A REMARK ON IDEALS OF c_0

B. BARNES

Theorem 1. *Let $\mathcal{A}$ be the Banach algebra c_0. Then the ideal $c_{00} \subseteq \mathcal{A}$ is not contained in a maximal ideal.*

Proof. Suppose, by way of contradiction, that $\mathcal{M}$ is a maximal ideal of $\mathcal{A}$ containing c_{00}. Let $y \in \mathcal{A} \backslash \mathcal{M}$. Define $x = |y|^{1/2}$. Note that $x \in \mathcal{A}$ (this is the key non-algebraic fact: $\mathcal{A}$ is square-root closed!)

Now the set

$$\mathcal{N} := \{ax + v : a \in \mathcal{A}, v \in \mathcal{M}\}$$

is an ideal of $\mathcal{A}$ which contains $\mathcal{M}$ and also y (write $y = u|y|$ where $|u| = 1$ and observe that $ux \in c_o$ and $y = (ux)x$). Hence $\mathcal{N} = \mathcal{A}$ and so $x \in \mathcal{N}$.

Therefore there are $a \in \mathcal{A}$ and $v \in \mathcal{M}$ so that $x = ax + v$, in other words $x - ax \in \mathcal{M}$. Since $a \in c_0$, there exists $n \in \mathbb{N}$ so that $|a(k) - 1| \geq 1/2$ for $k > n$. Now define

$$b(k) = \begin{cases} 0, & k \leq n \\ \frac{a(k)}{a(k)-1}, & k > n \end{cases}$$

Thus $|b(k)| \leq 2|a(k)|$ for all k, so $b := \sum b(k)e_k \in c_0$. Now

$$\sum_{k>n} a(k)x(k)e_k = b(ax - x) \in \mathcal{M}.$$

But since $c_{00} \subseteq \mathcal{M}$ we have $\sum_{k \leq n} a(k)x(k)e_k \in \mathcal{M}$ and so $ax \in \mathcal{M}$. It follows that $x = (x - ax) + ax \in \mathcal{M}$ which gives $y \in \mathcal{M}$, a contradiction.

Date: May 7, 2017, file: maxid.

From the lecture notes of A. Katavolos, *Banach spaces of analytic functions and Banach algebras*, Athens, 1992 (in Greek, unpublished).