Example (One-sided vs one-sided)

Let $X_1, \dots, X_\nu$ r.s. of $N(\mu, 1)$ and $H_0: \mu \le \mu_0$ vs $H_1: \mu > \mu_0$

Find u.m.p.t with error a.

Solution

By the standard method we get $C = \{(x_1, \dots, x_\nu) \mid \bar{X} > w\}$, so we have u.m.p.t. with $\sup\limits_{\mu \le \mu_0} P(\bar{X} > w \mid H_0) = a \Rightarrow$

$$\Rightarrow \sup_{\mu \le \mu_0} P\left(\frac{\bar{x}-\mu}{1/\sqrt{\nu}} > \frac{w-\mu}{1/\sqrt{\nu}}\right) = a \Rightarrow \sup_{\mu \le \mu_0} P\left(Z > \frac{w-\mu}{1/\sqrt{\nu}}\right) = a \Rightarrow$$

$$\Rightarrow \sup_{\mu \le \mu_0} \Phi\left(\frac{\mu - w}{1/\sqrt{\nu}}\right) = a \Rightarrow \Phi\left(\frac{\mu_0 - w}{1/\sqrt{\nu}}\right) = a \Rightarrow \frac{\mu_0 - w}{1/\sqrt{\nu}} = -Z_a \Rightarrow$$

$$\Rightarrow w = \mu_0 + Z_a \cdot \frac{1}{\sqrt{\nu}}$$ So the c.r. is $C = \{(x_1, \dots, x_\nu) : \bar{X} > \mu_0 + Z_a \frac{1}{\sqrt{\nu}}\}$

Randomized Tests

Example

Suppose we have a coin and we toss it 6 times. Let $S = \sum_{i=1}^{6} X_i$, where $X_i = \begin{cases} 0, & i\text{-toss heads} \\ 1, & i\text{-toss tails} \end{cases}$. Test $H_0: p = 0,5$ vs $H_1: p = 0,75$

Solution

We are looking for a c.r. $C: S > w$ with error $a = 0.05$, so $P(S > w \mid H_0) = a = 0,05$. We have:

$$P(S > w \mid H_0) = \sum_{k=w+1}^{6} \binom{6}{k} 0,5^k 0,5^{6-k} = 0,5^6 \cdot \sum_{k=w+1}^{6} \binom{6}{k} = a = 0,05$$

For $w = 5$: $P(S > w \mid H_0) = 0,5^6 \cdot \binom{6}{6} = 0,016 < 0,05$

$w = 4$: $P(S > w \mid H_0) = 0,5^6 \left(\binom{6}{6} + \binom{6}{5}\right) = 0,106 > 0,05$

We consider the following rule:

i) If $s > 5$ we reject H_0

ii) If $s < 5$ we don't reject H_0

iii) If $s = 5$ then we Keep (don't reject) H_0 with probability $1-r$, where $a = 0{,}05 = P(S=6) + r \cdot P(S=5) \Rightarrow 0{,}05 = 0{,}016 + r \cdot 0{,}093$
$\Rightarrow r = \frac{0{,}034}{0{,}093} = \frac{11}{30}$.

Hence, when $S=5$ we run a random experiment with probability $r = \frac{11}{30}$ to reject H_0 and probability $1-r = \frac{19}{30}$ to not reject H_0, so that $P(\text{Type I error}) = 0{,}05$