Example (Continuation from last lecture

$X_1, \dots, X_n \sim N(\mu, 36)$, $H_0: \mu = 50$ vs $H_1: \mu = 55$.

b) For $n = 16$, $C: \bar{X} \geq 53$, find a.

Solution

$$a = P(\text{Type I error}) = P(\bar{X} \geq 53 \mid H_0) = P\left(\frac{\bar{X}-50}{\sqrt{36/16}} > \frac{53-50}{\sqrt{36/16}} \right) =$$

$$= P(Z > 2) = 1 - P(Z \leq 2) = 1 - \Phi(2) = 0,0228$$

Example

Let $X_1, \dots, X_n$ r.s of $N(\theta, 1)$. $H_0: \theta = 0$ vs $H_1: \theta = -1$

Show that if $n = 25$, $a = 0,05$, then the power of the test is $0,999$

Solution

$$L(\theta) = \prod_{i=1}^{n} f(x_i; \theta) = (2\pi)^{-\frac{n}{2}} \cdot \exp\left\{ -\frac{1}{2} \sum_{i=1}^{n} (x_i - \theta)^2 \right\}$$

$$\frac{L_0}{L_1} = \frac{L(\theta_0)}{L(\theta_1)} = \frac{(2\pi)^{-n/2} \cdot \exp\left\{ -\frac{1}{2} \sum_{i=1}^{n} (x_i - 0)^2 \right\}}{(2\pi)^{-n/2} \cdot \exp\left\{ -\frac{1}{2} \sum_{i=1}^{n} (x_i + 1)^2 \right\}} = \exp\left\{ -\frac{1}{2} \left[\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} (x_i + 1)^2 \right] \right\} =$$

$$= \exp\left\{ -\frac{1}{2} \left[\sum x_i^2 - \sum x_i^2 - 2\sum x_i - n \right] \right\} = \exp\left\{ \frac{n}{2} + \sum x_i \right\}$$

so $\frac{L_0}{L_1} \leq k \Leftrightarrow \exp\left\{ \frac{n}{2} + \sum_{i=1}^{n} x_i \right\} \leq k \Leftrightarrow \frac{n}{2} + \sum x_i \leq \log k \Leftrightarrow$

$\Leftrightarrow \bar{X} \leq \frac{\log k}{n} - \frac{1}{2} = w$. So $C = \{(x_1, \dots, x_n) : \bar{X} \leq w\}$

$$\left. \begin{aligned} P(C \mid H_0) &= P(\bar{X} \leq w \mid \theta = 0) = P\left(\frac{\bar{X}-0}{\sqrt{1/25}} \leq \frac{w-0}{\sqrt{1/25}} \right) = 0,05 \\ &P(Z < -Z_{0,05}) = 0,05 \end{aligned} \right\} \Rightarrow$$

$\Rightarrow 5w = -Z_a \Rightarrow w = -0,329$

Power Function: $\Pi(\theta) = P(C \mid \theta) = P_\theta(C)$ $\begin{cases} \xrightarrow{H_0} a \\ \xrightarrow{H_1} 1-\beta \end{cases}$

$$\Pi(\theta_1) = P_{\theta_1}(C) = P(\bar{X} \leq -0,329 \mid \theta = -1) =$$

$$= P\left(\frac{\bar{X}+1}{\sqrt{1/25}} \leq \frac{-0,328+1}{\sqrt{1/25}} \right) = P(Z \leq 3,3) = \Phi(3,3) = 0,999$$

Generalization

H_0 and H_1 do not necessarily belong to the same distribution, nor are the random variables $X_1, \dots, X_v$ independent. Hence, if H_0 is the simple hypothesis that the joint PDF of the sample is $g(\underline{x})$ and H_1 is the simple hypothesis that the joint PDF is $h(\underline{x})$, then for the m.p.c.r. C of the test H_0 vs H_1 we have:

i) $\frac{g(\underline{x})}{h(\underline{x})} \leq k$ for $(x_1, \dots, x_v) \in C$

ii) $\frac{g(\underline{x})}{h(\underline{x})} > k$ for $(x_1, \dots, x_v) \notin C$

iii) $P((x_1, \dots, x_v) \in C \mid H) = a$

Example

Let $X_1, \dots, X_v$ r.s. with PDF $f(x) > 0$, $x = 0, 1, \dots$

Test: $H_0: g(x) = \frac{e^{-1}}{x!}$, $x = 0, 1, \dots$

$H_1: h(x) = (1/2)^{x+1}$, $x = 0, 1, \dots$

We have: $\frac{g(\underline{x})}{h(\underline{x})} = \frac{\prod_{i=1}^{v} e^{-1}/x_i!}{\prod_{i=1}^{v} (\frac{1}{2})^{x_i+1}} = \frac{(2e^{-1})^v \cdot 2^{\sum x_i}}{\prod_{i=1}^{v} x_i!} \leq k \Leftrightarrow$

$\Leftrightarrow v \log(2e^{-1}) + \sum_{i=1}^{v} x_i \log 2 - \log \prod_{i=1}^{v} (x_i!) \leq \log k \Leftrightarrow$

$\Leftrightarrow \sum_{i=1}^{v} x_i \cdot \log 2 - \sum_{i=1}^{v} \log(x_i!) \leq \log k - v \log(2e^{-1})$

Application: $k = 1$, $v = 1$

We have $\frac{g(x)}{h(x)} < 1 \Leftrightarrow \frac{2^{x_1}}{x_1!} \leq \frac{e}{2}$, so $C = \{x_1 : x_1 = 0, 3, 4, 5, \dots\}$

$a = P(\text{Type I error}) = P(C \mid H_0) = 1 - P(C' \mid H_0) = 1 - P(x_1 = 1, 2 \mid H_0) =$

$= 1 - e^{-1} - \frac{e^{-2}}{2!} = 0{,}445$

The power of the test is $P(C \mid H_1) = 1 - P(C' \mid H_1) =$

$= 1 - P(x_1 = 0, 1 \mid H_1) = 1 - \frac{1}{2^2} - \frac{1}{2^3} = 0{,}625$

Simple vs Complex

Definition

If we are testing $H_1 : \theta = \theta_0$ (simple) versus a complex H_1, the test is called uniformly most powerful test (u.m.p.t) if it is the most powerful test versus every single alternative.

Example (One-sided test)

Let $X_1, \dots, X_\nu$ r.s of $N(0,\theta)$, $\theta > 0$. Prove that there exists a u.m.p.t. with given error a for $H_0 : \theta = \theta_0$ vs $H_1 : \theta > \theta_1$

Solution

Suppose $\theta_1 > \theta_0$. We test $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$

$$L(\theta) = \prod_{i=1}^{\nu} f(x_i ; \theta) = \prod_{i=1}^{\nu} \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{1}{2\theta} x_i^2} = (2\pi\theta)^{-\nu/2} \cdot \exp\left\{-\frac{1}{2\theta} \sum_{i=1}^{\nu} x_i\right\}$$

N-P Lemma: $\frac{L_0}{L_1} \leq K \iff \dfrac{(2\pi\theta_0)^{-\nu/2} \exp\left\{-\frac{1}{2\theta_0} \sum_{i=1}^{\nu} x_i^2\right\}}{(2\pi\theta_1)^{-\nu/2} \exp\left\{-\frac{1}{2\theta_1} \sum_{i=1}^{\nu} x_i^2\right\}} =$

$$= \left(\frac{\theta_1}{\theta_0}\right)^{\nu/2} \exp\left\{\frac{-1}{2\theta_0} \sum x_i^2 - \frac{1}{2\theta_1} \sum x_i^2\right\} = \left(\frac{\theta_1}{\theta_0}\right)^{\nu/2} \exp\left\{-\underbrace{\frac{\theta_1 - \theta_0}{2\theta_1\theta_0}}_{>0} \sum_{i=1}^{\nu} x_i^2\right\} \leq K$$

$$\iff \frac{\nu}{2} \log \frac{\theta_1}{\theta_0} - \frac{\theta_1 - \theta_0}{2\theta_0\theta_1} \sum_{i=1}^{\nu} x_i^2 \leq \log K \iff$$

$$\iff \sum_{i=1}^{\nu} x_i^2 \geq \frac{2\theta_0\theta_1}{\theta_1 - \theta_0} \left[\frac{\nu}{2} \log\left(\frac{\theta_1}{\theta_0}\right) - \log K\right] = w$$

So $C = \{(x_1, \dots, x_\nu) : \sum_{i=1}^{\nu} x_i \geq w\}$ and we need:

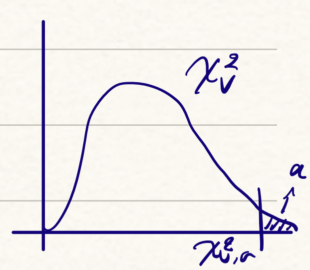
$P((x_1, \dots, x_\nu) \in C \mid H_0) = a \iff X_i \sim N(0,\theta) \iff \frac{X_i}{\sqrt{\theta}} \sim N(0,1)$

$\iff \frac{X_i^2}{\theta} \sim \chi_1^2 \implies Y = \sum_{i=1}^{\nu} \frac{X_i^2}{\theta} = \frac{\sum X_i^2}{\theta} \sim \chi_\nu^2$

and so $P(C \mid H_0) = P(\sum X_i^2 \geq w \mid H_0) = a \iff$

$\iff P\left(\frac{\sum X_i^2}{\theta} \geq \frac{w}{\theta} \,\middle|\, H_0\right) = a \iff P\left(\frac{\sum X_i^2}{\theta_0} \geq \frac{w}{\theta_0} \,\middle|\, H_0\right) = a \iff$

$$\left.\begin{aligned} &\iff P\left(Y \geq \frac{w}{\theta_0}\right) = a \\ &P(Y \geq \chi^2_{\nu,a}) = a \end{aligned}\right\} \implies \frac{w}{\theta_0} = \chi^2_{\nu,a} \implies w = \theta_0 \cdot \chi^2_{\nu,a}$$



Finally, $C = \{(x_1, \dots, x_\nu) : \sum_{i=1}^{\nu} x_i^2 \geq \theta_0 \chi^2_{\nu,\alpha}\}$

The c.r. C is independent of θ_1, hence the test is the u.m.p.t.