14.05.10 28° μάθημα

Λύσεις Φυλλάδιο 5°

② $f(x,y) = \begin{cases} c(y^2-x^2)e^{-y}, & y>0,\ -y<x<y \\ 0, & \text{αλλιώς} \end{cases}$

1. $c=;$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1 \Leftrightarrow \int_0^{\infty}\int_{-y}^{y} c(y^2-x^2)e^{-y}\,dx\,dy = 1 \Leftrightarrow \int_0^{+\infty} ce^{-y}\left(\int_{-y}^{y}(y^2-x^2)\,dx\right)dy = 1$$

$$\Leftrightarrow \int_0^{+\infty} ce^{-y}\left[y^2x-\frac{x^3}{3}\right]_{-y}^{y} dy = 1 \Leftrightarrow \int_0^{+\infty} ce^{-y}\,\frac{4}{3}y^3\,dy = 1 \Leftrightarrow \ldots \Leftrightarrow c = 1/8$$

2. $f_X, f_Y\ ;$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{|x|}^{+\infty} c(y^2-x^2)e^{-y}\,dy \overset{(|x|<y)}{=} c\int_{|x|}^{+\infty} y^2e^{-y}\,dy - cx^2\int_{|x|}^{+\infty} e^{-y}\,dy =$$

$$= \frac{1}{4}(1+|x|)e^{-|x|}$$

$$f_Y(y) = \int_{-y}^{y} c(y^2-x^2)e^{-y}\,dx = cy^2e^{-y}\int_{-y}^{y} 1\,dx - ce^{-y}\int_{-y}^{y} x^2\,dx = \frac{1}{6}y^3e^{-y}$$

3. $E[X]=;$

$$E[X] = \int_0^{+\infty} x f_X(x)\,dx = \int_{-\infty}^{+\infty}\frac{1}{4}x(1+|x|)e^{-|x|}\,dx = \int_{-\infty}^{+\infty} \quad + \int_0^{\infty}$$

4. $f_{X|Y}(x|y) = \dfrac{f(x,y)}{f_Y(y)}$

$f_{Y|X}(y|x) = \dfrac{f(x,y)}{f_X(x)}$

③ $f(x,y) = \begin{cases} \frac{6}{7}\left(x^2 + \frac{xy}{2}\right), & 0<x<1,\ 0<y<2 \\ 0, & \text{αλλιώς} \end{cases}$

1. επη; $\rightsquigarrow \int_0^1 \int_0^2 \dots = 1$.

2. $f_X(x) = \int_0^2 f(x,y)\,dy = \frac{6}{7}(2x^2 + |x|)$

$f_Y(y) = \int_0^1 f(x,y)\,dx = \frac{6}{7}\left(y^4 + \frac{1}{3}\right)$

5. $E[X] = \int_0^1 x\, f_X(x)\, dx = 1.$

6. $E[Y] = \int_0^2 y\, f_Y(y)\, dy = {}^{11}/_{28}$

7. $f_{Y|X}(y|x) = \dots$

3. $P[X>Y] = \int_0^2 \int_y^1 f(x,y)\,dx\,dy$

4. $P[Y>1/2,\ X<1/2] = \int_{1/2}^2 \int_0^{1/2} f(x,y)\,dx\,dy$

$P[Y>1/2 \mid X<1/2] = \frac{P[Y>1/2,\ X<1/2]}{P[X<1/2]} = \frac{\int_{1/2}^2 \int_0^{1/2} f(x,y)\,dx\,dy}{\int_0^{1/2} f_X(x)\,dx}$

⑤ $f(x,y) = \begin{cases} x+y, & 0<x<1,\ 0<y<1 \\ 0, & \text{διαφορετικά} \end{cases}$

1. ανεξάρτητες; Από θεώρημα: αν $f(x,y) = g(x)\cdot h(y)$ τότε X, Y ανεξάρτητες. Παρατηρώ ότι η $f(x,y)$ δεν μπορεί να γραφτεί ως γινόμενο μιας συνάρτησης του x με μια συνάρτηση του y, οπότε οι X,Y δεν είναι ανεξάρτητες.

2. $f_X(x) = \int_0^1 (x+y)\,dy = xy + \frac{y^2}{2}\Big]_0^1 = x + \frac{1}{2}$

$x+y<1 \Rightarrow x<1-y$

3. $P[X+Y<1] = \int_0^1 \int_0^{1-y} f(x,y)\,dx\,dy = \dots = 1/3$

4. $f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$

⑥ $f(x,y) = \begin{cases} 12xy(1-x) & , 0<x<1, 0<y<1 \\ 0, & \text{αλλιώς} \end{cases}$

1. ανεξάρτητες; Παρατηρώ $\underbrace{12x(1-x)}_{g(x)} \cdot \underbrace{y}_{h(y)}$ άρα είναι ανεξάρτητες.

2. $E[X] = \int_0^1 x f_X(x)\, dx = \ldots = 1/2$

3. $E[Y] = \int_0^1 y f_Y(y)\, dy = \ldots = 2/3$

4. $Var[X] = E[X^2] - (E[X])^2 = \int_0^1 x^2 f_X(x) - (E[X])^2 = \ldots = 1/20$

5. $Var[Y] = \ldots = 1/18$

⑧ $f(x,y) = \begin{cases} x \cdot e^{-x(y+1)} & , x>0, y>0 \\ 0 & , \text{αλλιώς} \end{cases}$

1. $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$ και $f_{Y|X} = \frac{f(x,y)}{f_X(x)}$ ①

$f_Y(y) = \int_0^{+\infty} x e^{-x(y+1)}\, dx = \int_0^{+\infty} x \left(\frac{e^{-x(y+1)}}{-(y+1)} \right)' dx = \left[-\frac{x e^{-x(y+1)}}{y+1} \right]_0^{+\infty} + \int_0^{+\infty} \frac{e^{-x(y+1)}}{y+1}\, dx$

$x e^{-x(y+1)} \overset{\infty \cdot 0}{=} \frac{e^{-x(y+1)}}{1/x} \overset{\text{de l'Hospital}}{=} \frac{e^{-x(y+1)}(y+1)}{1/x^2}$ δεν βοηθάει !

$\downarrow$

$\frac{x}{e^{x(y+1)}} \overset{\frac{\infty}{\infty}}{\underset{\text{de l'Hospital}}{=}} 0$ ⊛

$\overset{⊛}{=} 0 + 0 + \left[\frac{e^{-x(y+1)}}{-(y+1)^2} \right]_0^{\infty}$ άρα $f_Y(y) = \frac{1}{(y+1)^2}$

$f_X(x) = \int_0^{+\infty} x e^{-x(y+1)}\, dx = x e^{-x} \int_0^{+\infty} e^{-y}\, dy$ άρα $f_X(x) = e^{-x}$

Αντικαθιστώ τα $f_X(x), f_Y(y)$ στους τύπους ①

2. Να βρεθεί η κατανομή της $Z = X \cdot Y$

$F_Z(z) = P[Z \leq z] = P[XY \leq z] = \int_0^{+\infty} \int_0^{z/y} x e^{-x(y+1)}\, dx\, dy = \ldots = 1 - e^{-z} \Rightarrow f_Z(z) = e^{-z}$

⑩ X,Y ανεξάρτητες $\sim Bin(n,p)$

$$P[X=i] = P[Y\geq i] = \binom{n}{i} p^i (1-p)^{n-i}$$

$$1.\ P[X=i \mid X+Y=m] = \frac{P[X=i,\ X+Y=m]}{P[X+Y=m]} = \frac{P[X=i,\ Y=m-i]}{P[X+Y=m]} =$$

$$\overset{\text{ανεξ}}{=} \frac{P[X=i]\,P[Y=m-i]}{\underbrace{P[X+Y=m]}_{Bin(2n,p)}} = \frac{\binom{n}{i} p^i (1-p)^{n-i} \binom{n}{m-i} p^{m-i} (1-p)^{n-m+i}}{\binom{2n}{m} p^m (1-p)^{2n-m}} =$$

$$= \frac{\binom{n}{i}\binom{n}{m-i}}{\binom{2n}{m}}$$

2n φορές

X = # κ στις πρώτες n ζαριές ~ Bin

Y = # κ στις -ll- επόμενες ζαριές ~ Bin

⑨ Υπόδειξη ⊛ $P[\underbrace{X+Y}_{NegBin}=n]$

⑦ 2 ζάρια ; X = max, Y = min, $P_{Y|X}(y|x) = ;$

$$\xrightarrow{1/36} (4,4) \qquad \xrightarrow{2/36} (2,4) \qquad \xrightarrow{0} (4,2)$$

$$P[Y=j \mid X=i] = P[\min=j \mid \max=i] \overset{j\leq i}{=} \begin{cases} j=i \Rightarrow P[Y=j \mid X=j] = \dfrac{P[Y=j,X=j]}{P[X=i]} = \dfrac{1/36}{P[X=i]} \\ j<i \Rightarrow \dots = \dfrac{2/36}{P[X=i]} \end{cases}$$

Όμως $\sum_{j=1}^{i} P(Y=j \mid X=i) = 1 \iff$

$$\iff \underbrace{\frac{1/36}{P[X=i]}}_{j=i} + \sum_{j=1}^{i-1} \underbrace{\frac{2/36}{P[X=i]}}_{j<i} = 1 \iff \frac{1/36}{P[X=i]} + \frac{2/36\,(i-1)}{P[X=i]} = 1 \iff$$

$$\iff \frac{2i-1}{36\,P[X=i]} = 1 \iff P[X=i] = \frac{2i-1}{36}$$

άρα

$$P[Y=j \mid X=i] = \begin{cases} \dfrac{1}{2i-1}, & j=i \\ \dfrac{2}{2i-1}, & j<i \end{cases}$$