CI for σ^2 (Normal population)

Let $X_1, \dots, X_\nu$ r.s of $N(\mu, \sigma^2)$

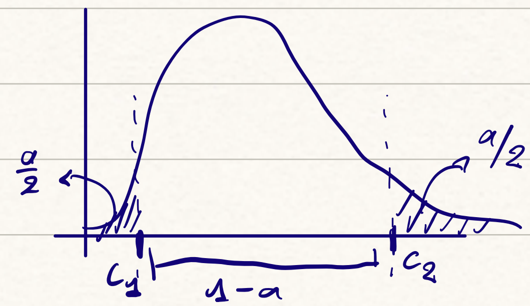
a) <u>If μ is known</u>: We can find the MLE $\hat{\sigma}^2 = \frac{1}{\nu}\sum_{i=1}^{\nu}(X_i - \mu)^2$

We are looking for a pivot function $\Psi(\hat{\sigma}^2, \sigma^2)$

We have $\frac{X-\mu}{\sigma} \sim N(0,1) \Rightarrow \left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2_1 \Rightarrow$

$$\Rightarrow \sum_{i=1}^{\nu}\left(\frac{X_i-\mu}{\sigma}\right)^2 = \frac{\nu \cdot \frac{1}{\nu} \cdot \sum (X_i-\mu)^2}{\sigma^2} = \underbrace{\frac{\nu\hat{\sigma}^2}{\sigma^2}}_{\Psi(\hat{\sigma}^2, \sigma^2)} \sim \chi^2_\nu \Rightarrow$$

$$\Rightarrow E\left[\frac{\nu\hat{\sigma}^2}{\sigma^2}\right] = \nu \Rightarrow E[\hat{\sigma}^2] = \sigma^2$$



$$P(c_1 < \Psi(\hat{\sigma}^2, \sigma^2) < c_2) = 1-a \Rightarrow$$

$$\Rightarrow P\left(c_1 < \frac{\nu\hat{\sigma}^2}{\sigma^2} < c_2\right) = 1-a \Rightarrow$$

$$\Rightarrow P\left(\frac{1}{c_2} < \frac{\sigma^2}{\nu\hat{\sigma}^2} < \frac{1}{c_1}\right) = 1-a \Rightarrow P\left(\frac{\nu\hat{\sigma}^2}{c_2} < \sigma^2 < \frac{\nu\hat{\sigma}^2}{c_1}\right) = 1-a \Rightarrow$$

$$\Rightarrow P\left(\frac{\sum(x_i-\mu)^2}{c_2} < \sigma^2 < \frac{\sum(x_i-\mu)^2}{c_1}\right) = 1-a$$

so the $100 \cdot (1-a)\%$ CI for σ^2 is $\left[\frac{\sum(x_i-\mu)^2}{\chi^2_{\nu,a/2}}, \frac{\sum(x_i-\mu)^2}{\chi^2_{\nu,-a/2}}\right]$.

b) <u>If μ is unknown</u>: In this case we have $S^2 = \frac{1}{\nu-1}\sum_{i=1}^{\nu}(x_i - \bar{x})^2$ as an u.e. of σ^2. We know that $\frac{(\nu-1)S^2}{\sigma^2} \sim \chi^2_{\nu-1}$, so $\Psi(S^2, \sigma^2) = \frac{(\nu-1)S^2}{\sigma^2}$ is our pivot function.

CI for variance ratio

Let $X_1, \dots, X_\nu$ r.s of $N(\mu_x, \sigma_x^2)$ and $Y_1, \dots, Y_m$ of $N(\mu_y, \sigma_y^2)$

We are looking for $\frac{\sigma_x^2}{\sigma_y^2}$.

a) If μ_x, μ_y are known: $\sigma_x^2 = \frac{1}{v}\sum_{i=1}^{v}(X_i - \mu_x)^2$ and $\sigma_y^2 = \frac{1}{m}\sum_{i=1}^{m}(Y_i - \mu_y)^2$

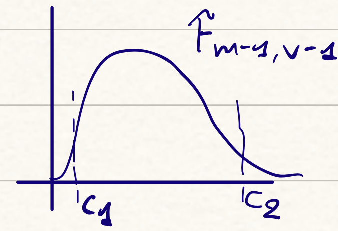
b) If μ_x, μ_y are unknown: $S_x^2 = \frac{1}{v-1}\sum_{i=1}^{v}(X_i - \bar{X})^2$ and $S_y^2 = \frac{1}{m-1}\sum_{i=1}^{m}(Y_i - \bar{Y})^2$.

We know that $\frac{(v-1)S_x^2}{\sigma_x^2} \sim \chi^2_{v-1}$, $\frac{(m-1)S_y^2}{\sigma_y^2} \sim \chi^2_{m-1}$

and, since X, Y are independent, S_x^2 S_y^2 are independent, so

$$\left(\frac{(v-1)S_x^2}{\sigma_x^2}\Big/ v-1\right)\Big/\left(\frac{(m-1)S_y^2}{\sigma_y^2}\Big/(m-1)\right) = \frac{S_x^2/S_y^2}{\sigma_x^2/\sigma_y^2} \sim F_{v-1,m-1} \Rightarrow$$

$$\Rightarrow \frac{\sigma_x^2/\sigma_y^2}{S_x^2/S_y^2} \sim F_{m-1,v-1}$$



We set $Y(S_x^2, S_y^2, \sigma_x^2, \sigma_y^2)$ and we have:

$$P\left(c_1 < \frac{\sigma_x^2/\sigma_y^2}{S_x^2/S_y^2} < c_2\right) = 1-a \Rightarrow P\left(\frac{S_x^2}{S_y^2}\cdot \underbrace{F_{m-1,v-1,-a/2}}_{1,5} < \frac{\sigma_x^2}{\sigma_y^2} < \frac{S_x^2}{S_y^2}\cdot \underbrace{F_{m-1,v-1,a/2}}_{3,5}\right) = 1-a$$

so a $100\cdot(1-a)\%$ CI for the variance ratio σ_x^2/σ_y^2 is $[1.5, 3.5]$

CI for percentage

Let $X_1, \ldots, X_v$ r.s. Bernoulli(p), where v is known.

The MLE is: $\hat{p} = \sum_{i=1}^{v} X_i / v$. We have $Y = \sum_{i=1}^{v} X_i \sim Bin(v, p)$ $\mu = vp$, $\sigma^2 = vp(1-p)$

If $v > 30$, from CLT (Central Limit Theorem) we have:

$$\frac{Y - E(Y)}{V(Y)} = \frac{Y - vp}{\sqrt{vp(1-p)}} \sim N(0,1)$$

So $P\left(Z_{-a/2} < \frac{Y - vp}{\sqrt{vp(1-p)}} < Z_{a/2}\right) = 1-a \Rightarrow P\left(-Z_{a/2} < \frac{\frac{Y}{v} - p}{\sqrt{\frac{p(1-p)}{v}}} < Z_{a/2}\right) = 1-a$

$$\Rightarrow P\left(\frac{Y}{v} - Z_{a/2}\sqrt{\frac{p(1-p)}{v}} < p < \frac{Y}{v} + Z_{a/2}\cdot\sqrt{\frac{p(1-p)}{v}}\right) = 1-a$$

↳ we set p equal to $\hat{p}$ in the bounds of the CI.

So the 100(1-a)% CI for q is $Y \pm Z_{a/2}\sqrt{\frac{Y/v \cdot (1-Y/v)}{v}}$

If v is small (<30):

$$P\left(\left|\frac{Y/v-q}{\sqrt{\frac{q(1-q)}{v}}}\right| \leq Z_{a/2}\right)=1-a \Rightarrow \left(\frac{Y}{v}-q\right)^2 - Z_{a/2}^2\frac{q(1-q)}{v} \leq 0 \Rightarrow$$

$$\Rightarrow \hat{q}^2+q^2-2\hat{q}q-\frac{Z_{a/2}^2 q}{2}\cdot q+\frac{Z_{a/2}^2}{2}\cdot q^2 \leq 0 \Rightarrow$$

$$\Rightarrow H(q)=\left(1+\frac{Z_{a/2}^2}{v}\right)q^2-\left(2\hat{q}+\frac{Z_{a/2}^2}{v}\right)q+\hat{q}^2$$

The 2 roots of $H(q)$ are $r_1, r_2 = \hat{q} \pm Z_{a/2}\sqrt{\frac{\hat{q}(1-\hat{q})}{v}}$

so for large v the CI is (r_1, r_2)

CI for percentage difference

Let $Y_1 \sim Bin(v_1, q_1)$, $Y_2 \sim Bin(v_2, q_2)$, then $E[Y_1]=v_1q_1$, $E[Y_2]=v_2q_2$, $V(Y_1)=v_1q_1(1-q_1)$, $V(Y_2)=v_2q_2(1-q_2)$ and for the r.v. $\frac{Y_1}{v_1}-\frac{Y_2}{v_2}$ we have $E\left[\frac{Y_1}{v_1}-\frac{Y_2}{v_2}\right]=q_1-q_2$, $V\left(\frac{Y_1}{v_1}-\frac{Y_2}{v_2}\right)=\frac{q_1(1-q_1)}{v_1}+\frac{q_2(1-q_2)}{v_2}$

For large v_1, v_2 from the CLT we get:

$$\left[\left(\frac{Y_1}{v_1}-\frac{Y_2}{v_2}\right)-(q_1+q_2)\right]\Big/\left(\sqrt{\frac{q_1(1-q_1)}{v_1}+\frac{q_2(1-q_2)}{v_2}}\right) \sim N(0,1)$$

So the 100(1-a)% CI is: $\left(\frac{Y_1}{v_1}-\frac{Y_2}{v_2}\right) \pm Z_{a/2}\sqrt{\frac{\frac{Y_1}{v_1}\left(1-\frac{Y_1}{v_1}\right)}{v_1}+\frac{\frac{Y_2}{v_2}\left(1-\frac{Y_2}{v_2}\right)}{v_2}}$

Example (Previous exam exercise)

Let $X_1, \dots, X_v$ r.s. with PDF $f(x;\theta)=\theta \cdot x^{\theta-1}$, $0<x<1$, $\theta>0$

a) Find an estimator for the moments of θ and the MLE of $\frac{1}{\theta}$

b) Find 100(1-a)% CI for θ

(Hint: Show that for the MLE δ, $Y=2v\theta\delta$ follows χ^2_{2v}.
For Gamma(v,θ) the PDF is $f(y,\theta)=\frac{\theta^v}{\Gamma(v)}y^{v-1}e^{-\theta y}$)

Solution

a) $E[X] = \int_0^1 x\theta x^{\theta-1} dx = \theta \frac{x^{\theta+1}}{\theta+1}\Big|_0^1 = \frac{\theta}{\theta+1}$

We equate $E[X] = \bar{X} \Rightarrow \frac{\theta}{\theta+1} = \bar{X} \Rightarrow \boxed{\hat{\theta} = \frac{\bar{X}}{1-\bar{X}}} \rightsquigarrow$ moments

MLE: $L(\theta) = \prod_{i=1}^{\nu} f(x_i; \theta) = \prod_{i=1}^{\nu} \theta x_i^{\theta-1} = \theta^{\nu}\left(\prod_{i=1}^{\nu} x_i\right)^{\theta-1} \Rightarrow$

$\Rightarrow \ell(\theta) = \log L(\theta) = \nu \log\theta + (\theta-1)\cdot \log \prod_{i=1}^{\nu} x_i =$

$= \nu\log\theta + (\theta-1)\cdot\sum_{i=1}^{\nu}\log x_i$

$\frac{\partial \ell(\theta)}{\partial\theta} = 0 \Rightarrow \frac{\nu}{\theta} + \sum_{i=1}^{\nu}\log x_i = 0 \Rightarrow \hat{\theta} = -\frac{\nu}{\sum_{i=1}^{\nu}\log x_i} \xrightarrow{\text{invariance property}}$

$\Rightarrow \delta = \frac{1}{\hat{\theta}} = -\frac{\sum \log x_i}{\nu}$

$\frac{\partial^2 \ell(\theta)}{\partial\theta^2}\Big|_{\theta=\delta} = -\frac{\nu}{\delta^2} < 0$ so δ is MLE.

b) We call $W = -\log X$ and it is $F_W(w) = P(W \le w) =$

$= P(-\log X \le w) = P(X > e^{-w}) = 1 - P(X \le e^{-w}) = 1 - F_X(e^{-w}) \Rightarrow$

$\Rightarrow f_W(w) = \frac{\partial}{\partial w}\left(1 - F_X(e^{-w})\right) = e^{-w} f_X(e^{-w}) = e^{-w}\theta e^{-w(\theta-1)} = \underbrace{\theta\cdot e^{-\theta w}}_{Exp(\theta)}$

So $-\log X_i \sim Exp(\theta) \longrightarrow T = -\sum_{i=1}^{\nu}\log X_i \sim Gamma(\nu, \theta)$

and $Y = 2\nu\theta\delta = 2\nu\theta\frac{T}{\nu} = 2\theta T$ so we have:

$F_Y(y) = P(Y \le y) = P(2\theta T \le y) = P(T \le \frac{y}{2\theta}) = F_T(\frac{y}{2\theta}) \Rightarrow$

$\Rightarrow f_Y(y) = \frac{1}{2\theta} f_T\left(\frac{y}{2\theta}\right) = \frac{1}{2\theta}\frac{\theta^{\nu}}{\Gamma(\nu)}\left(\frac{y}{2\theta}\right)^{\nu-1} e^{-\theta\cdot\frac{y}{2\theta}} =$

$= \frac{1}{2^{\nu}\Gamma(\frac{2\nu}{2})}\cdot y^{\nu-1}\cdot e^{-\frac{1}{2}y}$, which is the PDF of $\chi^2_{2\nu}$