EFFICIENCY

Definitions

If an estimator $T=T(\underline{X})$ of the parameter θ is unbiased and reaches the C-R lower bound then T is called a efficient estimator of θ.

$a(T)=\frac{CR-LB}{V(T)}$ is called efficiency of T and so T is efficient iff $a(T)=1$. Also:

- $a(T)=1 \Rightarrow T$ m.v.u.e
- T m.v.u.e $\not\Rightarrow a(T)=1$

If $U(\underline{X})$, $T(\underline{X})$ unbiased estimators of θ then $\frac{V(U)}{V(T)}$ is called the relative efficiency.

Example

Let $X_1, \ldots, X_v$ r.s of Poisson(θ). Find m.v.u.e of $g(\theta)=\theta^2$

Solution

We already know that $T=\sum_{i=1}^{v} X_i$ is sufficient and complete for θ and $T \sim$ Poisson$(v\theta)$, so $E[T]=v\theta$, $V(T)=v\theta$.

But $V(T)=E[T^2]-(E[T])^2 \Rightarrow E[T^2]=v\theta-v^2\theta^2$

so $E[T^2]-E[T]=v^2\theta^2 \Rightarrow E\left[\frac{T^2-T}{v}\right]=\theta^2$

Hence, $U=\frac{T^2-T}{v}$ u.e. of $g(\theta)=\theta^2$ and is a function the sufficient and complete estimator T, so U is m.v.u.e of θ^2.

We have: $V(U) = V\left(\frac{T^2-T}{\nu}\right) = \frac{1}{\nu^2}\left[E[(T^2-T)^2] - (E[T^2-T])^2\right] =$

$= \frac{1}{\nu^2}\left(E[T^4] - 2E[T^3] + E[T^2] - (E[T^2])^2 + 2E[T^2]\cdot E[T] - E[T]^2\right)$

and after calculating the moments $E[T]$, $E[T^2]$, $E[T^3]$, $E[T^4]$ we get $V(U) = \frac{4\theta^3}{\nu} + \frac{2\theta^3}{\nu^2}$ and so the CR-LB is: $\frac{(2\theta)^2}{\nu\cdot\frac{1}{\theta}} = \frac{4\theta^3}{\nu}$

Hence, $V(U) = \frac{4\theta^3}{\nu} + \frac{2\theta^3}{\nu^2} > \frac{4\theta^3}{\nu} =$ CR-LB

So U is m.v.u.e but <u>not</u> efficient.

<u>Example</u>

Let $X_1, \dots, X_\nu$ r.s of $N(\mu, \sigma^2 = 25)$.

a) Find $I_{\underline{X}}(\mu)$

b) Prove that $\bar{X}$ is an efficient estimator of μ.

<u>Solution</u>

a) $I_{\underline{X}}(\mu) = \nu I_X(\mu) = \nu \cdot E\left[\left(\frac{\partial}{\partial\mu}\log f(x;\mu)\right)^2\right] = -\nu\, E\left[\frac{\partial^2}{\partial\mu^2}\log f(x;\mu)\right]$

$f(x;\mu) = \frac{1}{\sqrt{2\pi 25}}\cdot e^{-\frac{1}{50}(x-\mu)^2} \Rightarrow \log f(x;\mu) = -\frac{1}{2}\log(50\pi) - \frac{1}{50}(x-\mu)^2$

So $\frac{\partial}{\partial\mu}\log f(x;\mu) = \frac{1}{50}\cdot 2(x-\mu) = \frac{x-\mu}{25}$

and $\frac{\partial^2}{\partial\mu^2}\log f(x;\mu) = -1/25$

Finally: $I_{\underline{X}}(\mu) = \nu I_X(\mu) = -\nu\cdot E[-1/25] = \frac{\nu}{25}$

b) We know that $\bar{X}$ is an u.e. of μ.

CR-LB: $\frac{1}{\nu I_X(\mu)} = \frac{1}{I_{\underline{X}}(\mu)} = \frac{1}{\nu/25} = \frac{25}{\nu}$

$V(\bar{X}) = V\left(\frac{1}{\nu}\sum_{i=1}^{\nu} X_i\right) = \frac{1}{\nu^2}\sum_{i=1}^{\nu} V(X_i) = \frac{1}{\nu^2}\cdot\nu\cdot 25 = \frac{25}{\nu}$

Hence, $\bar{X}$ is an efficient estimator of μ, $a(\bar{X}) = 1$.

Theorem

Let $X_1, \dots, X_\nu$ r.s of PDF $f(x;\theta)$ and suppose we are estimating $g(\theta)$. For $T = T(\underline{X})$ an u.e of $g(\theta)$, $V(T)$ is equal to the CR-LB iff there exists a real function of θ $k(\theta)$ so that:

$$\sum_{i=1}^{v} \frac{\partial}{\partial\theta} \log f(x_i;\theta) = k(\theta)\cdot(T(\underline{X}) - g(\theta))$$

$$\Leftrightarrow \frac{\partial}{\partial\theta} \prod_{i=1}^{v} \log f(x_i;\theta) = k(\theta)\cdot(T(\underline{X}) - g(\theta))$$

Proof

(We skip the proof, it is based on the equality in the C-S inequality standing true iff the 2 quantities are linearly dependent)

Example (continuation)

Let $X_1, \dots, X_v$ r.s of $N(\mu, 25)$

We have $\sum_{i=1}^{v} \frac{\partial}{\partial\mu} \log f(x_i;\mu) = \sum_{i=1}^{v} \frac{x_i - \mu}{25} = \frac{1}{25}\left(\sum_{i=1}^{v} X_i - v\mu\right)$ Theorem

$= \frac{v}{25}(\overline{X} - \underbrace{\mu}_{g(\theta)})$

Example

Let $X_1, \dots, X_v$ r.s of $N(\mu, \theta)$, where μ is known.

Solution

$f(x;\theta) = \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{1}{2\theta}(x-\mu)^2} \Rightarrow \log f(x;\theta) = -\frac{1}{2}\log 2\pi - \frac{1}{2}\log\theta - \frac{1}{2\theta}(x-\mu)^2$

$\Rightarrow \frac{\partial}{\partial\theta}\log f(x;\theta) = -\frac{1}{2\theta} + \frac{1}{2\theta^2}(x-\mu)^2 \Rightarrow$

$\Rightarrow \left[\frac{\partial}{\partial\theta}\log f(x;\theta)\right]^2 = \frac{1}{4\theta^2} + \frac{1}{4\theta^2}\left(\frac{x-\mu}{\sqrt{\theta}}\right)^4 - \frac{1}{2\theta^2}\left(\frac{x-\mu}{\sqrt{\theta}}\right)^2$

So $E\left[\left(\frac{\partial}{\partial\theta}\log f(x;\theta)\right)^2\right] = \frac{1}{4\theta^2} + \frac{1}{4\theta^2}E\left[\left(\frac{x-\mu}{\sqrt{\theta}}\right)^4\right] - \frac{1}{2\theta^2}E\left[\left(\frac{x-\mu}{\sqrt{\theta}}\right)^2\right]$

$\frac{x-\mu}{\sqrt{\theta}} \sim N(0,1) \longrightarrow \left(\frac{x-\mu}{\sqrt{\theta}}\right)^2 \sim \chi^2_1$, so $E\left[\left(\frac{x-\mu}{\sqrt{\theta}}\right)^2\right] = 1$

and $V\left(\left(\frac{x-\mu}{\sqrt{\theta}}\right)^2\right) = 2$. Also $E\left[\left(\frac{x-\mu}{\sqrt{\theta}}\right)^4\right] = V\left(\left(\frac{x-\mu}{\sqrt{\theta}}\right)^2\right) + \left(E\left(\left(\frac{x-\mu}{\sqrt{\theta}}\right)^2\right)\right)^2 = 3$

$I_X(\theta) = E\left[\left(\frac{\partial}{\partial\theta}\log f(x;\theta)\right)\right] = \frac{1}{4\theta^2} + \frac{3}{4\theta^2} - \frac{1}{2\theta^2} = \frac{1}{2\theta^2}$

So the CR-LB is : $\frac{1}{\nu I_X(\theta)} = \frac{1}{\nu \cdot 1/2\theta^2} = 2\theta^2/\nu$

We set $T = T(\underline{x}) = \frac{1}{\nu}\sum_{i=1}^{\nu}(X_i-\mu)^2$ and it is:

$E[T] = \frac{1}{\nu}\sum_{i=1}^{\nu} E[(X_i-\mu)^2] = \frac{1}{\nu}\cdot\nu\cdot\theta = \theta \Rightarrow T$ u.e. of θ

$V(T) = \frac{1}{\nu^2}\cdot\sum_{i=1}^{\nu} V((X_i-\mu)^2)$, and we have:

$\left(\frac{X-\mu}{\sqrt{\theta}}\right)^2 \sim \chi_1^2 \Rightarrow V\left(\left(\frac{X-\mu}{\sqrt{\theta}}\right)^2\right) = 2 \Rightarrow \frac{1}{\theta^2}\cdot V((X-\mu)^2) = 2 \Rightarrow$

$\Rightarrow V((X-\mu)^2) = 2\theta^2$

So $V(T) = \frac{1}{\nu^2}\cdot 2\theta^2\cdot\nu = \frac{2\theta^2}{\nu}$ and T is efficient.