CONSISTENCY

Let $X_1, \dots, X_n$ r.s. with PDF $f(x;\theta)$ and $T = T(\underline{X})$ u.e. of θ. Also we suppose $T_n = T_n(\underline{x}) \to \theta$ in one of the following ways:

a) $\lim_{n\to\infty} T_n = \theta$ with probability 1 : $P(\lim_{n\to\infty} T_n = \theta) = 1$

We denotate $T_n \xrightarrow{a.s.} \theta$ (almost surely)

b) $\lim_{n\to\infty} P(|T_n - \theta| < \varepsilon) = 1$ or $\lim_{n\to\infty} P(|T_n - \theta| \geq \varepsilon) = 0$.

We denotate $T_n \xrightarrow{P} \theta$.

Definition

The sequence of estimators T_n is "weakly consistent" for θ if $T_n \xrightarrow{P} \theta$ $\forall \theta \in \Theta$ and "strongly consistent" for θ if $T_n \xrightarrow{a.s.} \theta$ $\forall \theta \in \Theta$

Theorem

Let T_n sequence of estimators of θ. If:

i) T_n is an u.e. of θ and

ii) $\lim_{n\to\infty} V(T_n) \to 0$

then T_n is a weakly consistent (we will call that "consistent" from now on) estimator of θ.

Proof

We need to prove that $T_n \xrightarrow{P} \theta$

From Chebyshev inequality: $P(|T_n - E[T_n]| > \varepsilon) \leq \frac{V(T_n)}{\varepsilon^2}$
Since $E[T_n] = \theta$ (from (i)) and $V(T_n) \to 0$ (from (ii))
we have $P(|T_n - \theta| > \varepsilon) \to 0 \Rightarrow \lim_{n\to\infty} P(|T_n - \theta| > 0) = 0$
$\Rightarrow \lim_{n\to\infty} P(|T_n - \theta| \leq \varepsilon) = 1 \Rightarrow T_n \xrightarrow{P} \theta$

□

Generalization

Let T_n sequence of estimators of $g(\theta)$. If:
i) $\lim_{n\to\infty} E[T_n] = g(\theta)$
ii) $\lim_{n\to\infty} V(T_n) = 0$

then T_n is a consistent estimator of $g(\theta)$.

Example

Let $X_1, \dots, X_n$ r.s. with PDF $f(x;\theta)$, then $\bar{X}$ is <u>always</u> a consistent estimator of the population mean.
Indeed, we have: $E(\bar{X}) = \mu \Rightarrow \bar{X}$ u.e. of μ

$$V(\bar{X}) = \frac{\sigma^2}{n} = \frac{V(x)}{n} \xrightarrow[n\to\infty]{} 0$$

Note

The consistency criterion <u>can't</u> lead us to good estimators on its own. For example, if T_x consistent, then $T_n^* = T_n + Q(n)$, where $Q(n) \xrightarrow[n\to\infty]{} 0$ is also consistent

Example

Let $X_1, \dots, X$ r.s. of $N(\mu, \sigma^2)$. Prove that $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ is a consistent u.e.

<u>Solution</u>

$\frac{(\nu-1)S^2}{\sigma^2} \sim \chi^2_{\nu-1}$, so $E\left[\frac{(\nu-1)S^2}{\sigma^2}\right] = \nu - 1 \Rightarrow E[S^2] = \sigma^2 \Rightarrow S^2$ u.e

Also : $V\left(\frac{(\nu-1)S^2}{\sigma^2}\right) = 2(\nu-1) \Rightarrow \frac{(\nu-1)^2}{\sigma^4} V(S^2) = 2(\nu-1) \Rightarrow$

$\Rightarrow V(S^2) = \frac{2\sigma^4}{\nu-1} \xrightarrow[\nu\to\infty]{} 0$, so S^2 consistent estimator of σ^2.

Example

Let $X_1, \dots, X_\nu$ r.s. of $Bin(N, q)$ with PDF
$f(x ; q) = \binom{N}{q} q^x (1-q)^{N-x}$

a) Find m.v.u.e of q

b) Find the CR-LB

c) Efficiency

d) Consistency

Solution

a) We can show (in many ways, e.g. EFD) that $T = \sum_{i=1}^{\nu} X_i$ is a sufficient and complete statistic for q. We have:

$f(x ; q) = \binom{N}{q} q^x (1-q)^{N-x} = \binom{N}{x} \left(\frac{q}{1-q}\right)^x (1-q)^N =$

$= \exp\{x \log\left(\frac{q}{1-q}\right) + N \log(1-q)\} \cdot \binom{N}{q} =$

$= \exp\{T(x) \cdot \eta(q) - B(q)\} \cdot h(x)$

so, since the support is independent of q, T belongs to the EFD.

The joint PDF is: $f(\underline{x} ; q) = \exp\{\sum_{i=1}^{\nu} T(x_i) \cdot \eta(q) + \nu B(q)\} \cdot \prod_{i=1}^{\nu} h(x_i)$

so $T^*(x) = \sum_{i=1}^{\nu} T(x_i)$ is sufficient and complete for q.

Also $T^* = \sum T(x_i) \sim Bin(\nu N, q) \Rightarrow E[T^*] = \nu N q \Rightarrow$

$\Rightarrow E\left[\frac{T^*}{\nu N}\right] = q$. $\frac{T^*}{\nu N}$ is a function of a complete and sufficient statistic, so from the Lehmann-Scheffé theorem it is m.v.u.e of

d) $V\left(\frac{T^*}{\nu N}\right) = \frac{1}{\nu^2 N^2} V(T^*) = \frac{1}{\nu^2 N^2} \cdot \nu N q (1-q) = \frac{q(1-q)}{\nu N} \xrightarrow[\nu\to\infty]{} 0$

and $\frac{T^*}{\nu N}$ is an u.e. of q so it is consistent.

b) CR-LB: $\frac{1}{v I_X(q)}$, where $I_X(q) = E\left[\left(\frac{\partial}{\partial q}\log f(x,q)\right)^2\right] =$
$= -E\left[\frac{\partial^2}{\partial q^2}\log f(x,q)\right]$

We have $f(x,q) = \binom{N}{x} q^x (1-q)^{N-x} \Rightarrow$

$\Rightarrow \log f(x,q) = \log\binom{N}{x} + x\log q + (N-x)\log(1-q) \Rightarrow$

$\Rightarrow \frac{\partial}{\partial q}\log f(x,q) = \frac{x}{q} + \frac{N-x}{1-q} \Rightarrow \frac{\partial^2}{\partial q^2}\log f(x,q) = -\frac{x}{q^2} + \frac{N-x}{(1-q)^2}$

Hence, $-E\left[\frac{\partial^2}{\partial q^2}\log(x;q)\right] = -E\left[-\frac{x}{q^2} - \frac{N-x}{(1-q)^2}\right] =$

$= \frac{E[X]}{q} + \frac{N-E[X]}{(1-q)^2} = \frac{Nq}{q^2} + \frac{N-Nq}{(1-q)^2} = \frac{N}{q} + \frac{N}{1-q} = \frac{N(1-q)+Nq}{q(1-q)} =$

$= \frac{N}{q(1-q)}$.

So the CR-LB is: $\frac{1}{v \cdot \frac{N}{q(1-q)}} = \frac{q(1-q)}{vN}$

c) $V\left(\frac{T^*}{vN}\right) = \frac{q(1-q)}{vN} =$ CR-LB so $\frac{T^*}{vN}$ is also sufficient

---

Let $X_1, \ldots, X_v$ r.s with PDF $f(x;\theta)$. Then

$L(\theta;\underline{x}) = f(\underline{x};\theta) = f(x_1,\ldots,x_v;\theta) = \prod_{i=1}^{v} f(x_i;\theta)$

is called Likelihood function.