18ο Μάθημα

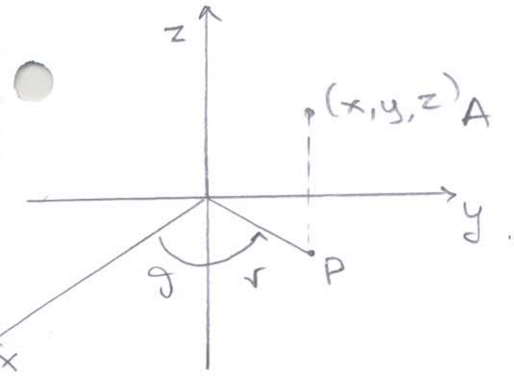
25/05/15

Για τον υπολογισμό τριπλών ολοκληρωμάτων χρησιμοποιούμε συντεταγμένες: **Καρτεσιανές** (x, y, z)

Κυλινδρικές $(x = r\,\text{συν}\theta,\ y = r\,\text{ημ}\theta,\ z)$

$r \in (0, +\infty),\ \theta \in [0, 2\pi)$

$(\vec{T}(r, \theta, z) = (r\,\text{συν}\theta, r\,\text{ημ}\theta, z)$, ορίζουσα $\underline{\underline{r}}$)



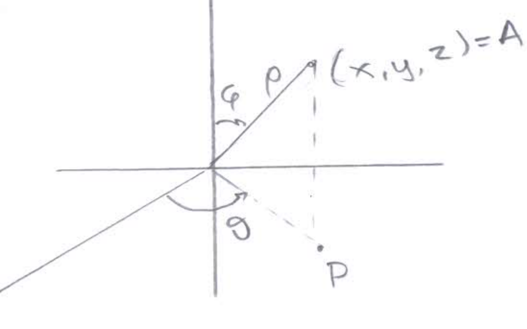
, $r = \sqrt{x^2 + y^2}$, $\theta = \measuredangle(O\vec{x}, O\vec{P})$

Σφαιρικές $(x = \rho\,\text{συν}\theta\,\text{ημ}\varphi,\ y = \rho\,\text{ημ}\theta\,\text{ημ}\varphi,\ z = \rho\,\text{συν}\varphi$

$\rho \in (0, +\infty),\ \theta \in [0, 2\pi),\ \varphi \in (0, \pi)$

$\vec{T}(\rho, \theta, \varphi) = (\rho\,\text{συν}\theta\,\text{ημ}\varphi, \rho\,\text{ημ}\theta\,\text{ημ}\varphi, \rho\,\text{συν}\varphi)$

(ορίζουσα $\rho^2\,\text{ημ}\varphi$)



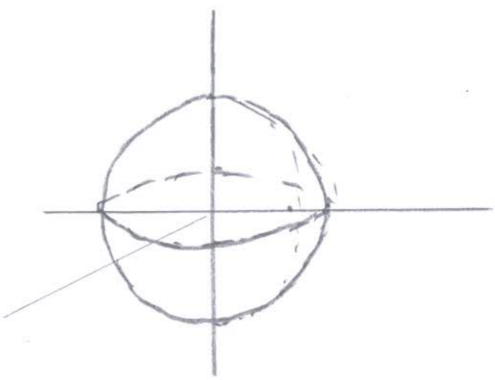
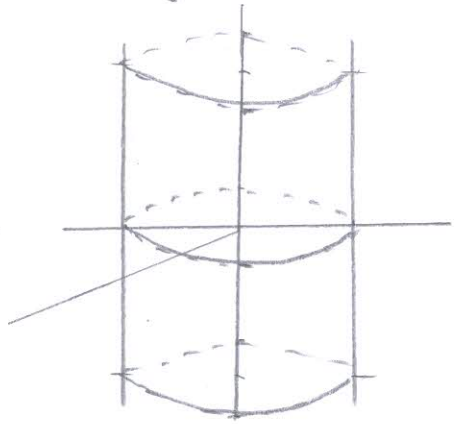
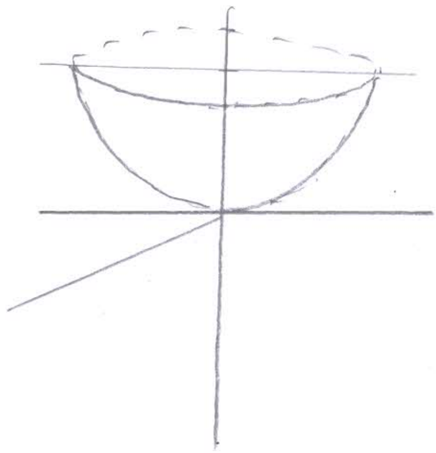
$\rho = \sqrt{x^2 + y^2}$

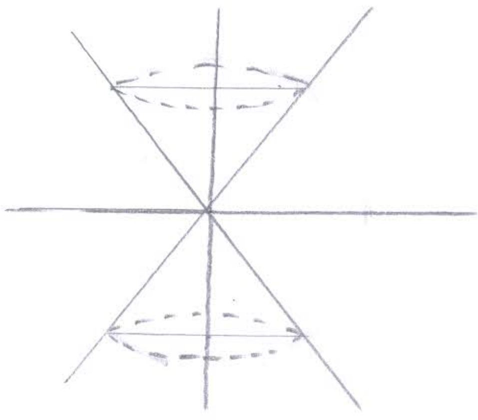
$\theta = \measuredangle(O\vec{x}, \vec{OP})$

$\varphi = \measuredangle(O\vec{z}, \vec{OA})$

Οι επιφάνειες που εμφανίζονται να περιβάλλουν στερεό B
Είναι συνήθως οι εξής:

- $\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2=a^2\}$ $(a>0)$ Σφαίρα
- $\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2=b^2,\ z\in\mathbb{R}\}$ $(b>0)$ Κύλινδρος
- $\{(x,y,z)\in\mathbb{R}^3 : z=c(x^2+y^2)\}$ $(c\neq 0)$ Παραβολοειδές
- $\{(x,y,z)\in\mathbb{R}^3 : z^2=c(x^2+y^2)\}$ $(c>0)$ Κώνος (Διπλός)



<u>Και</u> μεταφορές ή και στροφές αυτών

Ασκήσεις

1. $B_a = \{(x,y,z) : (x-a_1)^2 + (y-a_2)^2 + (z-a_3)^2 \leq a^2\}$ $(a>0)$

Σφαίρα κέντρου : (a_1, a_2, a_3) και ακτίνας a

Να υπολογιστεί ο όγκος της σφαίρας αυτής $(V(B_a))$

ΛΥΣΗ

$B((0,0,0), a)$

$V(B_a) = V(B(\vec{0}, a)) = a^3 V(B(\vec{0}, 1))$ $\left(V_a(\lambda D) = \lambda^d V_a(B)_{\lambda \geq 0} \right)$

Αρκεί να υπολογίσουμε τον $V(B_1)$, $B_1 = \{(x,y,z) : x^2+y^2+z^2 \leq 1\}$

i) Καρτεσιανές

$B_1 = \{(x,y,z) : -1 \leq x \leq +1, -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}, -\sqrt{1-x^2-y^2} \leq z \leq \sqrt{1-x^2-y^2}\}$

$$V(B_1) = \int_{-1}^{+1} \left[\int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} \left(\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} 1\, dz \right) dy \right] dx =$$

$$= \int_{-1}^{+1} \left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\sqrt{1-x^2-y^2}\, dy \right] dx =$$

(Απ. Λογισμός ΙΙ) $= \int \sqrt{\beta^2 - y^2}\, dy = \frac{1}{2}\left(\beta^2 \text{τοξημ} \frac{y}{\beta} + y\sqrt{\beta^2 - y^2} \right)$

$V(B_1) = \frac{4\pi}{3}$ //

ii) Κυλινδρικές

$x^2+y^2+z^2 = 1$ / $x = r\,\text{συν}\theta$, $y = r\,\text{ημ}\theta$, $z = z$

$r^2 + z^2 = 1 \Rightarrow z^2 = 1 - r^2$, $r \in [0,1]$

$$V(B_1) = \int_0^{2\pi} \int_0^1 \left[\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r\, dz \right] dr\, d\theta \xlongequal[\text{(Ασ/Μαθ17)}]{} \frac{4\pi}{3}$$

iii) Σφαιρικές

$x^2+y^2+z^2=1$, $x=\rho\,\sigma\upsilon\nu\theta\,\eta\mu\varphi$, $y=\rho\,\eta\mu\theta\cdot\eta\mu\varphi$, $z=\rho\,\sigma\upsilon\nu\varphi$

$\underline{\rho=1}$

$$V(B_1)=\int_0^{2\pi}\int_0^{\pi}\left(\int_0^1 \rho^2\eta\mu\varphi\, d\rho\right)d\varphi\, d\theta \underset{(\text{Ασκ}|_{17})}{=\!=\!=} \frac{4\pi}{3}.$$

iv) Ως όγκος εκ περιστροφής (τέλος Μάθημα 18)

v) Μέθοδος Αρχιμήδη (πχ. Νεγρεπόντης κ.ά, Εύδοξος)

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2) $V(\mathcal{E})$, $\mathcal{E}=\left\{(x,y,z):\left(\frac{x}{\alpha}\right)^2+\left(\frac{y}{\beta}\right)^2+\left(\frac{z}{\gamma}\right)^2\leq 1\right\}$, $(\alpha,\beta,\gamma>0)$

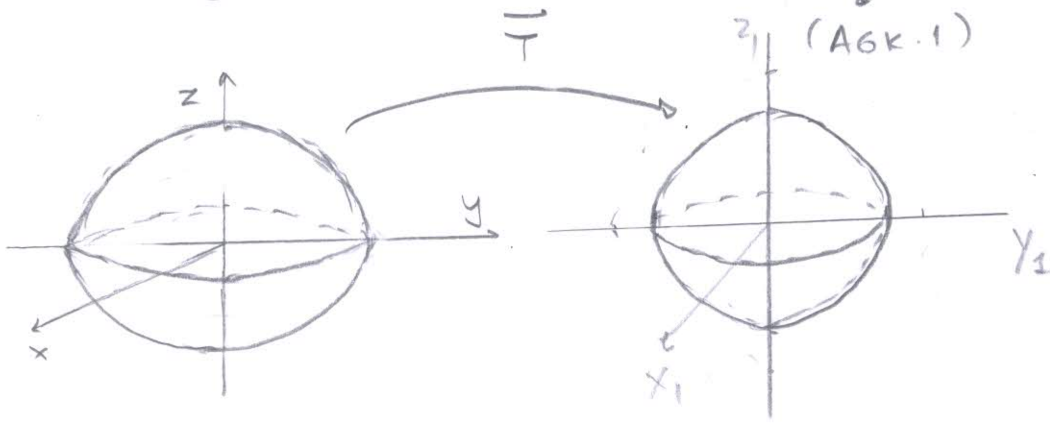
Θέτουμε, $x_1=\frac{x}{\alpha}$, $y_1=\frac{y}{\beta}$, $z_1=\frac{z}{\gamma}$ / $\det J_{\vec{T}}(x_1,y_1,z_1)=\alpha\beta\gamma$

$\vec{T}(x_1,y_1,z_1)=(\alpha x_1,\beta y_1,\gamma z_1)$

$$\mathcal{E}^*=\left\{(x_1,y_1,z_1):x_1^2+y_1^2+z_1^2\leq 1\right\}.$$

$$V(\mathcal{E})=\iiint_{\mathcal{E}^*}\alpha\beta\gamma\, dx_1dy_1dz_1=\alpha\beta\gamma\cdot\frac{4\pi}{3} \quad //$$

(Ασκ. 1)



3) $I = \iiint_D z^2 \, dx\,dy\,dz$, $D = \{(x,y,z) : x^2+y^2+z^2 \le 1, z \ge 0\}$

(Απειροστικός Λογ. ΙΙ $t^2 = \frac{y^2}{1-x^2}$)

Καρτεσιανές

$$\int (1-t^2)^{3/2} dt = t(1-t^2)^{3/2} + \frac{3}{8}\left[\text{τοξημ}\,t - \frac{1}{4}\,\text{ημ}(4\,\text{τοξημ}\,t)\right]$$

$$I = \frac{2\pi}{15}$$

Κυλινδρικές: $x = r\,\text{συν}\theta$, $y = r\,\text{ημ}\theta$, $z = z$ / $r^2+z^2=1$ Σφαίρα

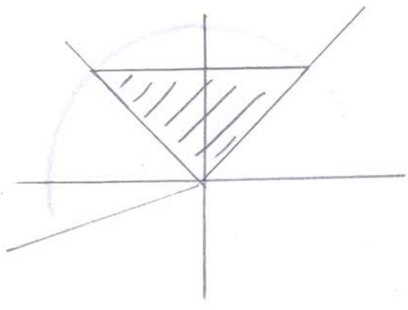
$$I = \int_0^{2\pi}\int_0^1\left(\int_0^{\sqrt{1-r^2}} r\cdot z^2 dz\right) dr\, d\theta = \frac{2\pi}{15}$$

Σφαιρικές $z \ge 0$, $\rho\,\text{σων}\varphi \ge 0$ κ' $\varphi \in [0,\pi] \Rightarrow \varphi \in [0, \frac{\pi}{2}]$

$$I = \int_0^{2\pi}\int_0^{\pi/2}\left[\int_0^1 (\rho^2\,\text{ημ}\varphi)(\rho\,\text{συν}\varphi)^2 d\rho\right] d\varphi\, d\theta = \frac{2\pi}{15}$$

4) $B = \{(x,y,z) : z \ge \sqrt{x^2+y^2}, x^2+y^2+z^2 \le 1\}$

$$I = \iiint_B (x^2+y^2)\, dV, \quad J = \iiint_B \sqrt{z}\, dV$$



Λυση

Κυλινδρικές

$x = r\,\text{συν}\theta$

$y = r\,\text{ημ}\theta$

$z = z$

Κώνος $z = \sqrt{x^2+y^2}$, $z = r$

Σφαίρα $z \ge 0$, $z = \sqrt{1-r^2}$

$(0 \le)\, r \le z \le \sqrt{1-r^2}$ $\left(0 \le z \le \sqrt{1-z^2},\ z^2 \le 1-z^2,\ 2z^2 \le 1, 0 \le z \le \frac{1}{\sqrt{2}}\right)$

$0 \le r \le \frac{1}{\sqrt{2}}$

$B^* = \{ (r,\vartheta,z) : 0 \le \vartheta \le 2\pi,\ 0 \le r \le \frac{1}{\sqrt{2}},\ r \le z \le \sqrt{1-r^2} \}$

$I = \int_0^{2\pi} \int_0^{\frac{1}{\sqrt{2}}} \left(\int_r^{\sqrt{1-r^2}} r \cdot r^2 \, dz \right) dr\, d\vartheta$

$J = \int_0^{2\pi} \int_0^{1/\sqrt{2}} \left(\int_r^{\sqrt{1-r^2}} \sqrt{z} \cdot r \, dz \right) dr\, d\vartheta$ /

Σφαιρικές

$x = \rho\sigma\upsilon\nu\vartheta\,\eta\mu\varphi$	Σφαίρα $\rho = 1$
$y = \rho\eta\mu\vartheta\,\eta\mu\varphi$	Κώνος (Μαθ 17) $\varphi = \frac{\pi}{4}$
$z = \rho\sigma\upsilon\nu\varphi$	

$B^* = \{ (\rho,\vartheta,\varphi) : 0 \le \vartheta \le 2\pi,\ 0 \le \varphi \le \pi/4,\ 0 \le \rho \le 1 \}$.

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5) $V(B)$

$B = \{ (x,y,z) : x^2+y^2+z^2 \le 5,\ \sqrt{3}\, z \ge \sqrt{x^2+y^2} \}$.

Λύση

Κυλινδρικές.

Σφαίρα, $(z \ge 0)$, $z = \sqrt{5-r^2}$

Κών, $z = \frac{1}{\sqrt{3}} r$

$B^* = \{ (r,\vartheta,z) : 0 \le \vartheta \le 2\pi,\ 0 \le r \le \frac{\sqrt{15}}{2},\ \frac{r}{\sqrt{3}} \le z \le \sqrt{5-r^2} \}$

(όπως Ασ 4 $\alpha \le z \le \frac{\sqrt{15}}{2}$)

$V(B) = \int_0^{2\pi} \int_0^{\frac{\sqrt{15}}{2}} \left(\int_{\frac{r}{\sqrt{3}}}^{\sqrt{5-r^2}} r \, dz \right) dr\, d\vartheta = \frac{5\sqrt{5}}{3}\pi.$

Σφαιρικές

Σφαίρα, $\rho=\sqrt{5}$

Κώνος, $\varphi=\frac{\pi}{3}$

$$V(B)=\int_0^{2\pi}\int_0^{\pi/3}\left(\int_0^{\sqrt5}\rho^2 \eta\mu\varphi \, d\rho\right) d\varphi \, d\theta=\frac{5\sqrt5}{3}\pi$$

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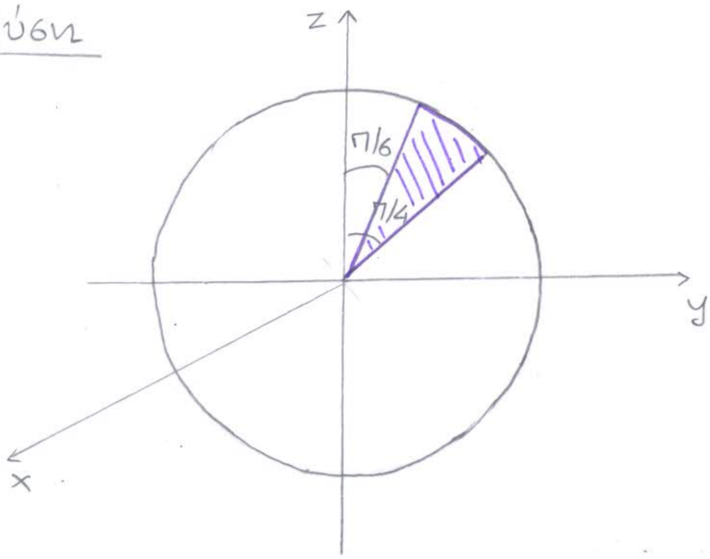
6) $$I=\iiint_K x^2\sqrt{z}\, dV$$

$$K=\{(x,y,z): x^2+y^2+z^2\le a^2,\ \sqrt{x^2+y^2}\le z\le\sqrt{3(x^2+y^2)}\}\quad (a>0)$$

Λύση



Κυλινδρικές

2 ολοκληρώματα.

Σφαιρικές

Σφαίρα $\rho=a$

Κώνος $z=\sqrt{x^2+y^2}$, $\varphi=\frac{\pi}{4}$

Κώνος $z=\sqrt{3(x^2+y^2)}$, $\varphi=\frac{\pi}{6}$

$$I=\int_0^{2\pi}\int_{\pi/6}^{\pi/4}\left[\int_0^a(\rho^2\eta\mu\varphi)(\rho\sigma\upsilon\nu\theta\,\eta\mu\varphi)^2\sqrt{\rho\sigma\upsilon\nu\varphi}\, d\rho\right]d\varphi\, d\theta$$

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7) $$I=\iiint_B (1-4x^2-9y^2-z^2)\, dV$$

$$B=\{(x,y,z): 4x^2+9y^2+z^2\le 1\}.$$

$x_1=2x$

$y_1=3y$

$z_1=z$

$\vec{T}(x_1,y_1,z_1)=\left(\frac{x_1}{2},\frac{y_1}{3},z_1\right)$ Ορίζουσα $\frac{1}{6}$.

$$B^* = \{ (x_1, y_1, z_1) : x_1^2 + y_1^2 + z_1^2 \leq 1 \}$$

$$I = \frac{1}{6} \iiint_{B^*} (1 - x_1^2 - y_1^2 - z_1^2)\, dV$$

$$I = \frac{1}{6} \int_0^{2\pi} \int_0^{\pi} \left[\int_0^1 (\rho^2 \eta\mu\varphi)(1 - \rho^2)\, d\rho \right] d\varphi\, d\theta = \frac{4\pi}{45}.$$

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8) $$I = \iiint_K e^{(x^2+y^2+z^2)^{3/2}}\, dV$$

$$K = \{ (x,y,z) : x^2 + y^2 + z^2 \leq a^2 \} \quad (a > 0)$$

Λύση

Κυλινδρικές. $\left(\int e^{(r^2+t^2)^{3/2}} dt \right)$ (Δεν υπολογίζεται !)

Σφαιρικές. $$I = \int_0^{2\pi} \int_0^{\pi} \int_0^a e^{\rho^3} (\rho^2 \eta\mu\varphi)\, d\rho\, d\varphi\, d\theta =$$

$$= \frac{4\pi}{3}(e - 1)$$

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9) $$I = \lim_{\varepsilon \to 0^+} \iiint_{\varepsilon^2 \leq x^2+y^2+z^2 \leq (1-\varepsilon)^2} \frac{dx\,dy\,dz}{\sqrt{(x^2+y^2+z^2)(1-x^2-y^2-z^2)}}$$

$(0 < \varepsilon < 1)$

Λυση $$I_\varepsilon = \int_0^{2\pi} \int_0^{\pi} \int_\varepsilon^{1-\varepsilon} \frac{\rho^2 \eta\mu\varphi}{\sqrt{\rho^2(1-\rho^2)}}\, d\rho\, d\varphi\, d\theta =$$

$$= 4\pi \int_\varepsilon^{1-\varepsilon} \frac{\rho}{\sqrt{1-\rho^2}}\, d\rho = 4\pi \left[-\sqrt{1-\rho^2} \Big|_\varepsilon^{1-\varepsilon} \right] = 4\pi \left[\sqrt{1-\varepsilon^2} - \sqrt{1-(1-\varepsilon)^2} \right]$$

$$= 4\pi$$

10) Για ποια $\lambda \in \mathbb{R}$ το $I = \lim\limits_{\varepsilon \to 0^+} \iiint\limits_{\varepsilon^2 \le x^2+y^2+z^2 < 1} \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{\lambda}} \in \mathbb{R}$;

Λύση

$$I_\varepsilon = \int_0^{2\pi}\int_0^{\pi}\left(\int_\varepsilon^1 \frac{\rho^2 \eta\mu\varphi}{\rho^{2\lambda}}\,d\rho\right)d\varphi\,d\theta = 4\pi\int_\varepsilon^1 \rho^{2-2\lambda}\,d\rho =$$

$$= \begin{cases} 4\pi \left.\dfrac{\rho^{3-2\lambda}}{3-2\lambda}\right|_\varepsilon^1, & \lambda \ne \frac{3}{2} \\ 4\pi \cdot \ln\rho \Big|_\varepsilon^1, & \lambda = \frac{3}{2} \end{cases}$$

$$= \begin{cases} 4\pi \cdot \dfrac{1}{3-2\lambda}\left(1-\varepsilon^{3-2\lambda}\right), & \lambda \ne \frac{3}{2} \\ -4\pi \cdot \ln\varepsilon, & \lambda = \frac{3}{2} \end{cases}$$

$0 < \varepsilon < 1$

Υπάρχει $\lim\limits_{\varepsilon \to 0^+} I_\varepsilon \iff 3-2\lambda > 0, \quad \lambda < \frac{3}{2}$

$$I = \frac{4\pi}{3-2\lambda}$$

11) $\iiint\limits_K z^2 \eta\mu(xz) \cdot e^{z^2} dV, \quad K = \{(x,y,z) : 0 \le x \le 1, \ x \le y \le z - x^2, \ -3 \le z \le +3\}$

Λύση

$f(x,y,z) = z^2 \eta\mu(xz)\, e^{z^2}, \quad x \ge 0, \quad -3 \le z \le +3$

$f(x,y,-z) = -f(x,y,z) \qquad z \in [-3,3] \qquad \underline{\underline{I = 0}}$

12)

$$I_1 = \int_{-2}^{+2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} (x+z^2)\, dz\, dx\, dy.$$

$$I_2 = \int_0^1 \int_x^{2-x^2} \int_{-3}^{+3} z^2 \eta\mu(xz)\, dz\, dy\, dx$$

Ένα από τα δύο ολοκληρωματα είναι 0. Ποιο είναι;

Λυση.

$I_2 = 0$ //

Συμπληρωματικές Ασκήσεις

$\mathbb{R}^2$, $K \subseteq \mathbb{R}^2$ και έχει εμβαδόν (πχ. απλό)

$(x,y) \in K$ πυκνότητα μάζας $\delta(x,y) > 0$

$m = \iint_K \delta\, dx dy$ / ΚΒ $(\bar{x},\bar{y}) = \frac{1}{m}\left(\iint_K x\delta\, dxdy, \iint_K y\delta\, dxdy\right)$

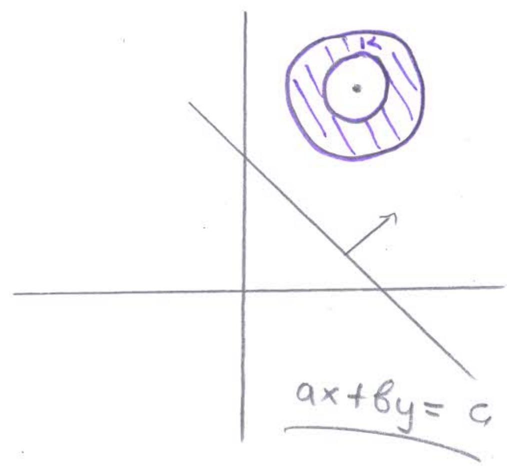
$\mathbb{R}^3$ Ανάλογα. —

Ασκηση

Έστω $K \subseteq \mathbb{R}^2$ και $ax+by \geq c$, $(x,y) \in K$.

$((a,b) \neq (0,0))$

ΝΔΟ, $a\bar{x} + b\bar{y} \geq c$.

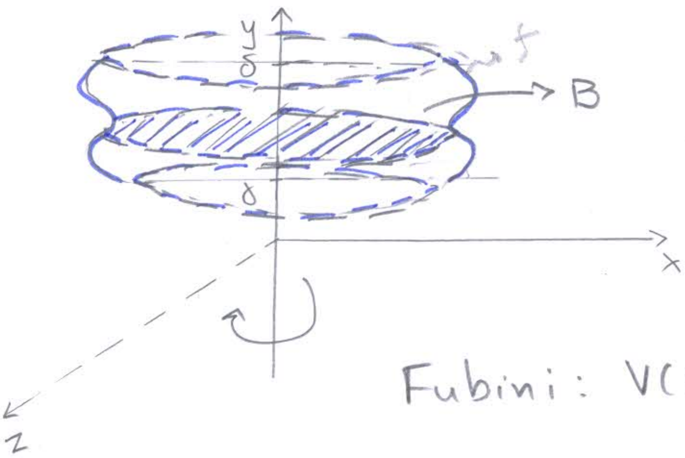


Λύση

$$a\bar{x} + b\bar{y} = a\,\frac{\iint_K x\delta\,dx\,dy}{m} + b\,\frac{\iint_K y\delta\,dx\,dy}{m} =$$

$$= \frac{\iint_K \overbrace{(ax+by)}^{\geq c}\delta(x,y)\,dx\,dy}{\iint_K \delta(x,y)\,dx\,dy} \geq \frac{\iint_K c\,\delta(x,y)\,dx\,dy}{\iint_K \delta(x,y)\,dx\,dy} = c.$$

Στερεό εκ περιστροφής

1) 

$x = f(y)$, $y \in [\gamma, \delta]$

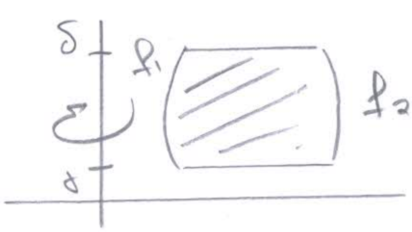
$f \geq 0$

Fubini: $V(B) = \int_\gamma^\delta E(y)\,dy$

$E(y)$: Εμβαδόν κύκλου

$(0,y,0)$, Ακτίνα $f(y)$ / $E(y) = \pi f^2(y)$

$$V(B) = \pi \int_\gamma^\delta f^2(y)\,dy$$

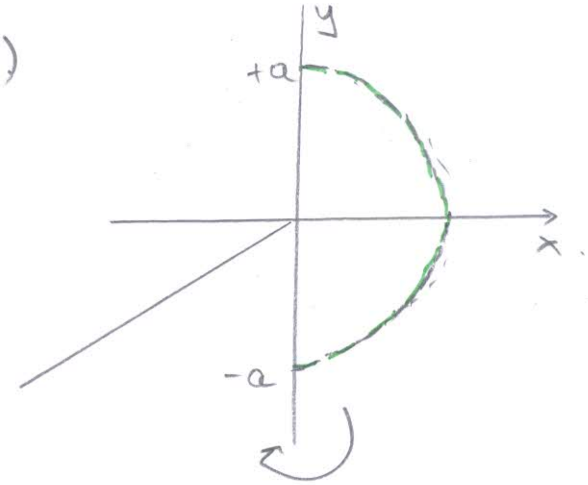
2) 

$D = \{(x,y) : \gamma \leq y \leq \delta,\ f_1(y) \leq x \leq f_2(y)\}$, $f_2, f_1 > 0$

$$V(B) = \pi \int_\gamma^\delta \left(f_2^2(y) - f_1^2(y)\right) dy$$

Εφαρμογές

1)

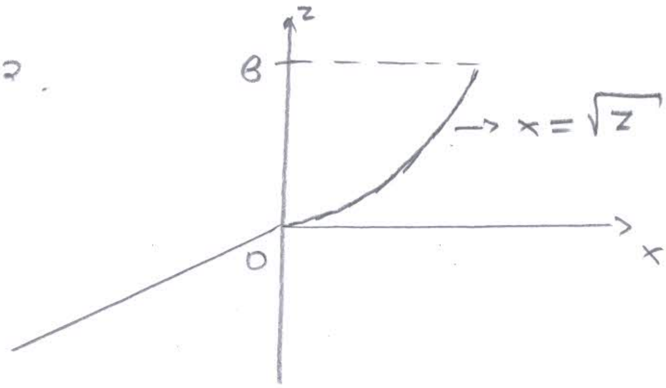


$x=\sqrt{a^2-y^2}=f(y)$, $y\in[-a,+a]$

Όγκος της $B((0,0,0,a))$

$$V(B)=\pi\int_{-a}^{+a}(a^2-y^2)\,dy=\frac{4\pi}{3}a^3$$

2) $z=x^2$.

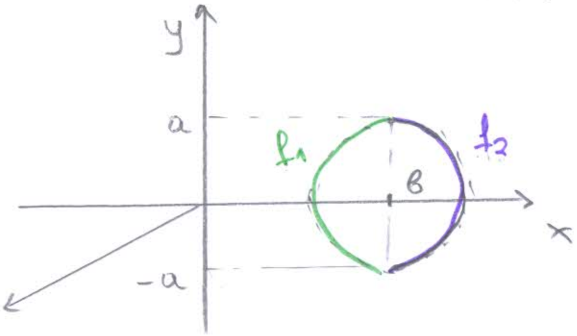


$z\in[0,\beta]$ $(\beta>0)$

$\rightarrow x=\sqrt{z}$

$$V(B)=\pi\int_0^{\beta}z\,dz=\frac{\pi\beta^2}{2}$$

3) Όγκος Ντόνατς / Λουκουμά / Torus



$(0<a<\beta)$

$f_2(y)=\beta+\sqrt{a^2-y^2}$

$f_1(y)=\beta-\sqrt{a^2-y^2}$ $y\in[-a,a]$

$$V(\Lambda)=\pi\int_{-a}^{+a}\left(f^2(y)-f_1^2(y)\right)dy=$$

$$=\pi\int_{-a}^{+a}\left(\left(\beta+\sqrt{a^2-y^2}\right)^2-\left(\beta-\sqrt{a^2-y^2}\right)^2\right)dy=$$

$$=\pi\int_{-a}^{+a}4\sqrt{a^2-y^2}\,\beta\,dy=2\beta\pi^2a^2$$