained a specific point $x = 0$, three of them changed their answer to the correct answer. Of the five students who did not give the correct answer as their first choice, four students said ' a ' in $f'(x) = ax^2 + b$ is the slope without using ' $x = 0$.' For example, one student chose (b) because ' a is the slope of $f'(x) = ax^2 + b \dots$ it has to be positive.' Two students integrated $f'(x)$ to find $f(x)$, and graphed $f(x)$ and the tangent line to decide on the slope. These two students did not apply their previous description of the derivative as the slope for Stages 1 or 3. Instead, the equation of $f'(x)$ seemed to remind them about a linear function $y = ax + b$ or integration.

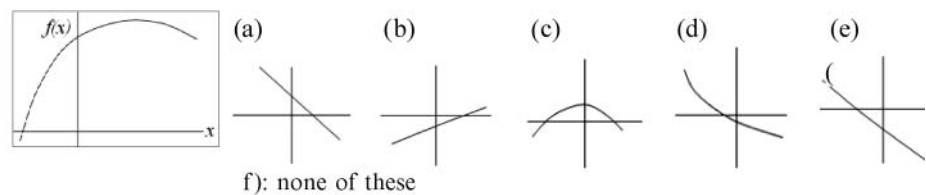
Survey Items 4 and 5

Item 4 asked students to find the graph of $f'(x)$ when the graph of $f(x)$ is given. Item 5 asked them to find the graph of $f(x)$ when the graph of $f'(x)$ is given (Figure 6).

9. The derivative of a function is $f'(x) = ax^2 + b$. What is required of the values of a and b so that the slope of the tangent line to f will be positive at $x = 0$?
- a) a and b must both be positive numbers.
b) a must be positive, while b can be any real number.
c) a can be any real number, while b must be positive.
d) a and b can be any real numbers.
e) None of these

Figure 5. Item 9.

4. Below is the graph of a function $f(x)$. Which choice a) to e) could be a graph of the first derivative, $f'(x)$?



5. Below is the graph of a derivative function $f'(x)$, which choice a) to e) could be a graph of the function $f(x)$?

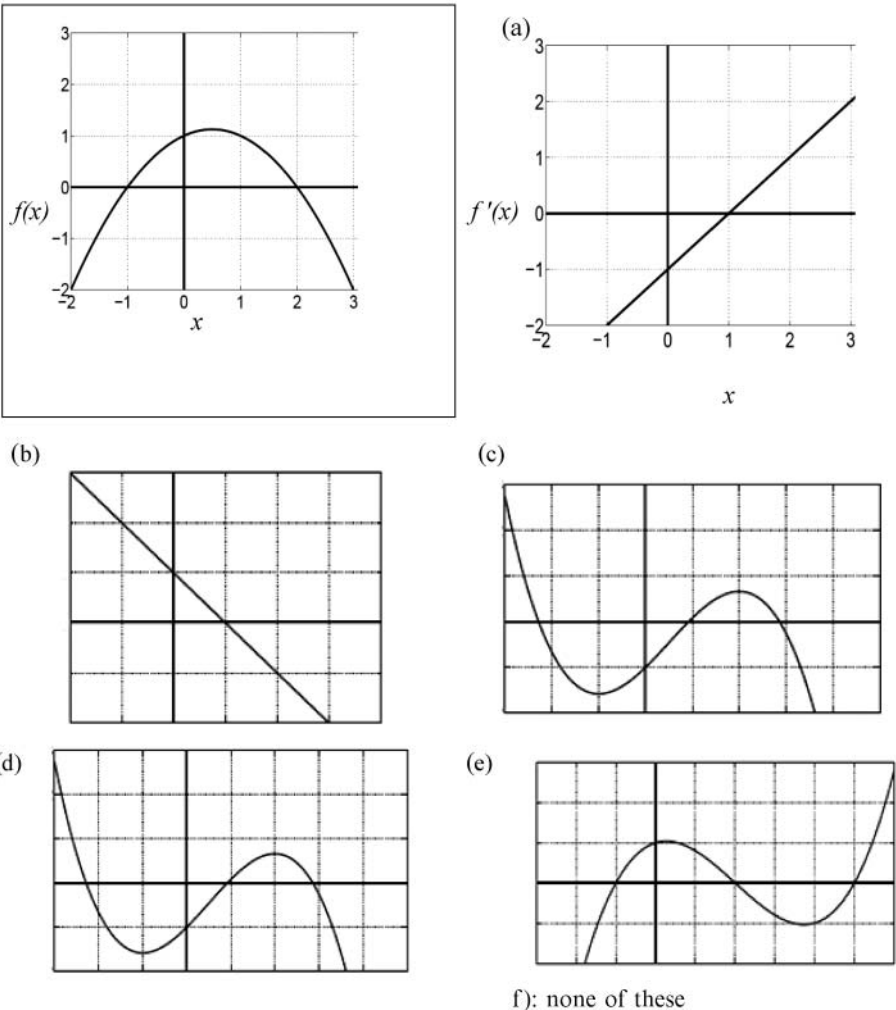


Figure 6. Items 4 and 5.

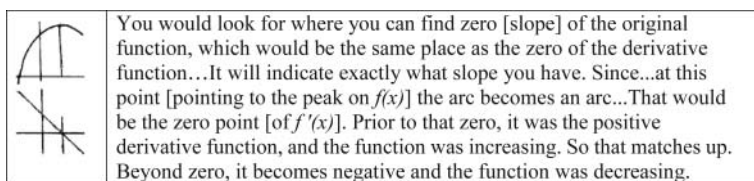


Figure 7. Neal's drawing and explanations of Item 4.

In Item 4, 10 students chose (a), one chose (c), and one chose (f). Of the students who chose (a), eight justified their choice with the word 'slope' at Stage 1 and Stage 3. Six of them justified their answer with how 'the slope' changes over the interval (Stage 3) and included the zero slope as its specific case. The other two students addressed Stage 1 while finding the zero of $f'(x)$ and then described the slopes on the left and right sides of the intervals. Figure 7 shows such an example.

Here, the student's use the term 'at this point,' indicates Stage 1. The terms, 'prior to the point' and 'beyond zero' indicate Stage 3. Other students using a similar method also used the terms indicating Stage 3 such as 'getting more and more negative.'

Among the four students, who did not use 'slope,' two students tried an algebraic method. They tried to find an equation of $f(x)$ and apply the rules for differentiation. However, their equation was incorrect. When they were asked to use their description in other problems, 'the derivative as the slope,' they were not able to use it (e.g., a student's assertion, 'I don't understand enough to use it.'). Two other students tried to find a similar shape of the graph for $f'(x)$.

In Item 5, nine chose (c), one chose (f), and two chose (d). Of the nine students, seven students explained their choice using Stage 3 (e.g., how slope changes over the interval) and Stage 1 while mentioning the critical points of $f(x)$ as special cases of Stage 3. They mainly used the same words and visual mediators involving the tangent lines that they used in Item 4. One student who found the correct answer by matching the zeros of $f'(x)$ and the critical points of $f(x)$ (Stage 1) did not justify her answer further on the interval between those critical points. One student, who correctly justified in Item 4, answered Item 5 differently. He chose (d) which shows 'the similar behaviour' of $f'(x)$. His justification did not include the word 'slope.' Another student also used the same method to find his answer. Two other students tried the algebraic method that they used in Item 4 but were not successful due to their lack of ability to find the correct equation of a curve.

Survey Items 6 and 7

Item 6 asked for the aspect of function that its derivative function describes, and Item 7 asked for the aspect that the derivative informs about its original function without giving specific representations of a function or the derivative function (Figure 8).

In Item 6, 10 among the 12 students chose (e) as their answer primarily using a graph as an example. Six of them used Stage 3 by completing the graph of the derivative of their example function or mentioning 'any points,' and the other three students used Stage 2 by checking the slope at various points. For example, Figure 9 shows one student's justification for her choice (e)).

Here, the student first used two points on the graph to show the derivative could be negative or positive (Stage 2), and then used a constant function to show 'the derivative'

6. If a function is always positive, then what must be true about its derivative function?

a) The derivative function is always positive.

b) The derivative function is never negative.

c) The derivative function is increasing.

d) The derivative function is decreasing.

e) You can't conclude anything the derivative function.

Why? _____.

7. The derivative of a function, $f(x)$ is negative on the interval $x = 2$ to $x = 3$. What is true for the function $f(x)$?

a) The function $f(x)$ is positive on this interval.

b) The function $f(x)$ is negative on this interval.

c) The maximum value of the function $f(x)$ over the interval occurs at $x = 2$.

d) The maximum value of the function $f(x)$ over the interval occurs at $x = 3$.

e) We cannot tell any of the above.

Figure 8. Items 6 and 7.

	If the function is above the x -axis...it would be like quadratic. From this example, the derivative could be negative (drawing a tangent line). So, a) and b) are out.
	The derivative, you can't say it's always increasing...You could have $\dots f'(x) = 6 \dots$. Its derivative would be zero...it's not increasing or decreasing.

Figure 9. Sara's explanation of Item 6.

is not increasing or decreasing (Stage 3). The other two students chose (a) based on the confusion between a function and the derivative function.

In Item 7, most students (11 of 12) chose (c) and justified the choice verbally and/or graphically. Their explanations mostly addressed Stage 3 as shown in Figure 10.

Here, the terms, 'going down,' and 'over the interval,' indicate Stage 3 by describing the behaviour of a function based on the sign of its derivative function. One of the 11 students did not justify her answer; she said 'it's a rule . . . can't remember why.' One student chose (b) based on the confusion between a function and its derivative function.

Survey Item 8

Item 8 asked for the relation between a function and its tangent line (Figure 11).

This item was added to explore students' thinking about Stage 1 because the tangent line is a well-known misconception about the derivative, e.g., [20]. Of 12 students, 9 chose (c), two chose (e), and one chose (a). Only eight drew the tangent line at $x = 1$, and compared the line with the curve. Four other students used the term 'tangent line' differently in their justification. One student integrated $y = \frac{1}{2}x + \frac{1}{2}$ to find ' $f(x) = \frac{1}{4}x^2 + \frac{1}{2}x + c$,' and compared $f(x)$ with ' $y = \frac{1}{2}x + \frac{1}{2}$.' Two other students used 'the tangent line' as 'the derivative' (Figure 12).

	From 2 to 3, 'the slope is negative' means that the function is going down ...The maximum value of the function over the interval occurs at $x = 2$...[The graph] can be curved, or it can be positive or negative.
---	--

Figure 10. Zack's graph and explanation of Problem 7.

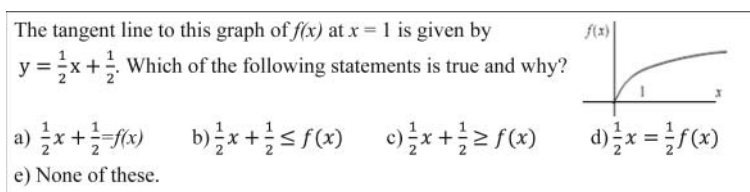
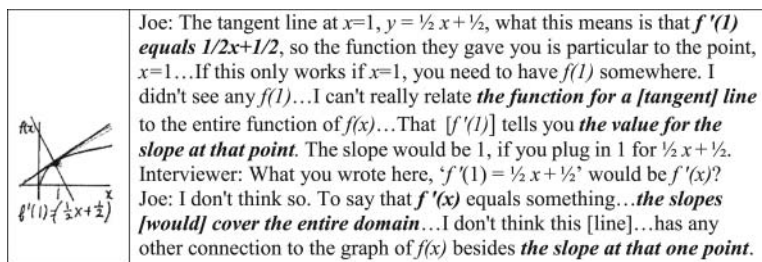


Figure 11. Item 8.

Here, the student wrote $f'(1)$ is equal to the equation of the tangent line and said that it is 'the function for [the] line.' Then, indicating the written notation $f'(1)$, he said it is 'the value for the slope at that point.' Here, he seemed to consider the right-hand side and the left-hand side of $f'(1) = \frac{1}{2}x + \frac{1}{2}$ differently: the former as a value and the latter as a function. Such confusion was also identified later; he incorrectly found the slope of $f(x)$ at $x = 1$ by substituting $x = 1$ in the equation of the tangent line not in $f'(x)$, and gave another slope $\frac{1}{2}$ from the equation, $y = \frac{1}{2}x + \frac{1}{2}$. When he was asked which one is correct, he said, 'the tangent line is a representative of the slope at this point ... I guess that this whole thing [pointing to $y = \frac{1}{2}x + \frac{1}{2}$] is the slope as opposed to just $\frac{1}{2}$... it might be pretty wrong,' and could not decide. This answer indicates that his 'derivative as slope' includes the derivative as a function (Stage 3), and he considers $f'(1)$ as point-specific (Stage 1) but not as a number. One student did not recall what 'tangent line' means although he frequently used the word 'slope.' He recalled it as 'sine over cosine.' One student used 'the tangent line as derivative' in relation to his narrative 'a function increases iff the derivative increases.' In Item 8, he said, ' $y = \frac{1}{2}x + \frac{1}{2}$ is the derivative of the function at that point' but 'was not sure ... to compare it to the whole function $f(x)$... It seems that it [pointing to $y = \frac{1}{2}x + \frac{1}{2}$] is always increasing, so is the function [pointing to $f(x)$].' He, here used 'the tangent line' and 'the derivative function at that point' synonymously.

These results can be summarized according to the four features of the discourse: word use, visual mediator, routines, and endorsed narratives. First, all students addressed the derivative as a point-specific value and as a function with keywords (e.g., derivative, slope, velocity, & rate of change) with attached phrases (e.g., at a point, any points, over the interval, & throughout the graph). However, while explaining their solution processes, they did not always apply these interpretations. For example, most of them explained C(2) in Item 1 as the change between C(2) and C(3) in dollars instead of the rate of change in dollars/mile. Though the change of a function between the two values is used as an approximation of the

Figure 12. Joe's graphs of a tangent line and $f'(x)$ and explanation.

derivative at a point, their word use and units did not imply that they made a connection between such change and the rate of change. Also, they often inconsistently used ‘slope’ and ‘derivative.’ For example, not all students, who used the slope at several points or on the interval (Stages 2 and 3) to find the graph of the derivative of the function in Item 4, used the same interpretation to find the graph of the original function when its derivative function graph was given as in Item 5. Two terms that were often connected were ‘the derivative’ and ‘tangent line.’ While solving problems, some students used the derivative as ‘the tangent line’ instead of its ‘slope’ (Item 8).

Second, during the warm-up session, 5 of 12 students used a tangent line as their visual mediator, whose slope was the derivative at a point. However, in the justification of their answers, several students’ written notations indicated their use of the tangent line as both the derivative at a point or the derivative of a function. For example, in Item 8, one student wrote ‘ $f'(1) = 1/2x + 1/2$ ’ and explained one side as a value and the other side as a function. This mixed notion was also found in other students’ work such as when two other students tried to find the equation of $f(x)$ by integrating the equation of tangent line. Similarly, when the equation of the derivative function $f'(x) = ax^2 + b$ was given, students used ‘ a ’ as the slope rather than their previous description about the derivative as the slope. This use also implies that their descriptions about the derivative involving the visual aid, the graph of the tangent, do not explicitly address Stage 1 or 3 because the tangent line itself is not either the derivative at a point (Stage 1) nor the derivative of a function (Stage 3).

Third, the most frequently identified endorsed narrative was ‘the derivative increases/decreases iff a function increases/decreases.’ When asked, most students switched to the derivative is positive/negative iff the function increases/decreases, but two of them consistently used the incorrect relation during the interview. For example, one student used the word and visual mediators for ‘the tangent line’ when comparing a function and the derivative at the point, which he visualized with the tangent line.

Fourth, two prominent routines were identified in students’ solution processes. First, when problems involved algebraic notations of a function, they tended to apply differentiation rules. However, they used the same routine when graphs were given. They tried to find the equation of the graph and differentiate it. For example, in Item 4, some students found a cubic polynomial for the given graph, differentiated the equation, and found the derivative graph. Second, when problems involved the equation of the derivative of a function, students substituted a number in x to find the derivative at a point. They did not, however, use this ‘plug-in’ routine when a problem asked to find ‘the slope’ at a point instead of ‘the derivative’ at a point as in Item 9. In Item 9, the word ‘slope’ seems to remind students about the slope-intercept form instead of the ‘plug-in’ routine. However, they used this ‘plug-in’ routine when the equation of the tangent line was given in Item 8. As using ‘the tangent line as the derivative function at a point,’ some students computed the y value of the equation of the tangent line to find the ‘slope’ or ‘the value of the derivative’ at the point.

5. Discussion and conclusion

As mentioned earlier, there is a rich body of research about students’ thinking about the derivative, mostly focusing on the misconceptions regarding the derivative at a point, e.g., Orton.[13] Though there has been an emphasis on studying the co-varying nature of the derivative, it has not been addressed in a systematic way. This study builds on existing studies by exploring features of students’ explanations about the derivative as a function through the lens of the developmental stages of thinking about function and explains some

of the well-known misconceptions. The students showed a mixed notion of the derivative as a function on an interval and as a point-specific object simultaneously, not fully appreciating their relation. Their uses of terms, 'slope' and 'derivative' with 'function at a point,' or 'represented by the tangent line' did not belong to Stages 1 or 3. Rather, such uses indicate a point-specific concept defined on an interval or an equation (e.g., tangent line). The results, which show that students' thinking about the derivative as a function was not fully developed even after the students had completed the derivative unit, lead to the following suggestions for instructors. First, the relation between the derivative of a function $f'(x)$ and the derivative at a point $f'(a)$ can be addressed by explaining that $f'(a)$ is a specific value of $f'(x)$. Reminding students about the function at a point $f(a)$ as a value of a function $f(x)$ would help them see the equivalence between how $f(x)$ and $f(a)$ are related, and how $f'(x)$ and $f'(a)$ are related. While graphing $f'(x)$, reminding them about the graphing process of $f(x)$ based on several values of the function would help them recognize the equivalence in the processes of graphing $f(x)$ and $f'(x)$. Second, the derivative function as a function can be explicitly addressed by reviewing properties of a function. Applying theorems for functions to $f'(x)$ and defining $f''(x)$ or $f'''(x)$ are good places to state $f'(x)$ as a function. Drawing parallels between $f(x)$ and $f'(x)$ would help students see both as functions. The derivative as a function can also be addressed by identifying the independent variable and exploring how the value of the derivative as the variable changes. To address their incorrect notion of the derivative as a tangent line, instructors may emphasize that each value of $f'(x)$ represents 'the slope' of the tangent line. They can show the process of plotting points for $f'(x)$ using the slopes of $f(x)$ at several points. The different units of $f(x)$ and $f'(x)$ can be shown by graphing them on separate planes and explaining the y axes having different quantities and units. To compare $f'(x)$ and $f(x)$, instructors can graph them on two transparent sheets and overlay them.

Although this study contributes to the field of mathematics education, it has some limitations. First, the students who participated in this study may not be a representative sample because they participated on a voluntary basis and came from only one university, and the sample size is small. Therefore, the results from the interview data cannot be generalized to the population. However, the 12 students were carefully chosen based on their answers with different performances on survey, and similarity with other students' answers. Second, students in this study used English. The ambiguity between the derivative at a point and the derivative of a function may come from the colloquial use of the derivative (e.g., 'Is the derivative positive?' 'Take the derivative'). If similar data were collected from other countries where the terms for these two concepts do not share a common word, the discourse on the derivative may show different aspects. As mentioned earlier, in Korean and Japanese, the terms for the two concepts do not share the same word, and thus the terms are not confusing but do not show the relation between the concepts. Data from these two countries could be used to understand further how word use is connected to students' thinking about the relation between the two concepts.

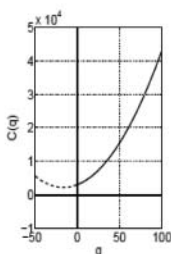
The results in this study show that students have difficulty with the derivative in general, and the derivative as a function in particular. Existing studies have shown that developing the idea of function on an interval from the point-wise view is *nontrivial* for students.[8] Then, the transition from the derivative at a point to the derivative of a function may also be *nontrivial*. If, as argued by many mathematics' educators, the derivative as a function is an important aspect of the derivative, addressing the derivative as a function based on how students previously developed their thinking about the function may make the concept more accessible to students.

References

- [1] Oehrtman MC, Carlson MP, Thompson PW. Foundational reasoning ability that promote coherence in students' function understanding. In: Carlson MP, Rasmussen C, editors. *Making the connection: research and practice in undergraduate mathematics*, (p. 150–171). Washington, DC: Mathematical Association of America; 2008.
- [2] Speer NM, Smith JP, Horvath A. Collegiate mathematics teaching: an unexamined practice. *J Math Behav.* 2010;19:99–114.
- [3] Zandieh M. A theoretical framework for analyzing students understanding of the concept of derivative. *CBMS Issues Math Educ.* 2000;8:103–127.
- [4] Fuson K, Kwon Y. Korean children's understanding multidigit addition and subtraction. *Child Dev.* 1992;63(2):491–506.
- [5] Sfard A. *Thinking as communicating*. Cambridge, UK: Cambridge University Press; 2008.
- [6] Sfard A, Lavie I. Why cannot children see as the same what grownups cannot see as different? – early numerical thinking revisited. *Cognit Instr.* 2005;23(2):237–309.
- [7] Breidenbach D, Dubinsky E, Hawks J, Nichols D. Development of the process conception of function. *Educ Stud Math.* 1992;23(3):247–285.
- [8] Monk GS. Students' understanding of functions in calculus courses. *Humanistic Math Netw J.* 1994;9:21–27.
- [9] Sfard A. Operational origins of mathematical objects and the quandary of reification-the case of function. In: Harel G, Dubinsky E, editors. *The concept of function: aspects of epistemology and pedagogy*, MAA notes 25, (p. 59–84). Washington, DC: MAA; 1992.
- [10] Dubinsky E, McDonald MA. APOS: A constructivist theory of learning in undergraduate mathematics education research. In: Holton D, editors. *The teaching and learning of mathematics at university level: an ICMI study*, (p. 273–280). Dordrecht, the Netherlands: Kluwer Academic Publishers; 2001.
- [11] National Governors Association. Common core state standards initiative. 2012. Retrieved 18 March 2013 from <http://www.corestandards.org/math>
- [12] Tall D, Vinner S. Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educ Stud Math.* 1981;12(2):151–169.
- [13] Tall D. Constructing the concept image of a tangent. In *The Eleventh Conference of the International Group for the Psychology of Mathematics Education. Proceedings*; Montreal. 1986;3:69–75.
- [14] Thompson P. Images of rate and operational understanding of the fundamental theorem of calculus. *Educ Stud Math.* 1994;26(2/3):229–274.
- [15] Lemke JL. *Talking science: language, learning and values*. Norwood, NJ: Ablex; 1990.
- [16] Gee JP. Literacy, discourse, and discourse: introduction. *J Educ.* 1989;171:5–17.
- [17] Epstein J. *The Calculus concept inventory*. Paper presented at the Conference on Research in Undergraduate Mathematics Education, Piscataway, NJ: 2006.
- [18] Strauss A, Corbin J. *Basics of qualitative research: grounded theory procedures and techniques*. Newbury Park, CA: Sage Publications; 1990.
- [19] Johnson RB, & Christensen LB. *Educational research: quantitative, qualitative, and mixed approaches*. Boston, MA: Allyn and Bacon; 2004.
- [20] Zandieh M. *The evolution of student understanding of the concept of derivative*. Corvallis, OR: Unpublished doctoral dissertation, Oregon State University; 1997.

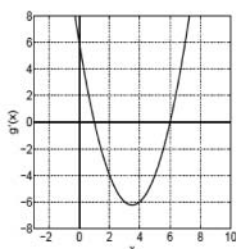
Appendix 1: Survey questions Please solve the following problems and show your work.

1. $C(q)$ is the total cost (in dollars) required to set up a new rope factory and produce q miles of the rope. If the cost satisfies the equation $C(q) = 3000 + 100q + 3q^2$, and the graph is given as follows.



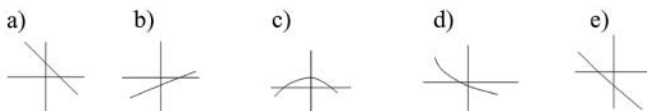
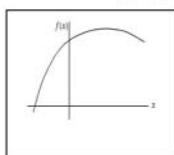
- Find the value of $C(2)$
- What are the units of 2 in (a)?
- What are the units of $C(2)$?
- What is the meaning of $C(2)$ in the problem context?
- Find the value of $C'(2)$.
- What are the units for 2 in (e)?
- What are the units of $C'(2)$?
- What is the meaning of $C'(2)$ in the problem context?

2. The derivative of a function f , is given as $f'(x) = x^2 - 7x + 6$. What is the value of $f'(2)$?
3. The graph of the derivative, $g'(x)$ of function g is given as follows. What is the value of $g'(2)$?



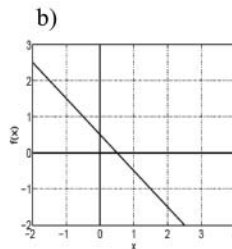
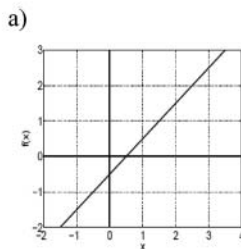
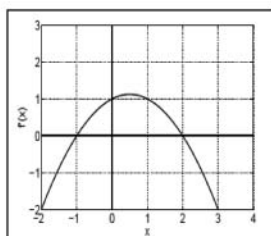
- 4
- 2
- 0
- 2
- 4

4. Below is the graph of a function $f(x)$, which choice a) to e) could be a graph of the derivative, $f'(x)$?

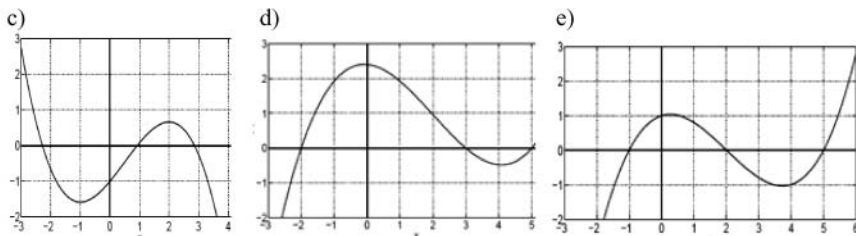


f); none of these.

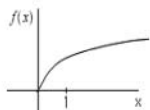
5. Below is the graph of the derivative $f'(x)$ of a function $f(x)$. Which choice a) to e) could be a graph of the function $f(x)$?



f); none of these.



6. If a function is always positive, then what must be true about its derivative function?
- The derivative function is always positive.
 - The derivative function is never negative.
 - The derivative function is increasing.
 - The derivative function is decreasing.
 - You can't conclude anything about the derivative function.
7. The derivative of a function $f(x)$ is negative on the interval $x=2$ to $x=3$. What is true for the function $f(x)$?
- The function $f(x)$ is positive on this interval.
 - The function $f(x)$ is negative on this interval.
 - The maximum value of the function $f(x)$ over the interval occurs at $x=2$.
 - The maximum value of the function $f(x)$ over the interval occurs at $x=3$.
 - We cannot tell any of the above.
8. Consider the graph below. The tangent line to this graph of $f(x)$ at $x=1$ is given by $y = \frac{1}{2}x + \frac{1}{2}$. Which of the following statements is true and why?



- $\frac{1}{2}x + \frac{1}{2} = f(x)$
 - $\frac{1}{2}x + \frac{1}{2} \leq f(x)$
 - $\frac{1}{2}x + \frac{1}{2} \geq f(x)$
 - $\frac{1}{2}x = \frac{1}{2}f(x)$
 - None of these
9. The derivative of a function, f , is $f'(x) = ax^2 + b$. What is required of the values of a and b so that the slope of the tangent line to the function f will be positive at $x=0$.
- a and b must both be positive numbers.
 - a must be positive, while b can be any real number.
 - a can be any real number, while b must be positive.
 - a and b can be any real numbers.
 - None of these
- Why?