Monotone Likelihood Ratio (MLR)

Definition

We say that a family of distributions $\mathcal{F} = \{f(x;\theta) : \theta \in \Theta\}$ has the MLR property, if:

a) the support S_f is independent of θ

b) for $\theta_1 \neq \theta_2$ the corresponding $f(x;\theta_1)$, $f(x;\theta_2)$ are not identical

c) there exists a statistic $T(\underline{x})$ such that for all $\theta_1 < \theta_2$ the ratio $\frac{f(\underline{x},\theta_1)}{f(\underline{x},\theta_2)}$ of the joint PDFs is an increasing function of $T(\underline{x})$

Example

For $\mathcal{F} = \{\text{PDFs of Poisson}(\theta)\}$: for all $\theta_1 < \theta_2$ we have

$$\frac{f(\underline{x},\theta_1)}{f(\underline{x},\theta_2)} = \frac{\prod_{i=1}^{n} e^{-\theta_2} \cdot \theta_2^{x_i}/x_i!}{\prod_{i=1}^{n} e^{-\theta_2} \cdot \theta_2^{x_i}/x_i!} = e^{-n(\theta_2-\theta_1)} \left(\frac{\theta_2}{\theta_1}\right)^{\sum_{i=1}^{n} x_i},$$

which is an increasing function of $T(\underline{x}) = \sum_{i=1}^{n} x_i$, so $\mathcal{F}$ has the MLR property.

Theorem

If the family of distributions $\mathcal{F} = \{f(x;\theta) : \theta \in \Theta\}$ has the MLR property with statistic $T(\underline{x})$, then the test

i) $T(\underline{x}) > w$ for $(x_1, \dots, x_n) \in G$

ii) $T(\underline{x}) < w$ for $(x_1, \dots, x_n) \in G'$

iii) $P((x_1, \dots, x_n) \in G \mid H_0) = a$

is u.m.p.t for $H_0: \theta = \theta_0$ vs $H_1: \theta > \theta_0$, with error a

Proof

For the test $H_0: \theta = \theta_0$ vs $H_1: \theta = \theta_1$, where $\theta_1 > \theta_0$, from the N-P lemma we have:

$\frac{L_0}{L_1} < K$ for $(x_1, \dots, x_n) \in C$. Given the MLR property, the ratio $\frac{L_0}{L_1}$ is a decreasing function of $T(\underline{x})$, so $\frac{L_0}{L_1} < K$ is equivalent to $T(\underline{x}) > w$ and $P(T(\underline{x}) > w \mid H_0) = a$.

The above relations are independent of θ, hence the test is u.m.p.t for $H_0: \theta = \theta_0$ vs $H_1: \theta > \theta_0$

Theorem

Let $f(x;\theta)$ belong to the single-parameter EFD with PDF

$$f(x;\theta) = B(\theta) \cdot \exp\{\eta(\theta) \cdot T(x)\} \cdot h(x)$$

i) If $\eta(\theta)$ is increasing then $f(\cdot)$ has the MLR property with $T(\underline{x}) = \sum_{i=1}^{n} T(x_i)$

ii) If $\eta(\theta)$ is decreasing then $f(\cdot)$ has the MLR property with $T(\underline{x}) = -\sum_{i=1}^{n} T(x_i)$

Proof

We have $f(\underline{x};\theta) = \prod_{i=1}^{n} f(x_i;\theta) = (B(\theta))^n \cdot \exp\{\eta(\theta) \cdot \sum_{i=1}^{n} T(x_i)\} \cdot \prod_{i=1}^{n} h(x_i) =$

$= B^*(\theta) \cdot \exp\{\eta(\theta) \cdot T(\underline{x})\} \cdot h^*(\underline{x})$

so $\frac{f(x;\theta_2)}{f(x;\theta_1)} = \frac{B^*(\theta_2)}{B^*(\theta_1)} \cdot \exp\{(\eta(\theta_2) - \eta(\theta_1))\, T(\underline{x})\}$

→ if $\eta(\theta)$ is increasing and $\theta_2 > \theta_1$ then $\eta(\theta_2) - \eta(\theta_1) > 0$ and the ratio is an increasing function of $T(\underline{x}) = \sum_{i=1}^{n} T(x_i)$

→ if $\eta(\theta)$ is decreasing and $\theta_2 > \theta_1$ then $\eta(\theta_2) - \eta(\theta_1) < 0$ and the ratio is an increasing function of $T(\underline{x}) = -\sum_{i=1}^{n} T(x_i)$

Hence, for the test $H_0: \theta = \theta_0$ vs $H_1: \theta > \theta_1$ we have:

Ⓐ i) $T(\underline{x}) > w$, if $(x_1,\dots,x_\nu) \in G$

ii) $T(\underline{x}) < w$, if $(x_1,\dots,x_\nu) \in G'$

iii) $P((x_1,\dots,x_\nu) \in G \mid H_0) = a$

if $\eta(\theta)$ increasing

Ⓑ i) $T(\underline{x}) > w$, if $(x_1,\dots,x_\nu) \in G$

ii) $T(\underline{x}) < w$, if $(x_1,\dots,x_\nu) \in G'$

ii) $P((x_1,\dots,x_\nu) \in G \mid H_0) = a$

if $\eta(\theta)$ decreasing

Example

Let $X_1,\dots,X_\nu$ of $Exp(\theta)$, $f(x;\theta) = \theta \cdot e^{-\theta x}$, $\theta > 0$, $x > 0$.

Test $H_0: \theta = \theta_0$ vs $H_1: \theta < \theta_0$

Solution

$Exp(\theta)$ belong to the EFD with $\eta(\theta) = -\theta$, which is decreasing, so for $T(\underline{x}) = \sum_{i=1}^{\nu} T(x_i) = \sum_{i=1}^{\nu} x_i$ the likelihood ratio will be decreasing as T increases. Hence, $G: T(x) = \sum_{i=1}^{\nu} x_i > w$

But $X_i \sim Exp(\theta) \Rightarrow \sum_{i=1}^{\nu} X_i \sim Gamma(\nu, \theta)$ and its PDF is

$$f(x;\theta) = \frac{\theta^\nu}{\Gamma(\nu)} \cdot x^{\nu-1} \cdot e^{-\theta x}$$

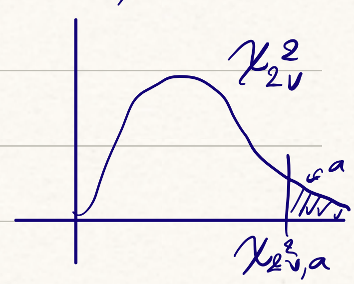
Also $2\theta T = 2\theta \sum_{i=1}^{\nu} X_i \sim Gamma\left(\frac{2\nu}{2}, \frac{1}{2}\right)$ and its PDF is

$$f(x) = \frac{(1/2)^\nu}{\Gamma(\nu)} \cdot x^{\nu-1} \cdot e^{-\frac{1}{2}x}$$

We want $P\left(\sum_{i=1}^{\nu} X_i > w \mid H_0\right) = a \Leftrightarrow P\left(2\theta_0 \sum_{i=1}^{\nu} X_i > 2\theta_0 \cdot w\right) > a \Leftrightarrow$

$\Leftrightarrow 2\theta_0 \cdot w = \chi^2_{2\nu,a} \Leftrightarrow w = \dfrac{\chi^2_{2\nu,a}}{2\theta_0}$

So $G = \left\{(x_1,\dots,x_\nu) : T(\underline{x}) = \sum_{i=1}^{\nu} x_i > \dfrac{\chi^2_{2\nu,a}}{2\theta_0}\right\}$



and the power function is:

$\Pi(\theta) = P_\theta(G) = P_\theta\left(\sum_{i=1}^{\nu} X_i > w\right) =$

$= P_\theta\left(2\theta T > \dfrac{\theta \chi^2_{2\nu,a}}{\theta_0}\right) = 1 - P\left(2\theta T \leq \dfrac{\theta \chi^2_{2\nu,a}}{\theta_0}\right) = 1 - F_{\chi^2_{2\nu}}\left(\dfrac{\theta}{\theta_0}\chi^2_{2\nu,a}\right)$