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ON THE DUAL NATURE OF MATHEMATICAL CONCEPTIONS: REFLECTIONS ON PROCESSES AND OBJECTS AS DIFFERENT SIDES OF THE SAME COIN

ABSTRACT. This paper presents a theoretical framework for investigating the role of algorithms in mathematical thinking. In the study, a combined ontological-psychological outlook is applied. An analysis of different mathematical definitions and representations brings us to the conclusion that abstract notions, such as number or function, can be conceived in two fundamentally different ways: *structurally* – as objects, and *operationally* – as processes. These two approaches, although ostensibly incompatible, are in fact complementary. It will be shown that the processes of learning and of problem-solving consist in an intricate interplay between operational and structural conceptions of the same notions.

On the grounds of historical examples and in the light of cognitive schema theory we conjecture that the operational conception is, for most people, the first step in the acquisition of new mathematical notions. Thorough analysis of the stages in concept formation leads us to the conclusion that transition from computational operations to abstract objects is a long and inherently difficult process, accomplished in three steps: *interiorization*, *condensation*, and *reification*. In this paper, special attention is given to the complex phenomenon of reification, which seems inherently so difficult that at certain levels it may remain practically out of reach for certain students.

INTRODUCTION

It is more than eight decades now, since the well-known French mathematician and philosopher Henri Poincaré wrote in obvious despair:

> One . . . fact must astonish us, or rather would astonish us if we were not too much accustomed to it. How does it happen that there are people who do not understand mathematics? If the science invokes only the rules of logic, those accepted by all well-formed minds . . . how does it happen that there are so many people who are entirely impervious to it? (Poincaré, 1952, p. 49; French original was published in 1908).

For all the knowledge accumulated by psychologists and educators since then, this question seems today as challenging and teasing as ever. The particular intricacy of mathematical thinking, the ubiquitous, sometimes insurmountable difficulty experienced by those who learn it, and the resulting persistent lack of success in teaching the subject – all these facts are not less puzzling than they are conspicuous. For the last several decades ever growing resources have been invested in a search for an improvement in mathematics teaching. The results, however, are still far from satisfactory – the sought-after solution seems to be as elusive as a cure for a common cold.

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There is probably much more to mathematics than just the rules of logic. It seems that to put our finger on the source of its ostensibly surprising difficulty, we must ask ourselves the most basic epistemological questions regarding *the nature of mathematical knowledge*. Indeed, since in its inaccessibility mathematics seems to surpass all the other scientific disciplines, there must be something really special and unique in the kind of thinking involved in constructing a mathematical universe. Saying what people usually say, namely that mathematics is the most *abstract* of sciences, does not help very much. Being almost a *cliché*, this claim has little explanatory power. The real question which should be asked here is qualitative rather than quantitative: How does mathematical abstraction differ from other kinds of abstraction in its nature, in the way it develops, in its functions and applications?

The question itself is certainly not new. The turn-of-the-century crisis in mathematics forced mathematicians themselves into philosophical discourse on the most fundamental questions regarding the nature of mathematical thought. Within the framework of Piaget's genetic epistemology, developed several decades later, it became possible to approach the same problems psychologically. But up to now, not enough has been done in the direction of unified theory which would address philosophy and psychology of mathematics simultaneously, and would take an equal care of mathematical thinking and of mathematical thought – of both the process and the product. In this context, the almost total neglect of advanced mathematics is especially regrettable, since the advanced topics are those in which the difference between mathematics and other sciences becomes most evident and the peculiarities of abstract thought can be observed in their purest form.

It seems that the philosophical insight into the nature of mathematical concepts is what we need in order to understand in depth the psychological processes in which such concepts emerge. In the suggested kind of investigation, epistemological and ontological analysis of "the stark, atemporal, formal universe of ideal [mathematical] knowledge" would hopefully shed some light on the roots of this overwhelming confusion which only too often seems to reign in "the organic, interior, processual universe of human knowing" (Kaput, 1979).

1. THE DUAL NATURE OF MATHEMATICAL CONCEPTIONS

Peculiarity of mathematical thinking investigated through reflections on the epistemological and ontological status of mathematical constructs is our subject in this paper. Depending on the point of view assumed at a given moment, two different words will be used to denote the building blocks of mathematics (or any other science, for this matter): the word "concept" (sometimes replaced by "notion") will be mentioned whenever a mathematical idea is concerned in its "official" form – as a theoretical construct within "the formal universe of ideal knowledge"; the whole cluster of internal representations and associations evoked by the concept – the concept's counterpart in the internal, subjective "universe of human knowing" – will be referred to as a "conception".

First, let us have a look at the world of mathematics, as it expresses itself through formal descriptions and representations.

As far as language is concerned, similarities between mathematics and other sciences seem more striking than the differences. Indeed, like physicists or biologists, the mathematicians use to talk about a certain universe, populated by certain objects. These objects have certain features and are subjected to certain processes governed by well defined laws. The mathematician describes properties of sets and numbers in much the same way as the scientist presents the structure of molecules and crystals. Utterances like "There exists a function such that ..." are as commonplace in modern mathematics as the claims about the existence of certain subatomic particles are in physics.

Unlike material objects, however, advanced mathematical constructs are totally inaccessible to our senses – they can only be seen with our mind's eyes. Indeed, even when we draw a function or write down a number, we are very careful to emphasize that the sign on the paper is but one among many possible *representations* of some abstract entity, which by itself can be neither seen nor touched. The mathematician would make claims about existence and properties of this intangible object without giving much thought to philosophical questions that his statements may evoke. Only rarely would an author of a textbook make an apologetic remark such as this: "We need not discuss how these abstract entities ... may be categorized from a philosophical point of view. For the mathematician ... it is important merely to know the rules or laws by which they may be combined" (Courant and John, 1962, p. 2). Being capable of somehow "seeing" these invisible objects appears to be an essential component of mathematical ability; lack of this capacity may be one of the major reasons because of which mathematics appears practically impermeable to so many "well-formed minds".

Even if this last claim is true (and I shall do my best in the sequel to convince the reader that it really is), the careful analysis of textbook definitions will show that treating mathematical notions as if they referred to some abstract *objects* is often not the only possibility. Although this kind of conception, which from now on will be called *structural*, seems to prevail in the modern mathematics, there are accepted mathematical definitions which reveal quite a different approach. Function can be defined not only as a set of ordered pairs, but also as a certain computational process or as a "method for getting from one system to another" (Skemp, 1971, p. 246). Symmetry can be conceived as a static property of geometric form, but also as a kind of transformation. The latter type of description speaks about *processes, algorithms and actions* rather than about objects. We shall say therefore, that it reflects an *operational conception* of a notion.

Seeing a mathematical entity as an object means being capable of referring to it as if it was a real thing – a static structure, existing somewhere in space and time. It also means being able to recognize the idea "at a glance" and to manipulate it as a whole, without going into details. Using Hadamard's metaphor (applied originally in a slightly different context), we can say that structural thinking endows a concept with "a kind of physiognomy", which allows a person to "think of it as a unique thing, however complicated it may be, just as we see a face of a man" (Hadamard, 1949, p. 65). In contrast, interpreting a notion as a process implies regarding it as a potential rather than actual entity, which comes into existence upon request in a sequence of actions. Thus, whereas the structural conception is static (or shall I say, after Frege, 1970, "timeless"), instantaneous, and integrative, the operational is dynamic, sequential, and detailed.

It is practically impossible to instantly pinpoint all the subtle aspects of the above distinction, let alone to formulate exact definitions of the structural and operational ways of thinking. At this point, it should already be quite evident that the former is more abstract, more integrated and less detailed than the latter, but it should also be clear that such a comparison leaves out at least as much as it comprises. Degrees of abstraction and of integration are but quantitative characteristics, whereas the crucial, qualitative, difference between the two modes of thinking lies in the basic, usually implicit, beliefs about the nature of mathematical entities. In other words, *there is a deep ontological gap between operational and structural conceptions*. It is the author's hope that as the discussion goes on, this fundamental but elusive aspect of the distinction will become more and more clear.

All this being said, it is very important to emphasize that operational and structural conceptions of the same mathematical notion are not mutually exclusive. Although ostensibly incompatible (how can anything be a process and an object at the same time?), they are in fact *complementary*. The

term "complementarity" is used here in much the same sense as in physics, where entities at subatomic level must be regarded both as particles and as waves to enable full description and explanation of the observed phenomena (for a fuller discussion of complementarity in the context of education see Otte, 1984, and Steiner, 1985). In the next section, I shall argue that in a similar way, the ability of seeing a function or a number both as a process and as an object is indispensable for a deep understanding of mathematics, whatever the definition of "understanding" is.

If we take a scrutinizing look at any mathematical concept, more often than not we shall find that it can be defined – thus conceived – both structurally and operationally. Some examples, chosen quite at random, are presented in Figure 1.

The dual nature of mathematical constructs can be noticed not only in verbal descriptions, but also through various kinds of symbolic representations. Although such property as structurality lies in the eyes of the beholder rather than in the symbols themselves, some representations appear to be more susceptible of structural interpretation than others. For instance, different approaches to the concept of function can be detected in

	Structural	Operational
Function	Set of ordered pairs (Bourbaki, 1934)	Computational process or Well defined method of getting from one system to another (Skemp, 1971)
Symmetry	Property of a geometrical shape	Transformation of a geometrical shape
Natural number	Property of a set or The class of all sets of the same finite cardinality	0 or any number obtained from another natural number by adding one ([the result of] counting)
Rational number	Pair of integers (a member of a specially defined set of pairs)	[the result of] division of integers
Circle	The locus of all points equidistant from a given point	[a curve obtained by] rotating a compass around a fixed point

Fig. 1. Structural and operational descriptions of mathematical notions.

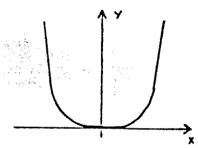
Graph	Algebraic expression	Computer program
	$y = 3x^4$	10 INPUT X 20 Y = 1 30 FOR I = 1 TO 4 40 Y = Y * X 50 NEXT I 60 Y = 3 * Y

Fig. 2. Different representations of a function.

the three distinct ways in which the mapping $y = 3x^4$ has been presented in Figure 2. The computer program seems to correspond to an operational conception rather than to a structural, since it presents the function as a computational process, not as a unified entity. In the graphic representation, on the other hand, the infinitely many components of the function are combined into a smooth line, so they can be grasped simultaneously as an integrated whole; the graph, therefore, encourages a structural approach. The algebraic representation can easily be interpreted both ways: it may be explained operationally, as a concise description of some computation, or structurally, as a static relation between two magnitudes (this duality of interpretation corresponds to the already widely noticed and discussed dual meaning of the equality sign: " = " can be regarded as a symbol of identity, or as a "command" for executing the operations appearing at its right side; see e.g. Behr et al., 1976; Kaput, 1979; Kieran, 1981).

Different kinds of conception – structural and operational – manifest themselves also in the special representations of which people avail themselves while processing knowledge mentally. According to what is already known on the internal encoding (see e.g. Paivio, 1971; Clements, 1981, Bishop, 1988; Eisenberg and Dreyfus, 1989), mathematical concepts are sometimes envisioned by help of "mental pictures", whereas on other occasions the same ideas are handled mainly through verbal representations. Mental images, being compact and integrative, seem to support the structural conception. Hadamard's introspective observations on the role of visualization reinforces this supposition: "I need [an image] in order to have a simultaneous view of all elements . . . to hold them together, to make a whole of them . . . ; to achieve synthesis . . . and give the concept its physiognomy" (Hadamard, 1949, p. 77). Visualization, therefore, makes abstract ideas more tangible, and encourages treating them almost as if they were material entities. Indeed, mental images can be manipulated

almost like real objects. Like in face recognition, the pictures would preserve their identity and meaning when "observed" from different points of view and in different contexts. Visual representation is holistic in its nature and various aspects of the mathematical construct may be extracted from it by "random access". In contrast, verbal encoding cannot be grasped "at one glance" and must be processed sequentially, so it seems more appropriate for representing computational procedures. Thus, the non-pictorial inner representation is more pertinent to the operational mode of thinking. [Please note: the above claims should not be interpreted as an attempt to say that there is a one-to-one correspondence between operational/structural conceptions and verbal/visual inner representations. The only suggestion I have been trying to make here is that some kinds of inner representations fit one type of conception better than the other.]

Before closing this introductory section it would be in point to notice that mathematical, psychological and philosophical literature teems with allusions to various dichotomies in mathematical universe. Distinctions between "two types of mathematical knowledge/thought/understanding" go like a thread of scarlet throughout all kinds of recent writings, and some of them may have certain bearings on the operational-structural duality suggested in this article.

Out of the long list of dichotomies proposed by different writers, let me mention only a few (for a fuller catalogue see Hiebert, 1985, pp. 1–2). According to some researchers, mathematics can be divided into *abstract* and *algorithmic* (see e.g. Halmos, 1985) or into *declarative* and *procedural* (Anderson, 1976). The names are almost self-explanatory, so even without any formal definitions the connection between these distinctions and the ideas presented in this paper should be quite obvious. The already mentioned observations on *process/product* duality of mathematical symbolism (Kaput, 1979; Davis, 1975), although much more restricted in scope, seem to go hand in hand with this kind of divisions. Another categorization, which has perhaps even more in common with our suggestions, is the one which splits mathematics into *dialectic* and *algorithmic* (Henrici, 1974). While algorithmic mathematics deals mainly with all kinds of computational processes, "dialectic mathematics is a rigorously logical science, where statements are either true or false, and where objects with specified properties either do or do not exist".

A certain kind of dichotomy has been observed also when psychological rather than philosophical aspects of mathematics were concerned. Two different modes of mathematical thinking have been distinguished by Piaget (1970, p. 14): *figurative*, which refers to seeing "states as momentary and static" and thus corresponds to our structural conception; and *operative*, which "deals... with transformations...", so has much to do with our operational approach. Incidentally, this distinction is deeply rooted in Piaget's theory of reflective abstraction which, especially in its further elaborations (Thompson, 1985; Dubinsky and Lewin, 1976), has touched explicitly upon the role of processes and objects in mathematical thinking. Even the widely accepted categorizations of mathematical understanding (or knowledge) into *conceptual* and *procedural* (see e.g. Lesh and Landau, 1983; Hiebert, 1985) or into *instrumental* and *relational* (Skemp, 1976) seem to be somehow in point here. Indeed, in the next section an attempt will be made at showing how our capability for developing operational and structural conceptions bears upon the type of understanding we achieve.

The classification suggested in this article puts us, therefore, in a good company. Moreover, even if our distinction still looks somewhat fuzzy, so do all the others. For instance, while talking about conceptual and procedural knowledge, Hiebert and Lefevre complained: "the relationship between these forms of knowledge is not yet well understood", and "the types of knowledge themselves are difficult to define; the core of each is easy to describe, but outside edges are hard to pin down" (ibid., p. 3). "Pinning down the edges", at least in the case of operational-structural distinction, is my objective in the remainder of this article.

To begin with, let me confront our division with those listed above. On the face of it, the idea of operational and structural conceptions may seem not much different from some of the dichotomies which have just been mentioned. For all the similarities, however, two fundamental characteristics of our distinction – its combined ontological-psychological nature and its complementarity – put it apart from the majority of other classifications. Firstly, most of those who suggested some kind of dichotomy rarely gave much attention to the question of tacit philosophical assumptions underlying any mathematical activity; rather, they referred either to certain more obvious aspects of the subject-matter (such as its structure or the role of its different components in problem-solving), or to the cognitive processes involved in handling the knowledge. In our classification, we tried to address the first and the last of these issues simultaneously by focusing on *the nature of mathematical entities* (ontological issue) as perceived by a thinker (psychological perspective). Secondly, whereas other distinctions lead to *decomposition* of mathematical knowledge into two separate components (e.g., concepts vs. procedures), our complementarian approach stresses its unity. True, recently the former position seems to be gradually abandoned. While referring to the issue of concepts and procedures,

Hiebert and Lefevre observed: "Historically, the two kinds of knowledge have been viewed as separate entities,... coexisting as disjoint neighbours... In contrast, there is a growing interest today in how concepts and procedures are related" (ibid., p. 2). Nevertheless, this new approach still cannot be regarded as complementarian: "current discussions treat the two forms of knowledge as distinct", even though "linked in critical, mutually beneficial way". Let me stress once more: unlike "conceptual" and "procedural", or "algorithmic" and "abstract", the terms "operational" and "structural" refer to inseparable, though dramatically different, facets of the same thing. Thus, we are dealing here with *duality* rather than *dichotomy*.

To complete the picture, let me make yet another remark regarding the normative approach some writers assume while talking about "different kinds of mathematics". "Algorithmic" and "abstract" mathematics, for example, are sometimes assessed and contrasted against each other as if a contest was held between them. While there seems to be a consensus that the "abstract" mathematics deserves the highest esteem, the algorithmic, procedural aspects are rather controversial. Recently, the dispute over the topic has become more heated than ever: provocatively exaggerated declaration that "algorithmic way of life is best" (Maurer, 1985) evoked angry reactions from those who feel that "algorithm drives out thought" (Stein, 1988). Even though everybody admits that "algorithmic" mathematics is important, the opinion seems to prevail that it is somehow second-rate.

Our complementarian approach strips this kind of discussion of any meaning. Whether the issue of applications or of education is concerned, the operational and structural elements cannot be separated from each other. Therefore, we may only repeat after Halmos (1985) that to "try to decide which component is more important is not much more meaningful than to debate whether for walking you need your right foot more than your left". This statement will be further substantiated in the next sections, where the mutual dependence and the necessity of both operational and structural conceptions will be carefully explained and illustrated. The thorough discussion of the role played by them in all kinds of cognitive processes will help us to grasp the reason for which abstract mathematics – the one more heavily based on a structural approach – is so highly-rated. Indeed, it will be shown that the real insight necessary for mathematical creation can hardly be achieved without the ability to "see" abstract objects, and that, on the other hand, the structural conception is very difficult to attain (that is probably why some people may feel, intuitively of course, that the special ability to develop a structural conception is what distinguishes mathematicians from "mere mortals"). For all the attention and respect given to the structural approach, the operational mode of thinking will also get its due share: we shall argue that a profound insight into the processes underlying mathematical concepts, maybe even a certain degree of mastery in performing these processes, should sometimes be viewed as a *basis* for understanding such concepts rather than as its *outcome*. Consequently, the "technical skills" will be rehabilitated, after they were unduly demoted in a somewhat exaggerated reaction to behaviorism. At last, we may even find ourselves in the position to offer a tentative answer to the vexing question so often asked by educators:

> Why is it that so many intelligent, well-trained, well-intentioned teachers put such a premium on developing students' skill in the routines of arithmetic and algebra despite decades of advice to the contrary from so-called experts? What is it that teachers know that others do not? (Kilpatrick, 1988).

2. THE ROLE OF OPERATIONAL AND STRUCTURAL CONCEPTIONS IN THE FORMATION OF MATHEMATICAL CONCEPTS – HISTORICAL OUTLOOK

Of the two kinds of mathematical definitions, the structural descriptions seem to be more abstract. Indeed, in order to speak about mathematical *objects*, we must be able to deal with *products* of some processes without bothering about the processes themselves. In the case of functions and sets (in their modern sense) we are even compelled to ignore the very question of their constructivity. It seems, therefore, that the structural approach should be regarded as the more advanced stage of concept development. In other words, we have good reasons to expect that *in the process of concept formation, operational conceptions would precede the structural.* Different kinds of evidence will be brought in this article to show that this statement is basically true whether historical development or individual learning is concerned.

Before any example will be given, however, it should be pointed out that the proposed model, although believed to be very broad in its scope, might be inadequate in certain cases. Geometric ideas, for example, for which the unifying, static graphical representations appear to be more natural than any other, can probably be conceived structurally even before full awareness of the alternative procedural descriptions has been achieved. The concept of a circle, for instance, could develop in the steps envisioned by M. Boole (Tahta, 1972): "The elementary geometrician who first conceived the idea of the circle caught his suggestion from looking at things whose forms were approximately round; but as soon as he discovered the law of roundness [the algorithm for obtaining a circle] within his own mind, he was able to express roundness in a new material". Even though this developmental scenario looks plausible, the ensuing historical analysis of other concepts will hopefully convince the reader that in computational mathematics, the majority of ideas originated in *processes* rather than in *objects*. Indeed, a close look at the history of such notions as number or function will show that they had been conceived operationally long before their structural definitions and representations were invented.

What follows now is a very brief and by no means exhaustive presentation of a long and turbulent history of some of the most central mathematical concepts. In this article I shall deal with only those historical facts and events which highlight the point I would like to make here (for a more detailed historical account see e.g. Cajori, 1985; Kleiner, 1989).

Let us begin our analysis with the notion of number. For a very long time the meaning of this term was restricted to what is known today under the name "natural number". This kind of number originates in the process of counting, so even before turning to the history, let us consider a certain well-known phenomenon observed in young children. It was noticed by researchers (see e.g. Piaget, 1952, p. 62) that when a child learns counting, there is a stage when he or she can already build a one-to-one mapping between the words "one", "two", "three", . . . and the objects in a given set, but would not use the last number-name used in this process as an answer to a question "How many objects are there?". Whenever asked, the child would just repeat the procedure of counting. This phenomenon clearly shows the operational roots of natural numbers: for the child, the process of counting itself, not its abstract product, is what is meant whenever the term "number" is mentioned.

The meaning of the term "number" has been generalized several times in the course of the last three thousand years. For long periods did mathematicians perform some special manipulations with already acknowledged kinds of numbers before they were able to sever an abstract product from these new processes and to accept the resulting entities as a new kind of mathematical objects. For instance, a ratio of two integers was initially regarded as a short description of a measuring process rather than as a number. Incidentally, some traces of purely operational approach to rationals were noticed by researchers (Carpenter et al., 1980) also in today's 13-year-old students, 50 percent of whom were found unable "to represent a division problem like 7 divided by 4 as a fraction". In these students, the division of integers was still only a process which could not yet be seen at will as a static entity.

For a long time the term "number" appeared mainly in the context of measuring processes. The Pythagorean discovery that in certain squares, the usual procedure for finding the length of the diagonal cannot be described in terms of integers and their ratios (because the diagonal and the sides have no common measure), was greeted with astonishment and bewilderment (Hippasus of Metapontum, the unfortunate discoverer of the incommensurability, was reportedly thrown to the sea for his "heresies"). Much time elapsed before mathematicians were able to separate the notion of number from measuring processes and to acknowledge the fact that the length of *any* segment represents a number even if it cannot be found in the "usual" way. Eventually, the set of numbers has been broadened again, to include positive irrationals along with integers and fractions.

This enlarged set, in its turn, gave birth to new kinds of computational processes, and then to new kinds of numbers. Cardan's prescriptions for solving equations of the third and fourth order, published in 1545, involved subtracting positive rationals from smaller ones and even finding roots of what is today called negative numbers. Despite the widespread use of these algorithms, however, mathematicians refused to accept their by-products and for some centuries referred to them as "absurd" or "imaginary". The term "negative number" and the symbol $\sqrt{-1}$ were initially considered nothing more than abbreviations for certain "meaningless" numerical operations. They came to designate a fully-fledged mathematical object only after mathematicians got accustomed to these strange but useful kinds of computation.

As to the operational origins of the negative and complex numbers, it would be most illuminating to look into the historical writings by the logician and philosopher P. E. Jourdain (1879–1919). In the following passage, Jourdain (1956, p. 27) says explicitly that negative number is nothing but a type of process:

Let $a - b$ be c. To get c from a we carry out the operation of taking away b. *This operation, which is the fulfillment of the order: "Subtract b", is a "negative number"*. Mathematicians call it a "number" and denote it by "$-b$" simply because of analogy: the same rules for calculation hold for "negative numbers" and "positive numbers".

Jourdain's explanation highlights the developmental gap between his conceptions of the "unsigned" and the negative numbers: while the former are already regarded by the author as "real", genuine objects, their operational origins totally forgotten, the latter are still identified with processes and allowed to be treated as static entities only by force of convention. Since the idea of complex numbers stems from certain manipulations performed on negatives, it should surprise nobody that Jourdain's interpretation of

the "imaginary" numbers is also purely operational: while referring to the number "i", he states that "it represents an operation, just as the negative numbers do, but of a different kind" (p. 30).

The most important thing to be learned from this short historical account is that the development of the notion of number was a cyclic process, in which approximately the same sequence of events could be observed time and again, whenever a new kind of number was being born. These iterations have been summarized schematically in Figure 3. Each recurrent segment of the schema represents a lengthy process, consisting of three phases:

(1) the preconceptual stage, at which mathematicians were getting used to certain operations on the already known numbers (or, as in the case of counting – on concrete objects); at this point, the routine manipulations

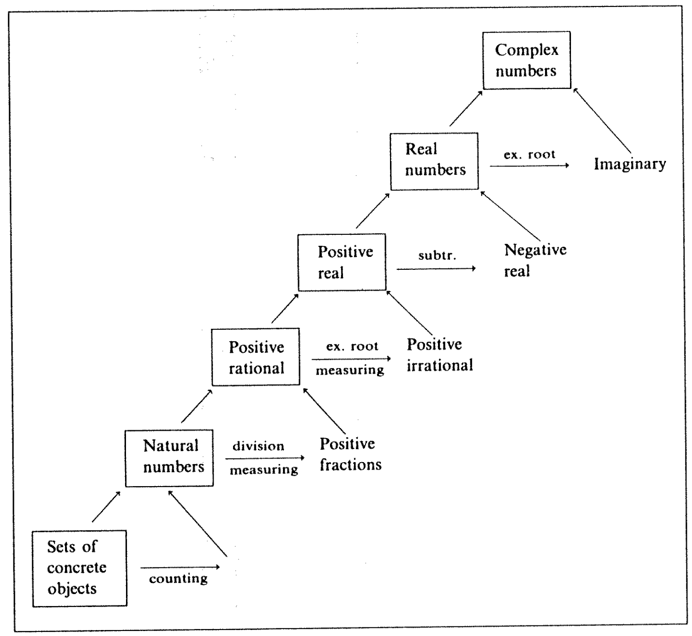


Fig. 3. Development of the concept of number.

were treated as they were: as processes, and nothing else (there was no need for new objects, since all the computations were still restricted to those procedures which produce the previously accepted numbers).

(2) a long period of predominantly operational approach, during which a new kind of number begun to emerge out of the familiar processes (what triggered this shift were certain uncommon operations, previously regarded as totally forbidden, but now accepted as useful, if strange); at this stage, the just introduced name of the new number served as a cryptonym for certain operations rather than as a signifier of any "real" object; the idea of a new abstract construct, although already in wide use, would still evoke strong objections and heated philosophical discussions;

(3) *the structural* phase, when the number in question has eventually been recognized as a fully-fledged mathematical object. From now on, different processes would be performed on this new number, thus giving birth to even more advanced kinds of numbers.

To sum up, the history of numbers has been presented here as a long chain of transitions from operational to structural conceptions: again and again, processes performed on already accepted abstract objects have been converted into compact wholes, or *reified* (from the Latin word *res* – a thing), to become a new kind of self-contained static constructs. Our conjecture is, that this model can be generalized to fit many other mathematical ideas.

For instance, the just presented pattern repeats itself in the history of function. This important idea, born in the end of seventeenth century (at least officially), was the result of a long search after a mathematical model for physical phenomena involving variable quantities. When the term "function" appeared for the first time (in a work by Leibniz, in 1692), the recently invented algebraic symbolism was gaining popularity and gradually entering every branch of mathematics. No wonder then, that the notion of function was initially tightly connected to algebraic processes. The new term was first used to denote "a quantity composed in any manner whatever of [a] variable and constant" (by Jean Bernoulli in 1718), or the so called "analytic expression" (by Euler in 1747). Thus, in a sense, the concept of function was for algebraic manipulations on variables what the idea of a negative number was for subtraction: something between the product and the process itself.

The main problem with the early definitions of function was that they leaned heavily on the concept of variable, which by itself was rather fuzzy

and escaped every attempt at reification. That is probably one of the reasons why in 1755, after a long discussion with d'Alembert, Euler suggested another definition, in which the term "variable" was not explicitly mentioned: "a quantity" should be called function only if it depends on another quantity "in such a way that if the latter is changed the former undergoes change itself". The operational flavor emanates from this description even more clearly than from the earlier versions.

All the further history of the notion may be seen as a long sequence of strenuous, if mostly failed, attempts at reification. Euler himself tried to arrive at a fully-fledged structural version by endowing his "changing quantities" with a "solidifying" graphic representation. Euler's idea was not very helpful, however, since neither he, nor any of his contemporaries were able to build a truly satisfactory bridge between the algebraic and graphic approach: each time a definition had been proposed which would fit the algebraic-operational intuition, after a while somebody would find an example showing that the new description fell short of the structural-graphic version; and vice versa (for a detailed account see Kleiner, 1989).

It should be pointed out that at a certain stage, mathematicians and philosophers became fully aware of what for some time had probably been done only intuitively – of their striving for reification, of their need of definition which would justify the common practice of referring to function as if it was a real "thing". For example, let us look at the following remark from the beginning of our century:

In recent times the word 'variable' is predominant in the definitions [of function]. Consequently Analysis would have to deal with a process in time, since it takes variables into consideration. But in fact it has nothing to do with time; its applicability to occurrences in time is irrelevant.... as soon as we try to mention a variable, we shall hit upon something that varies in time and thus does not belong to pure Analysis. And yet it must be possible to point to a variable that does not involve something alien to arithmetic, if variables are objects of Analysis at all. (Frege, 1970, p. 107; German original: 1904.)

Frege's call for elimination of time is an explicit request for reification. Also, this remark makes it clear how difficult the struggle for a structural version of the notion was.

The numerous failed attempts at translating operational intuition into structural definition led to Dirichlet's rebellion against the algorithmic approach, and eventually to the now widely accepted, purely structural Bourbaki's definition. This simple description presented function as a set of ordered pairs and made no reference whatsoever to any kind of computational process. Bourbaki's group solved the time-revered problem by eliminating the "unreificable" notion of variable and substituting it with purely structural set-theoretic concepts. Not surprisingly, this new definition, which had very little in common with its intuitive operational origin, evoked much criticism when first proposed. But when at a long, long last function – initially only a computational process – was converted into a mathematical object, our schema of concept development could repeat itself once more: on the new objects new operations could now be performed. These new operations are known today as functionals.

Let us summarize once again what has been observed thus far. In all our examples, the same phenomenon could be distinguished over and over again: various processes had to be converted into compact static wholes to become the basic units of a new, higher level theory. When we broaden our view and look at mathematics (or at least at its big portions) as a whole, we come to realize that it is a kind of hierarchy, in which what is conceived purely operationally at one level should be conceived structurally at a higher level. Such hierarchy emerges in a long sequence of reifications, each one of them starting where the former ends, each one of them adding a new layer to the complex system of abstract notions. In certain cases, of course, this picture would seem a little simplistic. The process of concept formation would look more intricate than implied by our unidirectional model. This model, however, is to be regarded as not more than a first approximation, indicating only the prevalent tendency.

3. THE ROLE OF OPERATIONAL AND STRUCTURAL CONCEPTIONS IN THE FORMATION OF MATHEMATICAL CONCEPTS – PSYCHOLOGICAL OUTLOOK

What really strikes the eyes in the already given examples is that the formation of a structural conception is a lengthy, often painfully difficult process. The question which now cries out to be asked concerns the sources of this difficulty. Naturally, this issue should be tackled from the psychological point of view, and this is exactly what will be done in the next section. In the remainder of the present part I shall restrict myself to the preliminary problem: is the proposed model of concept formation in force also when individual learning is concerned? Or, in other words, is it true that when a person gets acquainted with a new mathematical notion, the operational conception is usually the first to develop? The odds are that the answer to this question should be *yes*. Let me put it even more clearly: it seems that the scheme which was constructed on the basis of historical examples can be used also to describe learning processes.

At this point, some objections may be raised by a careful reader. Firstly, the above statements imply that there is some "natural" course of events in

processes, which can hardly be regarded as spontaneous. Indeed, mathematical learning, expecially at more advanced levels, cannot be expected to take place without external intervention (of a teacher, of a textbook), and may therefore be highly dependent on a kind of stimulus (of teaching method) which has been used. Moreover, given a specific teaching strategy, it is sometimes almost impossible to know to what extent the observed learning process has been influenced by this particular method, and how different it could be in other circumstances. The simplest way to deal with this kind of doubt would be to say that in the psychological context, the statement "operational before structural" should be understood merely as a *prescription* for teaching. Yet, although such interpretation should not be dismissed, it would not make full justice to the suggested model. Our whole argumentation is based on the assumption (which, incidentally, seems to underlie most of cognitive research since Piaget) that in the process of learning – any kind of learning! – certain constant characteristics can be identified which appear to be quite immune to changes in external stimuli. The precedence of the operational conceptions over structural is presented here as one of such invariants.

Secondly, it must be stressed right away that the suggested model of concept acquisition should by no means be viewed as a result of a light-minded, automatic projection from history to psychology. Claims about operational origins of mathematical notions were made by many writers, often without any reference to history. The pioneering work in this field has been done by Piaget, who wrote in his book on genetic epistemology (1970, p.16): "the [mathematical] abstraction is drawn not from the object that is acted upon, but from the action itself. It seems to me that this is the basis of logical and mathematical abstraction". For the last twenty years, this supposition has guided both theoretical and empirical research on mathematical thinking. Recent studies elaborated Piaget's original ideas and put into them new contents (see e.g. Thompson, 1985; Sinclair and Sinclair, 1986; Dubinsky and Lewin, 1986; Dőrfler, 1987, 1989). Our somewhat broader conjecture about the duality of mathematical thinking and the developmental precedence of the operational conceptions can be substantiated in many different ways. Some empirical evidence has already been woven into our historical account, and many other recent findings in the field of learning mathematics may serve as additional reinforcement (see Sfard, 1987, 1988, 1989). But first and foremost, there is a strong theoretical argument speaking for our thesis. If the structural approach is more abstract than the operational, if from the philosophical point of view numbers and functions are basically nothing but processes, if doing things is the only way to somehow "get in touch" with abstract constructs – if all this is true, then to expect that a person would arrive at a structural conception without previous operational understanding seems as unreasonable, as hoping that he or she would comprehend the two-dimensional scheme of a cube without being acquainted with its "real-life" three-dimensional model.

According to our scheme of historical development, three steps can be distinguished in the process of concept formation. These three stages correspond to three "degrees of structuralization" which may be named on the grounds of purely theoretical analysis of the relationship between processes and objects. In the light of the same analysis, our model of learning can now be refined along similar lines: if the conjecture on operational origins of mathematical objects is true, then first there must be a process performed on the already familiar objects, then the idea of turning this process into an autonomous entity should emerge, and finally the ability to see this new entity as an integrated, object-like whole must be acquired. We shall call these three stages in concept development *interiorization*, *condensation* and *reification*, respectively.

At this point, a detailed description of each stage should be given. Before this is done, however, we must be aware of a methodological difficulty stemming from the fact that we are dealing here with student's *implicit* beliefs about the nature of mathematical objects. Unable to investigate the problem in a direct way, how shall we diagnose the different stages in the conceptual development of a learner? It seems that we have no choice but to describe each phase in the formation of abstract objects in terms of such *external* characteristics as student's *behaviours*, *attitudes*, and *skills*. The resulting specification will hopefully be clear enough to serve as a tool for diagnosing, maybe even measuring, student's ability to think structurally about a concept at hand.

At the stage of *interiorization* a learner gets acquainted with the processes which will eventually give rise to a new concept (like counting which leads to natural numbers, subtracting which yields negatives, or algebraic manipulations which turn into functions). These processes are operations performed on lower-level mathematical objects. Gradually, the learner becomes skilled at performing these processes. The term "interiorization" is used here in much the same sense which was given to it by Piaget (1970, p. 14): we would say that a process has been interiorized if it "can be carried out through [mental] representations", and in order to be considered, analyzed and compared it needs no longer to be actually performed.

In the case of negative number, it is the stage when a person becomes skillful in performing subtractions. In the case of complex number, it is when the learner acquires high proficiency in using square roots. In the case of function, it is when the idea of variable is learned and the ability of using a formula to find values of the "dependent" variable is acquired.

The phase of *condensation* is a period of "squeezing" lengthy sequences of operations into more manageable units. At this stage a person becomes more and more capable of thinking about a given process as a whole, without feeling an urge to go into details. It is like turning a recurrent part of a computer program into an autonomous procedure: from now on the learner would refer to the process in terms of input-output relations rather than by indicating any operations. As in the case of computer procedures, a name might be given to this condensed whole. This is the point at which a new concept is "officially" born. Any difficulty with indicating the output of the underlying process (like in the case of subtracting a number from a smaller one while only unsigned numbers are known) will serve as an additional trigger for the idea of a new mathematical entity. Thanks to condensation, combining the process with other processes, making comparisons, and generalizing become much easier. A progress in condensation would manifest itself also in growing easiness to alternate between different representations of the concept.

In the case of the negative numbers, condensation may be assessed through student's proficiency in combining the underlying processes with other computational operations; or, in other words, in his or her ability to perform such arithmetic manipulations as adding or multiplying negative and positive numbers. In the case of complex numbers, condensation is what helps the learner to realize that reversing the operation of squaring may be useful as a part of lengthy calculations even if it would not, by itself, yield a legitimate mathematical object. The student may still treat such symbol as $5 + 2i$ as nothing but a shorthand for a certain procedure, but at this stage it would not prevent him from skillfuly using it as a part of a complex algorithm. When function is considered, the more capable the person becomes of playing with a mapping as a whole, without actually looking into its specific values, the more advanced in the process of condensation he or she should be regarded. Eventually, the learner can investigate functions, draw their graphs, combine couples of functions (e.g. by composition), even to find the inverse of a given function.

The condensation phase lasts as long as a new entity remains tightly connected to a certain process. Only when a person becomes capable of conceiving the notion as a fully-fledged object, we shall say that the concept has been reified. *Reification*, therefore, is defined as an ontological shift – a sudden ability to see something familiar in a totally new light. Thus, whereas interiorization and condensation are gradual, quantitative rather than qualitative changes, reification is an instantaneous quantum leap: a process solidifies into object, into a static structure. Various representations of the concept become semantically unified by this abstract, purely imaginary construct. The new entity is soon detached from the process which produced it and begins to draw its meaning from the fact of its being a member of a certain category. At some point, this category rather than any kind of concrete construction becomes the ultimate base for claims on the new object's existence. A person can investigate general properties of such category and various relations between its representatives. He or she can solve problems involving finding all the instances of the category which fulfill a given condition. Processes can be performed in which the new-born object is an input. New mathematical objects may now be constructed out of the present one. Penrose's (1989, p. 67) statement referring to Church's lambda-calculus may serve as a telling example: "one is concerned [here] with a 'universe' of objects, denoted by say $a, b, c, \ldots$, each of which stands for a mathematical operation or *function* ... The things on which these functions act – are other things of the same kind, i.e. also functions" (incidentally, this statement has another interesting aspect: by referring to function first as 'object' or 'thing', and then as 'mathematical operation', it clearly indicates the operational-structural duality of the author's approach).

The stage of reification is the point where an interiorization of higher-level concepts (those which originate in processes performed on the object in question) begins.

In the case of negative numbers, it is learner's ability to treat them as a subset of the ring of integers (without necessarily being aware of the formal definition of ring) which can be viewed as a sign of reification. Complex number may be regarded as reified when the symbol $5 + 2i$ is interpreted as a name of a legitimate *object* – as an element in a certain well-defined set – and not only (or even not at all) as a prescription for certain manipulations. In the case of function, reification may be evidenced by proficiency in solving equations in which "unknowns" are functions (differential and functional equations, equations with parameters), by ability to talk about general properties of different processes performed on functions (such as composition or inversion), and by ultimate recognition that computability is not a necessary characteristic of the sets of ordered pairs which are to be regarded as functions.

How and to what extent the just presented developmental scheme may be influenced by deliberate instructional actions is an important question

which must be dealt with in a separate article. There is one thing, however, which is much too essential to be passed over in silence. It is the potential role of names, symbols, graphs, and other representations in condensation and reification. Judging from the history, the importance of this factor can hardly be overestimated. For example, the introduction of the number line may be viewed as a final trigger for the reification of negative numbers, and the invention of what is known today as Argand plane can be regarded as a decisive step in turning complex numbers into legitimate mathematical objects. It seems reasonable to expect that representations would play a similar role in individual learning. This topic, although already studied in a similar context (see e.g. Dörfler, 1987), asks for more empirical investigation.

At this point it should already be clear that our three-phase schema is to be understood as a *hierarchy*, which implies that one stage cannot be reached before all the former steps are taken (see Figure 4). One word should be added, however, regarding certain side routes which can be taken by a learner. The student may manipulate a concept through a certain prototype (for example, the data collected by Markovitz et al., 1985, show that beginners tend to imagine linear mappings whenever the notion of function is mentioned) or, unable to come to terms with the invisible 'objects', he or she can develop a *debased*, quasi-structural approach, namely a tendency to identify the notion at hand with one of its representations (in the case of function: formula or graph). This stage, which is clearly a deviation from our scheme, may be transitory or permanent.

Let me conclude this part of the study with two remarks addressed to a critical reader. First, those who feel that at some points the author sounded somewhat too assertive should be reminded that the majority of statements made in this section were analytic rather than synthetic. For example, the hierarchical nature of the scheme is implicit in the definitions of interiorization, condensation and reification. More generally, the model of concept acquisition presented in this section has been deduced from one basic conjecture – from the thesis about the operational origins of mathematical objects. Like so many other ideas contrived by those who still believe in the possibility of a theoretical framework for cognitive research, our three-phase schema has a highly speculative character – and this is only natural. Hypothetical and simplified as it is, it has already started to prove itself useful as a tool for planning, integrating, and interpreting empirical research (see Sfard, 1987, 1989). Ultimately, it may lead to some important didactic implications (Sfard, 1988).

Secondly, the claim about the developmental priority of operational

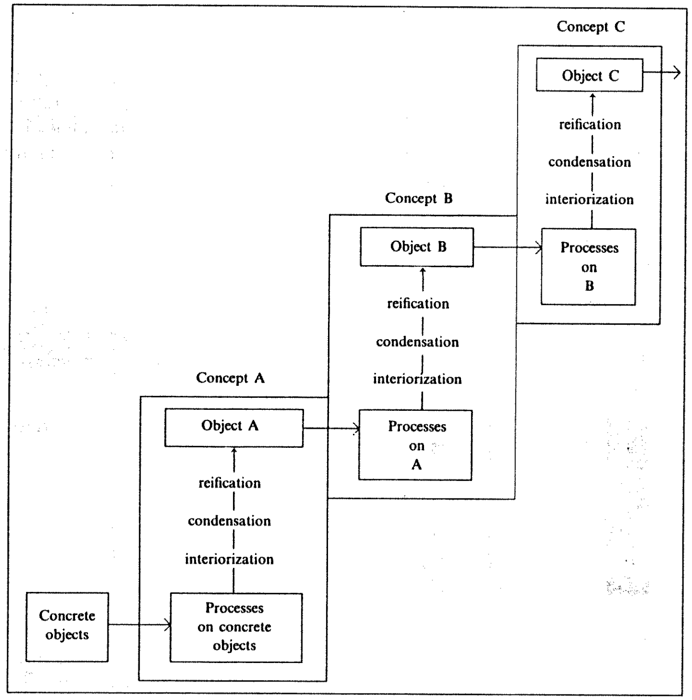


Fig. 4. General model of concept formation.

conceptions over structural, although historically sound and empirically demonstrable, may seem somehow at variance with the common practice of introducing new concepts by help of structural definitions, usually without an explicit reference to any kind of underlying processes. More often than not, a modern mathematical textbook would start its presentation of the concept of complex number with a simple statement: "Let us consider the set of all the pairs of real numbers with the following properties . . .". The contemporary mathematician would offer an entirely new idea in a form of a ready-made object, clearly believing that the abstract construct may be brought into being just by force of an appropriate definition. Thus, the possibility must be considered that, after all, structural conception may sometimes be the first. This can certainly be true in the case of professional

mathematicians – their well-trained minds can indeed be capable of manipulating abstract objects right away, without the mediation of computational processes. Even so, in the light of both theoretical arguments and experimental findings, our model does seem to present a *prevailing* tendency. In fact, this tendency may be so strong, that even if a new concept is introduced structurally, the student would initially interpret the definition in an operational way (much empirical evidence for this may be found in recent studies on algebra and on the concept of function; see Vinner and Dreyfus, 1989; Sfard, 1987, 1989).

4. THE ROLE OF OPERATIONAL AND STRUCTURAL CONCEPTIONS IN COGNITIVE PROCESSES

Before we try to find an explanation for the inherent difficulty of reification, let us tackle yet another more basic issue. The just suggested model of concept formation implies that certain mathematical notions should be regarded as fully developed only if they can be conceived both operationlly and structurally. The question arises, what is it that necessitates this dual outlook; or, in other words, what can be achieved with the ability of seeing a concept both operationally and structurally, which would not be attained if only one approach was always assumed.

4.1. *Operational Approach: Certainly Necessary, Sometimes also Sufficient*

Apparently, a purely operational way of looking at mathematics would be quite appropriate. Indeed, to an unprejudiced and insightful person, the very notion of "mathematical object" may appear superfluous: since processes seem to be the only real concern of mathematics, why bother about these elusive, philosophically problematic "things", such as infinite sets or "aggregates of ordered pairs"? Theoretically it would be possible to do almost all the mathematics purely operationally: we could proceed from elementary processes to higher-level processes and then to even more complex processes without ever referring to any kind of abstract objects.

As a matter of fact, a careful look at history would reveal that for a very long time big portions of mathematics were done almost exactly this way. As Davis and Hersh (1983, p. 182) notice, "The mathematics of Egypt, of Babylon, and of ancient Orient was all of algorithmic type . . . . It is only in modern times that we find mathematics with little or no algorithmic content, which we could call purely dialectical or existential". Indeed, the science of computation, known today under its relatively new name "algebra", has retained a distinctly operational character for thousands of years. The so called "rhetorical" algebra, which preceded the syncopated and symbolic algebras (the last one developed not before the 16th century!) dealt with computational processes as such, while the only kind of abstract objects permitted in the discourse were numbers. Even most complex sequences of numerical operations were presented by help of verbal prescriptions, which bore distinctly sequential character and did not stimulate condensation and reification. As long as the computational processes have been presented in the purely operational way, they could not be squeezed into static abstract entities, thus were not susceptible of being treated as objects.

4.2. *The Necessity of Structural Conceptions*

Twentieth-century mathematics, however, seems to be so deeply permeated with the structural outlook, that a modern mathematician had to be exceptionally open-minded – indeed, not himself at all – to realize that from a philosophical (not psychological!) point of view he could do without "mathematical objects". In his eyes, this notion is probably so inherent to mathematics as the idea of matter is to physics. (Can anybody imagine motion without physical bodies? Can anybody talk about computational processes performed on . . . nothing?). Why do we have this strong propensity for making abstractions in the image of the material word? This important question may be addressed at several levels. The most obvious, top-level answer is that our imagination is shaped by our senses. Probably that is why we have this overpowering feeling that we cannot perform a process, unless there is an object on which this process is carried out, and there is another object, which this process produces.

For a more profound explanation we shall turn in a moment to the theory of cognitive schemata. Before we do this, however, the reader is invited to perform an experiment, which will hopefully illuminate our subsequent claims in the most persuasive way.

Our exercise will be performed in two steps. Let us begin with mathematical definitions, presented in Figure 5. The first step, which should not be taken until the definition of stroll is learned, is to perform the three tasks listed in the box. The reader is invited to tackle the problems one by one, without changing the order.

Fulfilling the first two requirements, and especially the second, can be a rather tough job. Also, those who managed to solve the first problem might notice that responding to the now seemingly obvious third question is not at all as straightforward as it looked.

Definition: *Promenade* is the set P of all natural numbers from 1 to 25 together with the following four functions:

$S(x) = x + 5$, allowed only for $x \in P$, $x \leqslant 20$
$N(x) = x - 5$, allowed only for $x \in P$, $x > 5$
$E(x) = x + 1$, allowed only for $x \in P$, $x \bmod 5 \neq 0$
$W(x) = x - 1$, allowed only for $x \in P$, $x \bmod 5 \neq 1$

Any composition of the above functions is called *a stroll*. We say that *stroll s leads from a to b* iff $s(a) = b$.

Example: the stroll $S \circ W^3 \circ S^2$ leads from 5 to 17:

$$(S \circ W^3 \circ S^2)(5) = (W^3 \circ S^2)(10) = (W^2 \circ S^2)(9) = \ldots S^2(7) = S(12) = 17$$

Tasks:

1. Give an example of a stroll which would lead from 11 to 3.
2. Find all the numbers which can be reached by strolls from 9 without using the steps N and W.
3. Without looking into the answer you gave to the question 1 above, give an example of a stroll from 11 to 3 once again.

Fig. 5. The experiment – Part 1.

Even if not all three tasks have been successfully accomplished, we shall turn now to the second part of our experiment. The reader is invited to study a new description of the concept of stroll, as presented in Figure 6, and to solve the three problems once again.

If everything goes well, there should be a considerable difference between the first and the second trial at solving the problems. By introspection, the reader could probably find out what the author has observed with particular clarity whenever the experiment was carried out in a classroom: with the second kind of definition (the one presented in Figure 6) all three tasks become much easier. What is it about this new representation that makes such difference? It is probably its distinctly structural character: the "promenade", which was first regarded as nothing but a bunch of algorithms, has now been combined into an easily manipulable object-like structure; the "stroll", which according to the first definition was a computational procedure, can now be viewed as a path of a constant shape – as a polygonal line. It is important to notice that no information has been added when the shift from the operational to the structural approach was made: the computational processes were caught into a static construct just like water is frozen in a piece of ice.

In a quest for a more insightful explanation of the role of structural conceptions, it is time now to make use of the theory of cognitive schemata. Our example shows in a convincing, if somewhat simplified way, what could be figured out also on the grounds of purely theoretical considerations: the operationally conceived information, although absolutely indispensable and seemingly sufficient for problem-solving, cannot be easily processed. This kind of information can only be stored in unstructured, sequential cognitive schemata, which are inadequate for the rather modest dimensions of human working memory. Consequently, the purely operational ideas must be processed in a piecemeal, cumbersome manner, which may lead to a great cognitive strain and to a disturbing feeling of only local – thus insufficient – understanding. Naturally, such strain would be totally counter-productive for anybody trying to solve a complex problem. For instance, think about the difficulty you must have experienced when still endowed with only operational definitions and trying to perform the first two tasks in our experiment; or imagine how hard it would be to solve an advanced word problem in a "rhetorical" manner, without using algebraic symbols. It should also be pointed out that in the sequential cognitive schema there is hardly a place for assimilation of new knowledge, or for what is usually called meaningful learning. That is probably why even our third task, which required *recalling* the answer to the first question, seemed quite difficult as long as only the operational representation was available.

It is the static object-like representation which squeezes the operational information into a compact whole and turns the cognitive schema into a more convenient structure. To grasp the nature and the implications of such change, compare schemata A and B in Figure 7 (naturally, I am not trying to imply that these schemata are anything like faithful images of "real" mental structures in which our knowledge is stored; I use the pictures just as a convenient means for elucidating the technical aspects of our claims). From a purely technical point of view, the mathematical objects are the upper "nodes" in the hierarchical schema resulting from reification. Each of them serves as a single item in the catalogue of our mind. For a cognizing person, they function like simplified pictures or symbols, which can be seized at one glance and may be used instead of "the real thing" (the corresponding process) at certain stages of problem-solving. Naturally, more often than not, these abstract constructs can only be seen with our mind's eyes; most regrettably, very rarely are we as fortunate as in the case of our promenade and strolls, where it was possible to actually draw the "reifying" pictures and symbols on the paper.

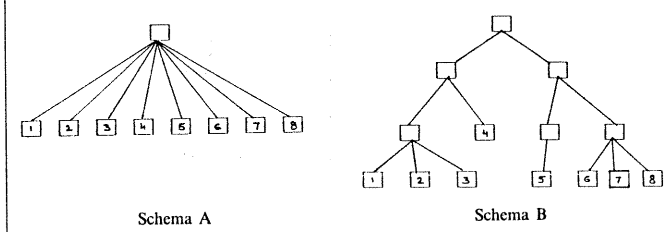
Definition: *Promenade P* is the graph presented below.

```
 1 — 2 — 3 — 4 — 5
 |   |   |   |   |
 6 — 7 — 8 — 9 —10
 |   |   |   |   |
11 —12 —13 —14 —15
 |   |   |   |   |
16 —17 —18 —19 —20
 |   |   |   |   |
21 —22 —23 —24 —25
```

If x is a node in P then $S(x)$, $N(x)$, $W(x)$ and $E(x)$ are the adjacent nodes, placed south, north, west, and east to x, respectively.

Stroll is defined like in box 5.

Fig. 6. The experiment – Part 2.



Any information can be stored in many different schemata. For example, the two schemata pictured in this figure contain the same information (represented by (1, 2, 3, 4, 5, 6, 7, 8)). Schema A is sequential, shallow and wide. As a result of reification it can be reorganized into a deeper and narrower structure, such as Schema B. With the new organization, all cognitive processes (retrieval and storing) become much faster.

Fig. 7. Different organizations of a hierarchical schema.

Both problem-solving and learning processes may be effectively "navigated" by the help of these compact, if not detailed, overall representations, just as movements of a ship are controlled and directed with maps.

While tackling a genuinely complex problem, we do not always get far if we start with concrete operations; more often than not it would be better to turn first to the structural version of our concepts. These upper-level representations provide us with a "general view", so we can use our system of abstract objects just like a person looking for information uses a catalogue; or like anybody trying to get to a certain street consults a map before actually going there. In other words, in problem-solving processes the compact abstract entities serve as pointers to more detailed information. Thus, almost any mathematical activity may be seen as an intricate interplay between the operational and the structural versions of the same mathematical ideas: when a complex problem is being tackled, the solver would repeatedly switch from one approach to the other in order to use his knowledge as proficiently as possible. Paraphrasing Henrici (1974), who spoke about dialectic and algorithmic mathematics, we may say that "the structural approach invites contemplation; the operational approach invites action; the structural approach generates insight; the operational approach generates result". An excellent illustration for the object-navigated problem-solving process can be found in Hadamard's account of his own thinking (Hadamard, 1949, pp. 76–7): in a most persuasive way this mathematician tells the reader how he recapitulates an argument for the existence of infinitely many prime numbers, through switching back and forth from objects (numbers and sets of numbers) to the underlying computations.

Let us list now the beneficial effects reification can have on *learning*. As explained before, formation of a structural conception means reorganizing the cognitive schema by adding new layers – by turning sequential aggregates into hierarchical structures. Obviously, the deeper and narrower the hierarchy, the greater the capacity of the schema. To have a better idea about the change resulting from the restructuring, let us compare the two schemata presented in Figure 7. In schema A, new items can hardly be added because the number of "sons" of the upper-level node is already quite large. In contrast, no node in schema B has yet been "saturated", since the average quantity of "sons" does not surpass the "magic number 7 ± 2" (according to Miller, 1956, this is the maximal number of chunks of information that can be kept simultaneously in our working memory). Thus, within the structural approach more room is available for inserting new information. As a result, learning becomes more effective, more meaningful. Also, the retrieval processes become faster when the necessary information is stored in hierarchical tree-shaped structures. The reader might be able to observe the difference while tackling again the third task in Figure 5.

We are now in a position to give a tentative answer to the question about the necessity of a structural approach asked at the beginning of this section: in the light of the explanations and examples given so far it seems that without the abstract objects all our mental activity would be more difficult. Since we are not super-computers, we just could not get along with very complex processes without breaking them into small pieces and without

squeezing each part into a more manageable whole. In other words, the distance between advanced computational processes and the concrete material entities which are the objects of the most elementary processes (such as counting) is much too large to be grasped by us in its totality. We overcome this difficulty by creating intervening abstract objects which serve us as a kind of way-stations in our intellectual journeys. These compact cognitive entities effectively shield our working memory against overflow. Abstract objects, neatly organized into a well-structured cognitive schema, were probably what allowed Poincaré (1952, p. 51) to make the following declaration: "... I can perceive the whole of the [lengthy mathematical] argument at a glance. [Thus] I need no longer be afraid of forgetting one of the elements; each of them will place itself naturally in the position prepared for it, without my having to make any effort of memory."

At certain stages of knowledge formation (or acquisition) the absence of a structural conception may hinder further development. As the amount of information grows, the old schema may become saturated and practically impervious to any enrichment. It was certainly not by pure chance that the transition from rhetorical to symbolic algebra – a transition from operational to structural approach in computational mathematics – occurred in the sixteenth century. And it was not just a historical accident that several different systems of symbols were invented almost simultaneously by independently working mathematicians. By that time, too great a complexity of computational processes brought the rhetorical algebra to a stalemate and practically put an end to its development. Looking back even further, we can venture a conjecture that the absence of structural representations (thus structural conceptions) was one of the factors that slowed down the development of computational science in Ancient Greece and caused algebra's falling behind geometry for centuries.

I have just presented the structural thinking as a very powerful weapon against the limitations of our working memory. At a less technical, more philosophical level, we can say that in mathematics, transition from processes to abstract objects enhances our sense of understanding mathematics. After all, reification increases problem-solving and learning abilities, so the more structural our approach, the deeper our confidence in what we are doing. At least some of the readers may be able to convince themselves about the accuracy of this claim by recalling the flash of enlightenment they probably experienced when presented with the structural definition of the concept of stroll. To sum up, structural conception is probably what underlies the relational understanding, defined by Skemp (1976) as "knowing both what and why to do", or having *both rules and reasons*. Purely operational approach would usually give no more than instrumental understanding, once presented by Skemp as having *rules without reasons*. It should immediately be stressed that this somewhat disdainful description (which apparently was revised by Skemp himself after a deeper thought was given to the subject) does not do full justice to the kind of understanding which is both valuable and inevitable at certain stages of learning. This claim will be clarified and substantiated in the last part of this section. Before I do this, however, I shall push Skemp's original idea a little further, and talk about a third kind of understanding: *reasons without rules*. What I mean by that is a kind of purely intuitive understanding, attained in those rare cases when the vague structural conception is achieved before the operational basis has fully developed. This is probably the kind of understanding mathematicians had when the earliest versions of the concept of function were introduced. As can be learned from the history, having "reasons without rules" may be not enough for creating a fully-fledged mathematical theory, but it is certainly most helpful in discovering theorems and in deciding about directions of future development.

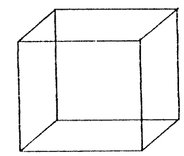
The roles and the features of operational and structural conceptions have been schematically summarized in Figure 9.

4.3. *The Inherent Difficulty of Reification*

After showing the importance of structural conceptions, we should now return to the question asked in the previous section: why is reification obviously so very difficult? Why mathematicians themselves needed several centuries to arrive at fully structural versions of the most basic concepts, such as number or function?

The problem will seem less puzzling if we remind ourselves that reification is an ontological shift, a qualitative jump. Such conceptual upheaval is always a rather complex phenomenon, especially when it is accompanied by subtle alternations of meaning and applications (which is usually the case; for example, the structural set-theoretic definition of function considerably extended the scope of the notion). The ability to see something familiar in a totally new way is never easy to achieve. The difficulties arising when a process is converted into an object are, in a sense, like those experienced during transition from one scientific paradigm to another; or – to make a less ambitious but perhaps more persuasive comparison – they are like the obstacles encountered by a person looking at a cube pictured on a paper and trying to perceive it as if it was presented from a different angle (see Figure 8).

Our three-phase schema of concept formation would shed more light on these difficulties. According to the model, reification of a given process occurs



If you feel that the cube presented here is viewed from above – try to convince yourself that it is seen from below; and vice versa.

Fig. 8. The cube.

simultaneously with the interiorization of higher-level processes. For example, in the case of negative numbers, the reification becomes most likely when the algebraic operations on this kind of numbers are, at least partially, interiorized. Indeed, what leads to recognizing such operations like $2-5$ and $0-6$ as numbers is the similarity of the algorithms involving these operations to those performed on more familiar numbers (namely the fact that adding and multiplying two negative or miscellaneous quantities is very much like adding or multiplying positive numbers). In order to notice the likeness, however, one has to achieve some mastery in the operations on negative quantities. Similarly, in order to see a function as an object, one must try to manipulate it as a whole: there is no reason to turn process into object unless we have some higher-level processes performed on this simpler process. But here is a vicious circle: on one hand, without an attempt at the higher-level interiorization, the reification will not occur; on the other hand, existence of objects on which the higher-level processes are performed seems indispensable for the interiorization – without such objects the processes must appear quite meaningless. In other words: *the lower-level reification and the higher-level interiorization are prerequisite for each other*!

It seems that from the psychological point of view, what has just been said can have quite important implications. The "vicious circle" of reification will help us now explain why for so many people "mathematics at school [is] a collection of unintelligible rules which, if memorized and applied correctly, [lead] to 'the right answer'"(Skemp, 1971, p. 3).

The question of the order in which different mathematical abilities should be developed has always drawn the attention of psychologists and educators. As Kilpatrick (1988) put it, "One of the most venerable and vexing issues in mathematics education concerns the trade-off between *proficiency* and *comprehension*, between promoting the smooth performance of mathematical procedure and developing understanding of how and why that procedure works and what it means . . .". Different answers have been given to this question by different psychological schools. The general stance taken by behaviourists implied that skills should be learned, whereas the more recent theories have seemed to prefer the view that "drill [should be] recommended [only] when ideas and processes already understood are to be practiced to increase proficiency" (Brownell, 1935, p. 19). According to our model of concept development, however, no clear order of abilities can be established. The thesis of the "vicious circle" implies that one ability cannot be fully developed without the other: on one hand, a person must be quite skillful at performing algorithms in order to attain a good idea of the "objects" involved in these algorithms; on the other hand, to gain full technical mastery, one must already have these objects, since without them the processes would seem meaningless and thus difficult to perform and to remember. For example, the concept of complex number cannot be reified until a person is able to make computations involving these numbers; at the same time, however, conceiving such constructs like i or $3+2i$ as fully-fledged numbers (and not just symbols for operations "without result") is a prerequisite for being proficient in manipulating them. This statement is in line with the results of a large-scale study carried out among 13- and 17-year olds in United States (Carpenter et al., 1980). According to the findings, "the development of a skill is closely tied to understanding the concept underlying the skill".

In the light of these claims it should not surprise us that ever so often, "[s]tudents appear to be learning many mathematical skills at a rote manipulation level and do not understand the concepts underlying the computation" (ibid.) For instance, pupils can be quite successful in computations involving fractions in spite of being unable to treat fractions as numbers. Because of the complex nature of their mutual dependence, it seems inevitable that in the process of learning, student's understanding – this feeling of competence and mastery which accompanies the ability of "seeing" abstract structures – will sometimes drop behind the technical proficiency. This implies that in some cases the learner must put up with a certain amount of "mechanical" drill accompanied by doubts about meaning and by a feeling of insufficient (instrumental only) understanding. Even professional mathematicians cannot escape this fate, and they sometimes complain about the necessity of struggling hard for meaning of ostensibly simple ideas. Halmos (1985a) recalls the times when he was a university student: in spite of "working furiously" on the concept of lambda-matrices, he "didn't really begin to understand what the subject was about till four or five years later" (pp. 40–1). We may conjecture that it was first condensation, and then reification that took so long.

More often than not, both students and teachers fail to acknowledge the fact which is one of the most important implications of our three-phase schema: insight cannot always be expected as an immediate reward for a person's direct attempts to fathom a new idea. The reification, which brings relational understanding, is difficult to achieve, it requires much effort, and it may come when least expected, sometimes in a sudden flash. In his pioneering book on the psychology of mathematics, Hadamard (1949) mentions an "illumination effect" which may occur after a period of intensive work followed by days of rest ("incubation period").

From the educational point of view, the main problem with this delay in reification and with the resulting periods of doubts about meaning is that they may bring a permanent harm – a life-long apprehension of mathematics and a conviction that it cannot be learned. Some people may be unable to recover from the shock caused already by the first encounter with the problematic situation. Those who are not prepared to actively struggle for meaning (for reification) would soon resign themselves to never understanding mathematics. The ability of orchestrating lower-level reification with higher-level interiorization in a subtle, painless manner may be one of the most important features which make a person capable of coping with mathematics. "Mathematically fittest", even if they feel at times somewhat shaky in their understanding, seem to have enough motivation, patience, and intellectual discipline to put up with this situation in a trustful anticipation of a salutary insight. It is certainly what can be learned from

	Operational conception	Structural conception
General characteristic	a mathematical entity is conceived as a product of a certain process or is identified with the process itself	a mathematical entity is conceived as a static structure – as if it was a real object
Internal representations	is supported by verbal representations	is supported by visual imagery
Its place in concept development	develops at the first stages of concept formation	evolves from the operational conception
Its role in cognitive processes	is necessary, but not sufficient, for effective problem-solving & learning	facilitates all the cognitive processes (learning, problem-solving)

Fig. 9. Operational and structural conceptions – summary.

history. As Jourdain (1956) put it, "when logically-minded men objected" to the "absurd" notions of negative and imaginary (complex) numbers, "mathematicians simply ignored them and said 'Go on; faith will come to you'". Those who could see the inner beauty of the idea thought that the new numbers, "though apparently uninterpretable and even self-contradictory, *must* have logic. So they [the numbers] were used with a faith that was almost firm and was justified much later" (pp. 29–30).

In the light of the "vicious circle" thesis, it seems that in the search for an improvement in mathematics education we should focus on the question what, and how much, can be done to unravel the harmful tangle and to stimulate reification. Also, we ought to ask ourselves what means should be used to ensure that the students go safely through these doubts-about-meaning periods, when they still feel certain uneasiness about the object they have to manipulate and sense, as a result, that their understanding is far from satisfactory. Another article will be devoted to these issues.

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APPLIED MATHEMATICAL PROBLEM SOLVING, MODELLING, APPLICATIONS, AND LINKS TO OTHER SUBJECTS – STATE, TRENDS AND ISSUES IN MATHEMATICS INSTRUCTION[1]

ABSTRACT. The paper will consist of three parts. In *part I* we shall present some background considerations which are necessary as a basis for what follows. We shall try to clarify some basic concepts and notions, and we shall collect the most important arguments (and related goals) in favour of problem solving, modelling and applications to other subjects in mathematics instruction. In the main *part II* we shall review the present state, recent trends, and prospective lines of development, both in empirical or theoretical research and in the practice of mathematics instruction and mathematics education, concerning (applied) problem solving, modelling, applications and relations to other subjects. In particular, we shall identify and discuss four major trends: a widened spectrum of arguments, an increased globality, an increased unification, and an extended use of computers. In the final *part III* we shall comment upon some important issues and problems related to our topic.

I. BACKGROUND CONSIDERATIONS

I.1. Clarification of Basic Concepts and Notions

We shall commence our paper by clarifying some basic concepts and notions such as "problem" and "problem solving", "model" and "modelling", "application", "applying" and "applied mathematics". By no means are we pretending to present an exhaustive epistemological treatment of these concepts. Rather does this section present a pragmatic attempt to give some working definitions which are not claimed to be very original but which are useful for the following parts of our paper.

By a *problem* we mean a situation which carries with it certain open questions that challenge somebody intellectually who is not in immediate possession of direct methods/procedures/algorithms etc. sufficient to answer the questions. This notion of a problem is apparently relative to the persons involved; so, what to one person is a problem may be an exercise to someone else. As to *mathematical* problems, there are two kinds: It is characteristic of an *applied* mathematical problem that the situation and the questions defining it belong to some segment of the real world and allow some mathematical concepts, methods and results to become involved. By *real world* we mean the "rest of the world" outside mathematics, i.e. school or university subjects or disciplines different from mathematics, or everyday life and the world around us. In contrast, with a *purely*

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