Example

Let $X \sim Bin(10, p)$, where $p \in \{\frac{1}{2}, \frac{1}{4}\}$

$H_0: p = \frac{1}{2}$ $\underline{vs}$ $H_1: p = \frac{1}{4}$. Test: $X \leq 3 \rightsquigarrow H_0$ is omitted

a) Find the power function

b) Find P(Type I error)

c) Find P(Type II error)

Solution

a) $\Pi(p) = P(C \mid p) = P(X \leq 3 \mid p) = \sum_{x=0}^{3} \binom{10}{x} p^x (1-p)^{10-x}$

b) $P(\text{Type I error}) = \Pi(\frac{1}{2}) = \sum_{x=0}^{3} \binom{10}{x} (1/2)^x (1/2)^{10-x} = \frac{11}{64} = a$

c) $P(\text{Type II error}) = 1 - \Pi(\frac{1}{4}) = 1 - \sum_{x=0}^{3} \binom{10}{x} (\frac{1}{4})^x (\frac{3}{4})^{10-x} = \beta$

Simple H_0 versus simple H_1

In order to find the best test, we stabilize $a = P(\text{Type I err})$ and attempt to minimize $b = P(\text{Type II error})$ or equivalently to maximize the power function at θ_1: $\Pi(\theta_1) = P(C \mid H_1) =$
$= 1 - P(\text{Type II error}) = 1 - \beta$

Definition

Let σ be a test, $H_0: \theta = \theta_0$ $\underline{vs}$ $H_1: \theta = \theta_1$. Let C be the critical range of magnitude $a = P(C \mid H_0) = P(\text{Type I error})$. Then C is the most powerful critical region of magnitude a if for every other c.r. of magnitude a, D, we have: $P(C \mid \theta_1) \geq P(D \mid \theta_1)$

Theorem (Neymann-Pearson Lemma)

Let $X_1, \dots, X_n$ r.s. with PDFs $f(x;\theta)$ and let $\Theta = \{\theta_1, \theta_2\}$.
Suppose we are testing $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$. If, for the likelihood function $L(\theta) = \prod_{i=1}^{n} f(x_i;\theta)$, there exists a constant k and region C so that:

i) $\frac{L(\theta_0)}{L(\theta_1)} \leq k$ if $(X_1, \dots, X_n) \in C$ and

ii) $\frac{L(\theta_0)}{L(\theta_1)} > k$ if $(X_1, \dots, X_n) \notin C$ $\quad (\Leftrightarrow (X_1, \dots, X_n) \in C')$

iii) $P((X_1, \dots, X_n) \in C \mid \theta_0) = \alpha$

Then C is the most powerful critical region (m.p.c.r) of magnitude α for the test H_0 vs H_1.

Example

Let $X_1, \dots, X_n$ r.s. of $N(\mu, \sigma^2 = 36)$. Test $H_0 : \mu = 50$ vs $H_1 : \mu = 55$.
Find m.p.c.r. of magnitude α.

Solution

$$L(\mu) = \prod_{i=1}^{n} f(x_i;\mu) = (72\pi)^{-n/2} \exp\left\{-\frac{1}{72}\sum_{i=1}^{n}(x_i-\mu)^2\right\}$$

so
$$\frac{L(50)}{L(55)} = \frac{L_0}{L_1} = \frac{\exp\left\{-\frac{1}{72}\sum_{i=1}^{n}(x_i-50)^2\right\}}{\exp\left\{-\frac{1}{72}\sum_{i=1}^{n}(x_i-55)^2\right\}} =$$

$$= \exp\left\{-\frac{1}{72}\left(\sum_{i=1}^{n} x_i^2 - 100\sum_{i=1}^{n} x_i + n\,50^2 - \sum_{i=1}^{n} x_i^2 - 110\sum_{i=1}^{n} x_i - n\,55^2\right)\right\} =$$
$$= \exp\left\{-\frac{1}{72}\left(10\sum_{i=1}^{n} x_i - (55^2 - 50^2)\cdot n\right)\right\} =$$
$$= \exp\left\{-\frac{1}{72}\left(10\sum_{i=1}^{n} x_i - 525n\right)\right\} \leq k \Leftrightarrow$$
$$\Leftrightarrow -\frac{1}{72}\left[10\sum x_i - 525n\right] \leq \log k \Leftrightarrow -10\sum x_i + 525n \leq 72\log k$$
$$\Leftrightarrow \bar{x} \geq -\frac{1}{10n}(72\log k - 525n)$$

According to the N-P lemma the m.p.c.r. is
$C = \{(x_1, \dots, x_n) : \bar{x} \geq \underbrace{-\frac{1}{10n}(72\log k - 525n)}_{w}\}$, where w

is chosen so that $P((x_1, \dots, x_v) \in C \mid \mu = 50) = a \Leftrightarrow$
$\Leftrightarrow P(\bar{X} \geq w \mid \mu = 50) = a$.
We know that $\bar{X} \sim N(\mu, \frac{36}{v})$. We will chose values for a and v and they will determine w.
$\longrightarrow a = 0,05 \quad , \quad v = 16$
$P(\bar{X} \geq w \mid \mu = 50) = a = 0,05 \Leftrightarrow P\left(\frac{\bar{X} - 50}{\sqrt{\frac{36}{16}}} \geq \frac{w - 50}{\sqrt{\frac{36}{16}}}\right) = 0,05 \Leftrightarrow$
$\Leftrightarrow P\left(Z > \frac{w-50}{6/4}\right) = 0,05 \Rightarrow \frac{w-50}{6/4} = 1,645 \Leftrightarrow w = 50 + \frac{3}{2} \cdot 1,645 =$
$= 52,47$