Example

Let $X_1, \dots, X_\nu$ r.s. of $N(\theta, 1)$. Prove that there exist <u>no</u> u.m.p.t. for $H_0: \theta = \theta_0$ <u>vs</u> $H_1: \theta \neq \theta_0$ with error α.

Solution

Let $\theta_1 \neq \theta_0$, from the N-P Lemma we have:

$$\frac{L_0}{L_1} \le k \Leftrightarrow \frac{(2\pi)^{-\nu/2} \cdot \exp\{-\frac{1}{2} \cdot \sum_{i=1}^{\nu}(x_i - \theta_0)^2\}}{(2\pi)^{-\nu/2} \cdot \exp\{-\frac{1}{2} \cdot \sum_{i=1}^{\nu}(x_i - \theta_1)^2\}} \le k \Leftrightarrow$$

$$\Leftrightarrow \exp\left\{-\cancel{\frac{\sum x_i^2}{2}} - \frac{\nu\theta_0^2}{2} + \theta_0 \sum_{i=1}^{\nu} x_i + \cancel{\frac{\sum x_i^2}{2}} + \frac{\nu\theta_1^2}{2} - \theta_1 \sum_{i=1}^{\nu} x_i\right\} \le k \Leftrightarrow$$

$$\Leftrightarrow -(\theta_1 - \theta_0)\sum_{i=1}^{\nu} x_i + \frac{\nu}{2}(\theta_1^2 - \theta_0^2) \le \log k \Leftrightarrow$$

$$\Leftrightarrow (\theta_1 - \theta_0) \cdot \sum_{i=1}^{\nu} x_i \ge -\log k + \frac{\nu}{2}(\theta_1^2 - \theta_0^2)$$

- If $\theta_1 > \theta_0$, then $\displaystyle\sum_{i=1}^{\nu} x_i \ge \frac{\frac{\nu}{2}(\theta_1^2 - \theta_0^2) - \log k}{\theta_1 - \theta_0} = W \Leftrightarrow \dots \Leftrightarrow$

$$\Leftrightarrow \frac{\sum x_i - \theta_0}{\sqrt{1/\nu}} \ge \frac{W - \theta_0}{\sqrt{1/\nu}} \Rightarrow W = \theta_0 + z_{\alpha/2} \cdot \sqrt{\frac{1}{\nu}}$$

z $\qquad$ $z_{\alpha/2}$

- If $\theta_1 < \theta_0$, then $\displaystyle\sum_{i=1}^{\nu} x_i \le \frac{\frac{\nu}{2}(\theta_1^2 - \theta_0^2) - \log k}{\theta_1 - \theta_0} = W \Leftrightarrow \dots \Leftrightarrow$

$$\Leftrightarrow W = \theta_0 - z_{\alpha/2} \cdot \sqrt{1/\nu}$$

The first case determines a m.p.c.r. for $H_0: \theta = \theta_0$ <u>vs</u> $H_1: \theta = \theta_1$ with $\theta_1 > \theta_0$ and the second for $\theta_1 < \theta_0$.
Hence, according to the definition, there exists no u.m.p.t.

Complex vs Complex

The N-P lemma cannot be used with complex hypotheses.

Let $H_0: \theta \in \Theta_0$ <u>vs</u> $H_1: \theta \in \Theta_1$, with $\Theta = \Theta_0 \cup \Theta_1$

Generalized likelihood ratio

$L_0 = \sup_{\theta \in \Theta_0} L(\theta)$ and $L_1 = \sup_{\theta \in \Theta_1} L(\theta)$

Instead of calculating $\frac{L_0}{L_1} = \frac{\sup_{\theta \in \Theta_0} L(\theta)}{\sup_{\theta \in \Theta_1} L(\theta)}$, we calculate $\frac{L_0}{L} = \frac{\sup_{\theta \in \Theta_0} L(\theta)}{\sup_{\theta \in \Theta_0 \cup \Theta_1} L(\theta)} \leq k$ and define the critical range C:

i) $\frac{L_0}{L} \leq k$ if $(x_1, \dots, x_v) \in C$

ii) $\frac{L_0}{L} > k$ if $(x_1, \dots, x_v) \in C$

iii) $P((x_1, \dots, x_v) \in C \mid H_0) = a$

Notes

⇝ The tests that are created via the generalized likelihood ratio are not necessarily u.m.p.t

⇝ If we have more than one parameters, then it is not obvious if a hypothesis is simple, e.g. if $H_0: \mu = \mu_0$ for $N(\mu, \sigma^2)$ is simple if σ^2 is known, but complex if not.

⇝ Since $\Theta = \Theta_0 \cup \Theta_1$, $L = \sup_{\theta \in \Theta_0 \cup \Theta_2} L(\theta) = \sup_{\theta \in \Theta} L(\theta) = L(\hat{\theta})$, where $\hat{\theta}$ the MLE.

Example

Let $X_1, \dots, X_v$ r.s. of $N(\mu, \sigma^2)$, where σ^2 is known. We want to find a critical range with error a for $H_0: \mu = \mu_0$ vs $H_1: \mu \neq \mu_0$

Solution

$$L_0 = L(\mu_0) = (2\pi\sigma^2)^{-\nu/2} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{\nu}(X_i-\mu_0)^2\right\}$$

$$L = \sup_{\mu\in\mathbb{R}} L(\mu) = L(\hat{\mu}) = L(\bar{X}) = (2\pi\sigma^2)^{-\nu/2} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{\nu}(X_i-\bar{X})^2\right\}$$

So $$\frac{L_0}{L} \le k \Leftrightarrow \frac{(2\pi\sigma^2)^{-\nu/2}\exp\left\{-\frac{1}{2\sigma^2}\sum(X_i-\mu_0)^2\right\}}{(2\pi\sigma^2)^{-\nu/2}\exp\left\{-\frac{1}{2\sigma^2}\sum(X_i-\bar{X})^2\right\}} \le k \Leftrightarrow$$

$$\Leftrightarrow \exp\left\{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{\nu}(X_i-\mu_0)^2 - \sum_{i=1}^{\nu}(X_i-\bar{X})^2\right)\right\} \le k \Leftrightarrow$$

$$\Leftrightarrow -\frac{1}{2\sigma^2}\left(\cancel{\sum_{i=1}^{\nu}X_i^2} + \nu\mu_0^2 - 2\mu_0\sum_{i=1}^{\nu}X_i - \cancel{\sum_{i=1}^{\nu}X_i^2} - \nu\bar{X}^2 + 2\bar{X}\sum_{i=1}^{\nu}X_i\right) \le \log k$$

$$\Leftrightarrow \nu(\mu_0^2 - \bar{X}^2) - 2\sum_{i=1}^{\nu}X_i(\mu_0 - \bar{X}) \ge -2\sigma^2\log k \Leftrightarrow$$

$$\Leftrightarrow (\mu_0 - \bar{X})(\nu\mu_0 - \nu\bar{X} - 2\nu\bar{X}) \ge -2\sigma^2\log k \Leftrightarrow$$

$$\Leftrightarrow (\mu_0-\bar{X})^2 \ge -\frac{2\sigma^2\log k}{\nu} \Leftrightarrow |\mu_0 - \bar{X}| \ge \sqrt{\frac{2\sigma^2\log k}{\nu}} = w$$

so $G = \{(X_1,\dots,X_\nu) : |\bar{X}-\mu_0| \ge w\}$

and we want $P(G \mid H_0) = a \Rightarrow P(|\bar{X}-\mu_0| \ge w) = a$ (1),

but $\bar{X} \sim N(\mu, \sigma^2/\nu)$, so (1) $\Rightarrow P\left(\left|\frac{\bar{X}-\mu_0}{\sqrt{\sigma^2/\nu}}\right| \ge \frac{w}{\sqrt{\sigma^2/\nu}}\right) = a \Rightarrow$

$\Rightarrow P\left(|Z| \ge w/\sqrt{\sigma^2/\nu}\right) = a$

We know that $Z \ge Z_{a/2}$ or $Z \le -Z_{a/2}$, so we reject H_0

if $\left|\frac{\bar{X}-\mu_0}{\sqrt{\sigma^2/\nu}}\right| = |Z| \ge Z_{a/2} \Rightarrow \bar{X} \ge \mu_0 + Z_{a/2}\frac{\sigma}{\sqrt{\nu}}$ or $\bar{X} \le \mu_0 - Z_{a/2}\frac{\sigma}{\sqrt{\nu}}$