ΛΥΣΕΙΣ ΑΣΚΗΣΕΩΝ 4ου ΦΥΛΛΑΔΙΟΥ

1 $\mu_x = \dfrac{x}{40(40+x)}, \quad x > 0$

α) $s(x)$ β) P(ο 40άρης να πεθάνει στο $[50,60)$) γ) $E[T(40)]$ = μέση υπολειπόμενη ζωή του 40άρη

Λύση

α) $s(x) = \exp\left(-\int_0^x \mu_t\,dt\right) = \exp\left(-\int_0^x \dfrac{t}{40(40+t)}\,dt\right) = \exp\left(-\dfrac{1}{40}\int_0^x \dfrac{t}{40+t}\,dt\right)$

και $\displaystyle\int_0^x \frac{t}{40+t}\,dt \overset{u=40+t}{=} \int_{40}^{40+x} \left(\frac{u-40}{u}\right) du = \int_{40}^{40+x}\left(1-\frac{40}{u}\right) du =$

$= \Big[u - 40\log u\Big]_{40}^{40+x} = 40+x-40\log(40+x) - 40 + 40\log 40$

$= x - 40\log\dfrac{40+x}{40}$

άρα $s(x) = \exp\left(-\dfrac{1}{40}\left(x - 40\log\dfrac{40+x}{40}\right)\right) = \exp\left(-\dfrac{x}{40} + \log\left(1+\dfrac{x}{40}\right)\right)$

$\Leftrightarrow \quad s(x) = \left(1+\dfrac{x}{40}\right)\cdot\exp\left(-\dfrac{x}{40}\right)$

β) $P(50 \le X < 60 \mid X > 40) = \dfrac{s(50)-s(60)}{s(40)} = \dfrac{\left(1+\frac{5}{4}\right)e^{-5/4} - \left(1+\frac{6}{4}\right)e^{-6/4}}{2e^{-1}}$

$= \dfrac{\frac{9}{4}e^{-\frac{1}{4}} - \frac{10}{4}e^{-\frac{1}{2}}}{2} = \dfrac{9e^{-\frac{1}{4}} - 10e^{-\frac{1}{2}}}{8}$

γ) $E[T(40)] = \displaystyle\int_0^\infty {}_uP_{40}\,du$ με ${}_uP_{40} = \dfrac{s(40+u)}{s(40)} = \dfrac{\left(2+\frac{u}{40}\right)\exp\left(-1-\frac{u}{40}\right)}{2e^{-1}}$

$${}_uP_{40} = \left(1+\frac{u}{80}\right)\cdot e^{-\frac{u}{40}}$$

άρα $$E[T(40)] = \int_0^\infty \left(1+\frac{u}{80}\right) e^{-\frac{u}{40}}\,du = \int_0^\infty e^{-\frac{u}{40}}\,du + \int_0^\infty \frac{u}{80} e^{-u/40}\,du$$

$$= 40\underbrace{\int_0^\infty \frac{1}{40} e^{-\frac{u}{40}}\,du}_{Exp(40)} + \frac{1}{80}\underbrace{\int_0^\infty \frac{(1/40)^2}{\Gamma(2)} u^{2-1}\cdot e^{-\frac{1}{40}u}\,\frac{\Gamma(2)}{(1/40)^2}\,du}_{Gamma(2,\frac{1}{40})}$$

$$= 40 + \frac{1}{80}\frac{\Gamma(2)}{(1/40)^2} = 40 + \frac{40^2}{80} = 60 \qquad (\Gamma(2)=1)$$

(2) $\mu_x = \dfrac{b}{a-x}$, $0\le x< a$, $a,b>0$ και T_x υπόλοιπη ζωή του x, $0<x<a$

(α) $s(x)$, (β) $E(T_x)$ (γ) $Var(T_x)$ (δ) a,β ώστε $2Var(T_x) = (ET_x)^2$ για $x\in(0,a)$

Λύση

(α) $s(x) = \exp\left(-\int_0^x \mu_t\,dt\right) = \exp\left(-\int_0^x \frac{b}{a-t}\,dt\right)$

και $-\int_0^x \frac{b}{a-t}\,dt \overset{u=a-t}{=} b\int_a^{a-x}\frac{1}{u}\,du = b\log u\Big|_a^{a-x} = \log\left(\frac{a-x}{a}\right)^b = \log\left(1-\frac{x}{a}\right)^b$

άρα $s(x) = \left(1-\frac{x}{a}\right)^b$, $0\le x\le a$, $a,b>0$

(β) $ET_x = \int_0^\infty {}_tP_x\,dt$. όμως ${}_tP_x = \dfrac{s(x+t)}{s(x)} = \dfrac{\left(1-\frac{x+t}{a}\right)^b}{\left(1-\frac{x}{a}\right)^b} = \left(1-\frac{t}{a-x}\right)^b$

$0\le t\le a-x$

άρα. $$ET_x = \int_0^{a-x} \left(1-\frac{t}{a-x}\right)^b dt \overset{u=\frac{t}{a-x}}{=}$$

$$= \int_0^1 (1-u)^b \cdot (a-x)\, du = -(a-x) \cdot \frac{(1-u)^{b+1}}{b+1}\Big|_0^1 \Rightarrow$$

$$\boxed{ET_x = \frac{a-x}{b+1}}$$

β) $$ET_x^2 = \int_0^{a-x} 2t \cdot {}_tp_x\, dt = 2\int_0^{a-x} t\left(1-\frac{t}{a-x}\right)^b dt \overset{u=\frac{t}{a-x}}{=}$$

$$= 2(a-x)^2 \int_0^1 u(1-u)^b du = 2(a-x)^2\left(\frac{1}{b+1}-\frac{1}{b+2}\right)$$

άρα $$Var(T_x) = ET_x^2 - (ET_x)^2 = 2(a-x)^2 \cdot \frac{1}{(b+1)(b+2)} - \frac{(a-x)^2}{(b+1)^2}$$

$$= \frac{(a-x)^2}{(b+1)^2(b+2)}\left[2(b+1)-(b+2)\right] = \frac{b(a-x)^2}{(b+1)^2(b+2)}$$

γ) $$2Var(T_x) = (ET_x)^2 \Leftrightarrow \frac{2b(a-x)^2}{(b+1)^2(b+2)} = \frac{(a-x)^2}{(b+1)^2} \Leftrightarrow$$

$$\frac{2b}{b+2} = 1 \Leftrightarrow b=2,\ a>0$$

3 $\mu_x = \dfrac{1}{x}$, για $30 \le x \le 49$ διάστημα $[30, 49]$

α) ${}_{19}P_{30}$ (β) Δεδομένης 30άρης έχει $\tilde{\mu}_x = \begin{cases} \dfrac{2}{x}, & x \in [30, 36] \\ \dfrac{1}{2x}, & x \in [36, 49] \end{cases}$

υπολογίστε ${}_{19}\tilde{p}_{30}$

Λύση α) $${}_{19}P_{30} = \frac{s(49)}{s(30)} = \frac{\exp\left(-\int_0^{49} \mu_x \, dx\right)}{\exp\left(-\int_0^{30} \mu_x \, dx\right)} = \exp\left(-\int_{30}^{49} \mu_x \, dx\right)$$

$$= \exp\left(-\int_{30}^{49} \frac{1}{x} \, dx\right) = \exp\left(-\log x \Big|_{30}^{49}\right) = \frac{30}{49}$$

όπου είναι η πιθανότητα επιβίωσης των 30άρηδων στο πληθυσμό.

β) Δεδομένου 30άρης έχει $\tilde{\mu}_x = \begin{cases} \dfrac{2}{x}, & x \in [30, 36] \\ \dfrac{1}{2x}, & x \in [36, 49] \end{cases}$

Τότε η δική του πιθανότητα επιβίωσης είναι:

$${}_{19}\tilde{p}_{30} = \frac{\tilde{s}(49)}{\tilde{s}(30)} = \exp\left(-\int_{30}^{49} \tilde{\mu}_x \, dx\right) = \exp\left(-\left(\int_{30}^{36} \frac{2}{x} \, dx + \int_{36}^{49} \frac{1}{2x} \, dx\right)\right)$$

$$= \exp\left(-\left(2 \log x \Big|_{30}^{36} + \frac{1}{2} \log x \Big|_{36}^{49}\right)\right) = \exp\left(-\log\left(\frac{30}{36}\right)^2 \cdot \left(\frac{36}{49}\right)^{1/2}\right)$$

$$= \left(\frac{5}{6}\right)^2 \cdot \frac{6}{7} = \frac{25}{42} < \frac{30}{49} = {}_{19}P_{30}.$$

(4) $\mu_x = \frac{1}{10}$, $x \geq 0$

(α) ET_x β) $Var(T_x)$

Λύση $ET_x = \int_0^\infty {}_tp_x \, dt$. με ${}_tp_x = \frac{s(x+t)}{s(x)} = \frac{\exp\left(-\int_0^{x+t} \mu_u \, du\right)}{\exp\left(-\int_0^{x} \mu_u \, du\right)}$

$$\Rightarrow {}_tp_x = \exp\left(-\int_x^{x+t} \mu_u \, du\right) = \exp\left(-\frac{1}{10}\int_x^{x+t} du\right) = \exp\left(-t/10\right)$$

άρα $ET_x = \int_0^\infty {}_tp_x \, dt = \int_0^\infty \exp\left(-\frac{t}{10}\right) dt = 10$

$$ET_x^2 = 2\int_0^\infty t \, {}_tp_x \, dt = 2\int_0^\infty t \exp\left(-t/10\right) dt = 20 \underbrace{\int_0^\infty \frac{t}{10} \exp\left(-t/10\right) dt}_{E\left[Exp\left(t/10\right)\right]}$$

$$ET_x^2 = 20 \cdot 10 = 200$$

άρα $Var(T_x) = ET_x^2 - (ET_x)^2 = 200 - 100 = 100$.

(5) $\mu_x = \frac{\theta}{x+1}$, $x \geq 0$, $\theta > 1$ και T_x = υπόλοιπος χρόνος ζωής ατόμου (x)

(α) $s(x)$ (β) ${}_tp_x$ (γ) ET_x (δ) θ αν $ET_3 = 80$

Λύση (α) $s(x) = \exp\left(-\int_0^x \mu_t \, dt\right) = \exp\left(-\int_0^x \frac{\theta}{t+1} dt\right) = \exp\left(-\theta \cdot \log(t+1) \Big|_0^x\right) = (x+1)^{-\theta}$

άρα $s(x) = \frac{1}{(x+1)^\theta}$, $x \geq 0$, $\theta > 1$.

(β) $${}_tp_x = \frac{s(x+t)}{s(x)} = \frac{\frac{1}{(x+t+1)^\theta}}{\frac{1}{(x+1)^\theta}} = \left(\frac{x+1}{x+t+1}\right)^\theta$$

(γ) $$ET_x = \int_0^\infty {}_tp_x\,dt = \int_0^\infty \left(\frac{x+1}{x+1+t}\right)^\theta dt = (x+1)^\theta \int_0^\infty \left(\frac{1}{x+1+t}\right)^\theta dt$$

$$\overset{u=x+1+t}{=} (x+1)^\theta \int_{x+1}^\infty \left(\frac{1}{u}\right)^\theta du = (x+1)^\theta \left.\frac{u^{-\theta+1}}{-\theta+1}\right|_{x+1}^\infty = \frac{(x+1)^\theta}{(\theta-1)(x+1)^{\theta-1}}$$

$$\Rightarrow ET_x = \frac{x+1}{\theta-1}$$

(δ) $$ET_3 = 80 \Leftrightarrow \frac{4}{\theta-1} = 80 \Leftrightarrow \boxed{\theta = \frac{21}{20}}$$