Μάθημα 13ο   Ε14. Διακριτά Δυναμικά Συστήματα   27/11/19

$A \in \mathbb{R}^{n\times n}$ $\varphi(\lambda) = \det(\lambda I_n - A) = (\lambda-\lambda_1)^{c_1}\dots(\lambda-\lambda_p)^{c_p}$

Ορίζουμε $d_i = n - \underbrace{\mathrm{Rank}[\lambda_i I_n - A]}_{r_{i1}}$ $i = 1,\dots,p$

Ορίζουμε $r_{ij} = \mathrm{Rank}([\lambda_i I_n - A]^j)$ $r_{ij} \geq r_{i,j+1}$ και l_i τον ελάχιστο δείκτη έτσι ώστε $r_{i1} \geq r_{i2} \geq \dots \geq r_{il} = r_{il+1} = \dots$

Παράδειγμα $A = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$ $\lambda_1 = 1$ $\lambda_1 I - A = \begin{pmatrix} \lambda_1 - 1 & 0 \\ 0 & \lambda_1+2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}$

$r_{11} = \mathrm{Rank}(\lambda_1 I - A) = 1$

$r_{12} = \mathrm{Rank}([\lambda_1 I - A]^2) = 1$

Χαρακτηριστική Segré

$S_i = [n - r_{i1},\ r_{i1} - r_{i2},\ \dots,\ r_{il_i-1} - r_{il_i}]$

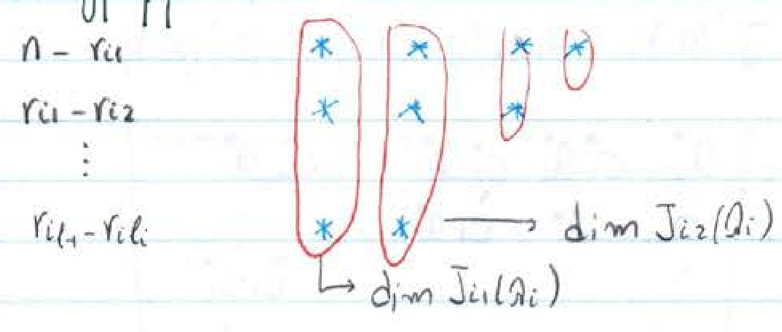
$d_i = n - r_{i1}$ ιδιοδιανύσματα   (γενικευμένα 1ης τάξης)

$r_{i1} - r_{i2}$ —"—   (γενικευμένα 2ης τάξης)

$\vdots$

$r_{il_i-1} - r_{il_i}$ —"—   (γενικευμένα l_iης τάξης)

Διάγραμμα Fernét



$$J_{ij}(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & & \bigcirc \\ & \lambda_i & 1 & \\ \bigcirc & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$$

Παράδειγμα

Έστω $A \in \mathbb{R}^{10\times 10}$, $\varphi(\lambda) = (\lambda-\lambda_1)^9(\lambda-\lambda_2)$

$\tau_1 = 9$

$r_{11} = 5$, $r_{12} = 3$, $r_{13} = 2$, $r_{14} = 1$, $r_{15} = 1$ $\longleftarrow$ $l_1 = 4$

Sergé : $[n-r_{11},\ r_{11}-r_{12},\ r_{12}-r_{13},\ r_{13}-r_{14}] = [5,\ 2,\ 1,\ 1]$

Fernet :

$n-r_{11}$	*	*	*	*	*
$r_{11}-r_{12}$	*	*			
$r_{12}-r_{13}$	*				
$r_{13}-r_{14}$	*				

$A\,[\,u_{11}^{(1)}\ u_{12}^{(1)}\ u_{13}^{(1)}\ u_{14}^{(1)} \mid u_{21}^{(1)}\ u_{22}^{(1)} \mid u_{31}^{(1)} \mid u_{41}^{(1)} \mid u_{51}^{(1)} \mid u_{1}^{(2)}\,]$

$$\begin{bmatrix} \lambda_1 & 1 & & & & & & & & \\ & \lambda_1 & 1 & & & & & & & \\ & & \lambda_1 & 1 & & & & \oslash & & \\ & & & \lambda_1 & & & & & & \\ & & & & \lambda_1 & 1 & & & & \\ & & & & & \lambda_1 & & & & \\ & & \oslash & & & & \lambda_1 & & & \\ & & & & & & & \lambda_1 & & \\ & & & & & & & & \lambda_1 & \\ & & & & & & & & & \lambda_2 \end{bmatrix}$$

$\Longrightarrow\ A u_{11}^{(1)} = \lambda_1 u_1$

$A u_{12}^{(1)} = \lambda_1 u_{12}^{(1)} + u_{11}^{(1)}$

$\vdots$

Έστω $AP = PJ \Longrightarrow A = PJP^{-1} \Longrightarrow A^k = PJ^kP^{-1}$

$\varphi(A) = (\lambda-\lambda_1)^{c_1} \dots (\lambda-\lambda_p)^{c_p}$

$J = \mathrm{bdiag}\{J_1(\lambda_1), J_2(\lambda_2), \dots, J_p(\lambda_p)\} \Longrightarrow J^k = \mathrm{bdiag}\{J_1^k(\lambda_1), \dots, J_p^k(\lambda_p)\}$

$J_i(\lambda_i) = \mathrm{bdiag}\{J_{i1}(\lambda_i), \dots, J_{ie_i}(\lambda_i)\} \Longrightarrow J_i^k(\lambda_i) = \mathrm{bdiag}\{J_{i1}^k(\lambda_i), \dots, J_{ie_i}^k(\lambda_i)\}$

$$\mathbb{C}^{m_{ij}\times m_{ij}} \ni J_{ij}^k(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & & \oslash \\ & \lambda_i & 1 & \\ \oslash & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}^k = \begin{bmatrix} \lambda_i^k & C_1^k\lambda_i^{k-1} & C_2^k\lambda_i^{k-2} & \dots & C_{m_{ij}}^k\lambda_i^{k-m_{ij}-1} \\ 0 & \lambda_i^k & C_1\lambda_i^{k-1} & & \vdots \\ \vdots & & & \ddots & C_1^k\lambda_i^{k-1} \\ 0 & \dots & \dots & 0 & \lambda_i^k \end{bmatrix}$$

όπου $C_p^k = \binom{k}{p}$

Ε14. Διακριτά Δυναμικά Σύστημα

27/11/2019

Παράδειγμα:

$$A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \qquad A^k = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}^k = (\lambda I_3 + H)^k$$

$$\overset{Newton}{=} \lambda^k I_3 + C_1^k \lambda^{k-1} H + C_2^k \lambda^{k-2} H^2 \underbrace{+ \dots + H^k}_{\emptyset} \qquad \text{όπου} \quad H = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \lambda^k \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + k\lambda^{k-1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \frac{k(k-1)}{2} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad H^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ H^m = \emptyset, \ m \geq 3$$

$$= \begin{bmatrix} \lambda^k & k\lambda^{k-1} & \frac{1}{2}k(k-1)\lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{bmatrix}$$

Παράδειγμα: $A \in \mathbb{R}^{3\times 3}$ $\qquad \varphi(\lambda) = (\lambda-\lambda_1)^2(\lambda-\lambda_2)$ $\qquad x_{k+1} = A x_k \Rightarrow x_k = A^k x_0$

$\tau_1 = 2$, $d_1 = 1$ $\qquad AP = PJ \longrightarrow PJ^kP^{-1}x_0$

$\tau_2 = 1$, $d_2 = 1$ $\qquad A^k = PJ^kP^{-1}$

$$x_k = \underset{P}{\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}} \left[\begin{array}{cc|c} \lambda_1^k & k\lambda_1^{k-1} & 0 \\ 0 & \lambda_1^k & 0 \\ \hline 0 & 0 & \lambda_2^k \end{array}\right] \underset{P^{-1}}{\begin{bmatrix} u_1^T \\ u_2^T \\ u_3^T \end{bmatrix}} x_0 = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} \lambda_1^k \overbrace{\langle u_1, x_0\rangle}^{\gamma_1} + k\lambda_1^{k-1} \overbrace{\langle u_2, x_0\rangle}^{\gamma_2} \\ \lambda_1^k \langle u_2, x_0 \rangle \\ \lambda_2^k \underbrace{\langle u_3, x_0\rangle}_{\gamma_3} \end{bmatrix}$$

$$= \gamma_1 \lambda_1^k u_1 + k\gamma_2 \lambda_1^{k-1} u_1 + \gamma_2 \lambda_1^k u_2 + \gamma_3 \lambda_2^k u_3$$

Παράδειγμα: $x_{k+1} = A x_k$

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix} \qquad x_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\varphi(\lambda) = (\lambda-2)^3$

$c = 3$, $\lambda = 2$

$d \in \{1, 2, 3\}$

Ιδιοδιανύσματα

$$(\lambda I - A)u = 0 \longrightarrow \begin{bmatrix} \lambda-2 & 1 & 3 \\ 0 & \lambda-2 & -1 \\ 0 & 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = 0$$

$$\left.\begin{array}{r} y_1 + 3z_1 = 0 \\ z_1 = 0 \end{array}\right\} \Rightarrow y_1 = z_1 = 0 \qquad u_1 = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} Au_1 &= \lambda u_1 \\ Au_2 &= \lambda u_2 + u_1 \\ Au_3 &= \lambda u_3 + u_2 \end{aligned}\right\} \rightsquigarrow \left.\begin{aligned} (\lambda I - A)\, u_1 &= 0 \\ (\lambda I - A)\, u_2 &= -u_1 \\ (\lambda I - A)\, u_3 &= -u_2 \end{aligned}\right\} \xRightarrow[\text{τα } u_2, u_3]{\text{έτσι βρίσκω}} \text{ άρα } u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, u_3 = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$$

$$\Rightarrow J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Άρα $A^k = P J^k P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2^k & k2^{k-1} & \frac{1}{2}k(k-1)2^{k-2} \\ 0 & 2^k & k2^{k-1} \\ 0 & 0 & 2^k \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix}$

Άρα $x_k = A^k x_0 = A^k \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \ldots$

Μάθημα 14ο   Ε14. Διακριτά Δυναμικά Συστήματα   3/12/2019

Μιγαδικές Ιδιοτιμές

Έστω $A \in \mathbb{R}^{2\times 2}$, $\varphi(\lambda) = (\lambda - 6)^2 + \omega^2$, $\omega \neq 0$ $\lambda_{1,2} = 6 \pm i\omega$

Έστω $\left.\begin{array}{l}\lambda_1 = 6 + i\omega \\ \bar{\lambda}_1 = \lambda_2 = 6 - i\omega\end{array}\right\}$

Τα ιδιοδιανύσματα $A u_1 = \lambda_1 u_1$

$\bar{A}\bar{u}_1 = \bar{\lambda}_1 \bar{u}_1$, $\bar{A} = A$, $\bar{\lambda}_1 \to \lambda_2$, $\bar{u}_1 \to u_2$

Θέλω να το εκφράσω με πραγματικές τιμές

Έστω $u_1 = x + iz \implies A(x+iz) = (6+i\omega)(x+iz)$

$$\left.\begin{array}{ll}\implies Ax = 6x - \omega z & \leftarrow Re() \\ Az = \omega x - 6z & \leftarrow Im()\end{array}\right\} \implies A\underbrace{[x \,\vdots\, z]}_{P_r} = [x \,\vdots\, z]\begin{bmatrix} 6 & \omega \\ -\omega & 6 \end{bmatrix}$$

Ισχύει ότι (x,z) γραμμικά ανεξάρτητα άρα P_r αντιστρέψιμος

$$\implies AP_r = P_r\begin{pmatrix} 6 & \omega \\ -\omega & 6\end{pmatrix} \implies A = P_r\begin{pmatrix} 6 & \omega \\ -\omega & 6\end{pmatrix}P_r^{-1} \text{ τότε } A^k = P_r\begin{pmatrix} 6 & \omega \\ -\omega & 6\end{pmatrix}^k P_r^{-1}$$

$$\text{Έστω } \left.\begin{array}{l} 6 = Re(\lambda_1) = \rho\cos\theta \\ \omega = Im(\lambda_1) = \rho\sin\theta\end{array}\right\} \implies A^k = \rho^k P_r\begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix}^k P_r^{-1}$$

$$P_r^{-1}A^k P_r = \rho^k \underbrace{\begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix}}_{Rot(\theta)\ (\text{πίνακας αντιστροφής})}^k = \begin{bmatrix}\rho^k\cos(k\theta) & \rho^k\sin(k\theta) \\ -\rho^k\sin(k\theta) & \rho^k\cos(k\theta)\end{bmatrix}$$

Ισχύει $(Rot(\theta))^k = Rot(k\theta)$

$$P_r^{-1}e^{At}P_r = \begin{pmatrix} e^{6t}\cos(\omega t) & e^{6t}\sin(\omega t) \\ -e^{6t}\sin(\omega t) & e^{6t}\cos(\omega t)\end{pmatrix}$$

$x' = Ax$ $x(0) = x_0$

$x(t) = e^{At}x_0$

Παράδειγμα Έστω A: $\varphi(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda^2 - 2\rho\cos\theta\,\lambda + \rho^2)$,

$\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_{3,4} = \rho e^{\pm i\theta}$ Έστω $P = [u_1\ u_2\ x\ z]$, όπου $u_3 = x + iz$, $u_4 = \bar{u}_3 = x - iz$

$$A^k P = P\begin{bmatrix}\lambda_1^k & 0 & 0 & 0 \\ 0 & \lambda_2^k & 0 & 0 \\ 0 & 0 & \rho^k\cos(\theta k) & \rho^k\sin(\theta k) \\ 0 & 0 & -\rho^k\sin(\theta k) & \rho^k\cos(\theta k)\end{bmatrix}$$

Παράδειγμα: $A^{4\times 4}$ $\quad \varphi(\lambda) = [(\lambda-6)^2 + w^2]^2$ $\quad \lambda_1 = 6+iw$, $c_1 = 2$, $d_1 = 1$

$\lambda_2 = 6 - iw$, $c_2 = 2$, $d_2 = 1$

$$\left.\begin{array}{l} Au_1 = \lambda_1 u_1 \\ Au_2 = \lambda_1 u_2 + u_1 \end{array}\right\} \qquad \left.\begin{array}{l} A\bar{u}_1 = \bar{\lambda}_1 \bar{u}_1 \\ A\bar{u}_2 = \bar{\lambda}_1 \bar{u}_2 + \bar{u}_1 \end{array}\right\}$$

Έστω $\quad u_1 = x_1 + iz_1 \qquad A(x_1+iz_1) = (6+iw)(x_1+iz_1)$

$\quad u_2 = x_2 + iz_2 \qquad A(x_2+iz_2) = (6+iw)(x_2+iz_2) + (x_1+iz_1)$

$$A\,\overbrace{[x_1 \;\; z_1 \;\; x_2 \;\; z_2]}^{P_r} = [x_1 \;\; z_1 \;\; x_2 \;\; z_2]\begin{bmatrix} 6 & w & 1 & 0 \\ -w & 6 & 0 & 1 \\ 0 & 0 & 6 & w \\ 0 & 0 & -w & 6 \end{bmatrix} = \begin{bmatrix} W & I \\ 0 & W \end{bmatrix}$$

$$P_r^{-1} A P_r = J \;\rightsquigarrow\; P_r^{-1} A^k P_r = J^k = \begin{bmatrix} W & I \\ 0 & W \end{bmatrix}^k = \begin{pmatrix} W^k & C_1^k W^{k-1} \\ 0 & W^k \end{pmatrix} = \begin{pmatrix} W^k & kW^{k-1} \\ 0 & W^k \end{pmatrix}$$

$\binom{k}{1} = k$

$$W^k = \begin{bmatrix} p^k \cos(k\theta) & p^k \sin(k\theta) \\ -p^k \sin(k\theta) & p^k \cos(k\theta) \end{bmatrix}$$

Έστω $\quad \varphi(\lambda) = [(\lambda-6)^2 + w^2]^m \qquad \lambda = 6+iw = pe^{i\theta}$, $\bar{\lambda} = 6 - iw = pe^{-i\theta}$

$$\lambda = \lambda_1, \quad c_1 = m, \quad d = 1 \qquad J = \begin{bmatrix} W & I & 0 & \dots & 0 \\ 0 & W & I & & \vdots \\ \vdots & & & & I \\ 0 & \dots & 0 & & W \end{bmatrix} \Longrightarrow$$

$$J^k = \begin{bmatrix} W^k & C_1^k W^{k-1} & \dots & C_{k-m-1} C^{k-m-1} \\ & W^k & C_1^k W^{k-1} & \vdots \\ & 0 & & C_1^k W^{k-1} \\ & & & W^k \end{bmatrix}$$

Ορισμός: $x_{k+1} = Ax_k$ λέγεται ασυμπτωτικά ευσταθές αν $\forall x_0 \in \mathbb{R}^n$

$x_k = A^k x_0 \xrightarrow{k\to\infty} 0 \iff \rho(A) = \max\{|\lambda_i| : i = 1, 2, \dots, n\} < 1$

Ε14. Διακριτά Δυναμικά Συστήματα

3/12/2019

Ορισμός: Έστω
$$\left.\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k \end{aligned}\right\}$$

Το σύστημα λέγεται εξωτερικά ευσταθές αν $\forall \{u_k\}_{k\in\mathbb{N}_0}$ με $\|u_k\| \le 1$
$\Rightarrow \sum_{k=0}^{\infty} \|y_k\| < \infty$ (με $x_0 = 0$)

Το σύστημα εξωτερικά ευσταθές αν $\sum_{k=0}^{\infty} \|G(k)\| < \infty$ όπου $G(k) = \begin{cases} D, & k=0 \\ CA^{k-1}B, & k>0 \end{cases}$

Συνάρτηση Συχνοτήτων

Έστω
$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \qquad x_0 \in \mathbb{R} \\ y_k &= Cx_k + Du_k \end{aligned}$$

Έστω $B \in \mathbb{R}^{m\times 1}$, $C \in \mathbb{R}^{1\times p}$, $D \in \mathbb{R}$, Έστω $\rho(A) < 1$, Έστω $u_k = e^{i\omega k}$ $(k \ge 0)$

Τότε $y_k = \hat{G}(e^{i\omega})u_k + \xi_k$, $\xi_k \xrightarrow{k\to\infty} 0$

Έστω $\hat{G}(e^{i\omega}) = |G(e^{i\omega})| e^{i\omega}$ $\qquad \left| y_k - |G(e^{i\omega})| e^{i(\text{[illegible]})} \right| \longrightarrow 0$

Μάθημα 15: Ε14. Διακριτά Δυναμικά Συστήματα 4/12/2019

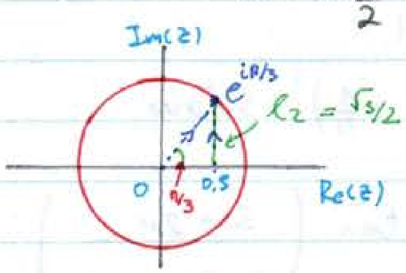
Παράδειγμα: $\hat{G}(z) = \dfrac{z}{z-0.5}$

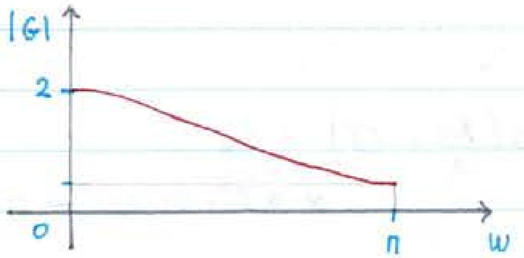
$$\left\{u_k = \cos\left(\tfrac{k\pi}{3}\right)\right\}_k \longrightarrow \boxed{\hat{G}(z)} \longrightarrow \{y_k\}$$

$z = e^{i\pi/3}$ $\quad \hat{G}(e^{i\pi/3}) = \dfrac{e^{i\pi/3}}{e^{i\pi/3}-0.5} = \dfrac{0.5 + i\frac{\sqrt{3}}{2}}{0.5 + i\frac{\sqrt{3}}{2} - 0.5} = 1 - i\dfrac{1}{\sqrt{3}}$

$|\hat{G}| = \sqrt{1 + \frac{1}{3}} = 2/\sqrt{3}$

$\Phi = -\tan^{-1}(1/\sqrt{3}) = -\pi/6$





Επομένως $y_k = \dfrac{2}{\sqrt{3}} \cos\left(\dfrac{k\pi}{3} - \dfrac{\pi}{3}\right) + \underset{\downarrow\, 0}{\xi_k}$

$\mathcal{Z}\{\cos wk\} = \dfrac{z^2 - z\cos w}{z^2 - 2z\cos w + 1}$, $\quad \mathcal{Z}\{\sin wk\} = \dfrac{z \sin w}{z^2 - 2z\cos w + 1}$

$\hat{u}(z) = \dfrac{z^2 - 0.5z}{z^2 - z + 1} \quad \rightsquigarrow \quad \hat{y}(z) = \dfrac{z}{z-0.5} \dfrac{(z-0.5)z}{z^2 - z + 1} = \dfrac{z^2}{z^2 - z + 1}$ (*)

$\mathcal{Z}\left\{\cos\dfrac{k\pi}{3}\right\} = \dfrac{z^2 - 0.5z}{z^2 - z + 1}$, $\quad \mathcal{Z}\left\{\sin\dfrac{k\pi}{3}\right\} = \dfrac{z\sqrt{3}/2}{z^2 - z + 1}$

$(*) = \dfrac{z^2 - 0.5z}{z^2 - z + 1} + \dfrac{1}{2}\dfrac{\frac{\sqrt{3}}{2}z}{z^2 - z + 1} \longrightarrow y_k = \cos\dfrac{k\pi}{3} + \dfrac{1}{3}\sin\dfrac{k\pi}{3} = \dfrac{2}{3}\cos\left(\dfrac{k\pi}{3} - \dfrac{\pi}{3}\right)$

Παράδειγμα: $\left(\cos\frac{k\pi}{2}\right)_{k\in\mathbb{N}_0} \xrightarrow{\hat{u}(z)} \boxed{\dfrac{1}{z^2-0{,}25}} \xrightarrow{\hat{y}(z)}$

$$\mathcal{Z}\{\cos wk\} = \frac{z(z-\cos w)}{z^2-2z\cos w+1} \qquad \mathcal{Z}\{\sin wk\} \quad \frac{z\sin w}{z^2-2z\cos w+1}$$

$$\hat{G}(e^{iw}) = \frac{1}{e^{2iw}-0{,}25} = \frac{1}{(\cos 2w-0{,}25)+i\sin 2w}$$

$$|G|^2 = \frac{1}{(\cos 2w+\frac{1}{4})^2+\sin^2 2w} = \frac{1}{\frac{17}{16}-\frac{1}{2}\cos 2w}$$

$$\arg[G(e^{iw})] = -\tan^{-1}\left(\frac{\sin 2w}{\cos 2w-0{,}25}\right)$$

$$\left.\begin{array}{l} |G(e^{i\pi/2})| = 4/5 \\ \phi = 0^\circ \end{array}\right\} \qquad y_k = \frac{4}{5}\cos\left(\frac{k\pi}{2}+0^\circ\right) + \underbrace{\xi_k}_{\alpha\cdot 0.5^k+\beta(-0{,}5)^k \to 0}$$

Ελεγξιμότητα

Έστω
$$x_{k+1} = Ax_k + Bu_k \qquad x_0 \in \mathbb{R}^n$$
$$y_k = Cx_k + Du_k \qquad A\in\mathbb{R}^{n\times n},\ B\in\mathbb{R}^{n\times m}$$

$$x_k = A^k x_0 + \sum_{j=0}^{k-1} A^{k-1-j} B u_j$$

Ορισμός: (A,B) είναι πλήρως ελέγξιμο ("reachable") αν $\forall (x_\alpha, x_\beta) \in \mathbb{R}^n \times \mathbb{R}^n$ τότε $\exists\ k \in \mathbb{N}_0$ και ακολουθία $(u_k)_{j=0}^{k-1}$ έτσι ώστε
$$x_\beta = A^k x_\alpha + \sum_{j=0}^{k-1} A^{k-1-j} B u_j$$

Θεώρημα: (C-H) Αν $A \in \mathbb{R}^{n\times n}$, $\varphi(\lambda) = \det(\lambda I_n - A) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \dots + \alpha_0$
τότε $A^n + \alpha_{n-1}A^{n-1} + \dots + \alpha_0 I_n = \varphi(A) = 0$

Ε14. Διακριτά Δυναμικά Συστήματα

4/12/2019

$A^k \in \langle I_n, A, \dots, A^{n-1} \rangle$

Θεώρημα: (A,B) πλήρως ελέγξιμο αν και μόνο αν $\mathrm{Rank}(\Gamma_c) = n$, όπου $\Gamma_c = [B \,\vdots\, AB \,\vdots\, A^2B \,\vdots\, \dots \,\vdots\, A^{n-1}B] \in \mathbb{R}^{n \times mn}$

Απόδειξη:

$(\Leftarrow)$. Έστω $\mathrm{Rank}(\Gamma_c) = n$ δηλαδή $\mathcal{R}(\Gamma_c) = \mathbb{R}^n$

Αν x_α, x_β αυθαίρετα τότε $x_\beta - A^n x_\alpha = \Gamma_c \xi$ για κάποιο $\xi \in \mathbb{R}^{n \times m}$

$$\xi = \begin{bmatrix} u_{n-1} \\ u_{n-2} \\ \vdots \\ u_0 \end{bmatrix} \begin{matrix} \}m \\ \}m \\ \\ \}m \end{matrix} \qquad \text{Τότε} \qquad x_\beta = A^n x_\alpha + [B \,\vdots\, AB \,\vdots\, \dots \,\vdots\, A^{n-1}B] \begin{bmatrix} u_{n-1} \\ \vdots \\ u_0 \end{bmatrix}$$

$$\Longrightarrow x_\beta = A^n x_\alpha + \sum_{j=0}^{n} A^{n-j-1} B u_j \Rightarrow (A,B) \text{ πλήρως ελέγξιμο}$$

$(\Longrightarrow)$ Έστω (A,B) πλήρως ελέγξιμο, θα δείξουμε ότι $\mathrm{Rank}(\Gamma_c) = n$

Έστω (για αντίφαση) ότι $\mathrm{Rank}(\Gamma_c) < n$

Άρα $\exists\, \underset{\neq 0}{\xi} \in \mathbb{R}^n : \ \xi^T \Gamma_c = 0 \implies \xi^T [B \,\vdots\, AB \,\vdots\, \dots \,\vdots\, A^{n-1}B] = 0$

$$\implies \xi^T A^k B = 0 \quad \forall\, k = 0, 1, \dots, n-1$$

και επομένως και $\forall\, k \in \mathbb{N}_0$

Έστω ότι $\exists\, k \in \mathbb{N}_0$ και $\{u_j\}_{j=0}^{k-1}$ τέτοια ώστε:

$$\xi = \sum_{j=0}^{k-1} A^{k-j-1} B u_j \qquad \text{Τότε}$$

$$\xi^T \xi = \|\xi\|^2 = \sum_{j=0}^{k-1} \xi A^{k-j-1} B u_j = 0 \implies \xi = 0 \qquad \text{Άτοπο.}$$