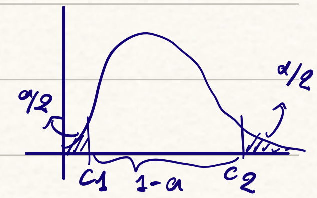
Example (Continuation from last lecture)

We have $Y = 2\nu\theta\bar{T} = Y(\hat{\theta}, \theta)$ and $Y \sim \chi^2_{2\nu}$ so:



$P(c_1 < Y < c_2) = 1 - a \Rightarrow$

$\Rightarrow P(\chi^2_{2\nu, 1-a/2} < Y < \chi^2_{2\nu, a/2}) = 1 - a \Rightarrow$

$\Rightarrow P\left(\frac{\chi^2_{2\nu, 1-a/2}}{2\bar{T}} < \theta < \frac{\chi^2_{2\nu, 1-a/2}}{2\bar{T}}\right) = 1 - a$

so this is the $100(1-a)\%$ CI.

Hypothesis Tests

We have 2 results that are either expressions (e.g. PDFs) S_1, S_2 of which one is "correct" (compliant with the data)

A statistician creates an experiment in order to check if there is enough information to choose the hypothesis he is testing.

Method: We consider the results as values of random variables and suppose that S_1, S_2 belong to 2 distributions (hypotheses). Which one will we choose?

Example

We toss a coin and check if it is fair. At first we suppose that it is fair ($p = 0,5$)

H_0: Null hypothesis (the one we make at first)

H_1: Alternative hypothesis

Let's suppose that we toss the coin 6 times, then:

$X_i \sim \text{Bernoulli}(p) \Longrightarrow S = \sum_{i=1}^{6} X_i \sim \text{Bin}(6, p)$, $f(x, p) = \binom{6}{x} \cdot p^x (1-p)^{6-x}$

each toss   total heads

$H_0 : q = 0,5$

$H_1 : q = 0,75$

S	H_0		H_1
0	$1/64$	$>$	$1/4096$
1	$6/64$	$>$	
2	$15/64$	$>$	$135/4096$
3	$20/64$	$>$	$500/4096$
4	$15/64$	$<$	$105/4096$
5	$6/64$	$<$	$1458/4096$
6	$1/64$	$<$	$729/4096$

C'

C

Definition

If the hypothesis completely defines the distribution, then it is called **simple hypothesis.** If it doesn't define it completely the it is called **complex hypothesis.**

If $S = 0, 1, 2, 3$ then we choose H_0,
if $S = 4, 5, 6$ then we choose H_1

Wrong choice cost

If $S = 4, 5, 6$ and so we chose H_1, then the region C is called **critical region.**

a) **Type I error**: $P(S=4,5,6 \mid H_0 \text{ is correct}) = P(C \mid H_0 \text{ is correct}) =$
$= P(C \mid q = 1/2) = P_{H_0}(C) = P(S=4 \mid q = 1/2) + P(S=5 \mid q = 1/2) + P(S=6 \mid q = \frac{1}{2})$
$= \frac{22}{64} \simeq 0,34$

"a" $P(H_0 \text{ is omitted} \mid H_0 \text{ is correct})$ is called **Type I error probability**

b) **Type II error**: $P(S=0,1,2,3 \mid H_1 \text{ is correct}) = P(C' \mid H_1 \text{ correct}) =$
$= P(S=0 \mid q=0,75) + P(S=1 \mid q=0,75) + P(S=2 \mid q=0,75) + P(S=3 \mid q=0,75) \simeq 0,17$

"β" $P(H_0 \text{ is not omitted} \mid H_1 \text{ is correct})$ is called **Type II error probability**

	H_0 correct	H_1 correct
H_0 omitted	a	✓
H_1 not omitted	✓	β

Definition

A criterion that determines a subset C of the sample space such that: if $(X_1, \dots, X_v) \in C$ then H_0 is omitted in favor of H_1 and otherwise we don't omit, is called test of H_0 with respect to H_1 and C is called the critical region of the test.

Definition

If H_0, H_1 are simple hypotheses, then a, b take specific values. The test is then called of magnitude a if:

$H_0 : \theta \in \Theta_0 = \{\theta_0\}$ or $H_1 : \theta \in \Theta_1 = \{\theta_1)$

Then $P(C \mid H_0 \text{ is correct}) = P(C \mid \theta \in \Theta_0) = a$

If H_0 is complex then a is defined as:

$$a = \sup_{f \in H_0} P(H_0 \text{ is omitted} \mid f) \quad \text{and is called significance level.}$$

$f \in H_0 \to$ distributions that are derived from hypothesis H_0

Definition

The function $\Pi(\theta) = P_\theta(H_0 \text{ is omitted}) = P(C \mid \theta) = P_\theta(C)$ as a function of θ is called power function of the test.

If H_0 is correct: $\Pi(\theta_0) = P(C \mid H_0 \text{ correct}) = a$

If H_1 is correct: $\Pi(\theta_1) = P(C \mid H_1 \text{ correct}) = 1 - P(C' \mid H_1 \text{ correct}) =$

$= 1 - \beta$

$\boxed{\text{Power of the test}}$