Example

Let $X_1, \dots, X_\nu$ r.s of distribution with PDF $f(x;\lambda) = 2\lambda^2 x^{-3}$, $x > \lambda$, $\lambda > 0$.

i) Prove that $T = X_{(1)} = \min\{X_i, i = 1, \dots, \nu\}$ is a sufficient and complete statistic for λ.

ii) Find m.v.u.e of λ^{-k}

Solution

i) - Sufficiency: $f(x;\lambda) = 2\lambda^2 x^{-3} \cdot I(\lambda < x < \infty)$

So $f(\underline{x};\lambda) = \prod_{i=1}^{\nu} f(x_i;\lambda) = \prod_{i=1}^{\nu} 2\lambda^2 x_i^{-3} I(\lambda < x_i < \infty) =$

$= (2\lambda^2)^\nu \cdot \frac{1}{\prod_{i=1}^{\nu} x_i^3} \cdot \prod_{i=1}^{\nu} I(\lambda < x_i < \infty)$

We need: $\prod_{i=1}^{\nu} I(\lambda < x_i < \infty) = 1 \iff I(\lambda < x_i < \infty) = 1 \quad \forall i \in \{1, \dots, \nu\}$

$\iff \lambda < x_1, \dots, x_\nu < \infty \iff \begin{cases} \lambda < x_{(1)} < \infty \\ 0 < x_{(\nu)} < \infty \end{cases} \iff I(\lambda < x_{(1)} < \infty) \cdot I(0 < x_{(\nu)} < \infty) = 1$

So $f(\underline{x};\lambda) = (2\lambda^2)^\nu \frac{1}{\prod_{i=1}^{\nu} x_i^3} I(\lambda < x_{(1)} < \infty) \cdot I(0 < x_{(\nu)} < \infty) =$

$= g(T(\underline{x}), \lambda) \cdot u(\underline{x})$, where $g(T(\underline{x}), \lambda) = (2\lambda^2)^\nu \cdot I(\lambda < x_{(1)} < \infty)$

and $u(\underline{x}) = \frac{1}{\prod x_i^3} I(0 < x_{(\nu)} < \infty)$ so $T(\underline{x}) = X_{(1)}$ is sufficient for λ.

- Distribution of $T(\underline{x})$: $F_T(t) = P(T \le t) = P(X_{(1)} \le t) =$

$= 1 - P(X_{(1)} > t) = 1 - P(X_1, \dots, X_\nu > t) = 1 - \prod_{i=1}^{\nu} P(X_i > t) =$

$= 1 - (P(X > t))^\nu = 1 - (1 - P(X \le t))^\nu = 1 - [1 - F_X(t)]^\nu$

and we have: $F_X(t) = \int_\lambda^t f_X(x)\,dx = \int_\lambda^t 2\lambda^2 x^{-3}\,dx = 2\lambda^2 \frac{1}{-2x^2}\Big|_\lambda^t =$

$= 2\lambda^2 \left[\frac{1}{-2t^2} - \frac{1}{-2\lambda^2}\right] = 1 - \left(\frac{\lambda}{t}\right)^2, \quad t > \lambda$

So $F_T(t) = 1 - \left(\frac{\lambda}{t}\right)^{2\nu} \overset{\frac{\partial}{\partial t}}{\Rightarrow} f_T(t) = \frac{2\nu\lambda^{2\nu}}{t^{2\nu+1}}, \quad t > \lambda$

- Completeness: Let $g(T)$ function of T. Then:

$$E[g(T)]=0\ \forall\lambda \Rightarrow \int_\lambda^\infty g(t) f_T(t)\,dt = 0\ \forall\lambda \Rightarrow$$
$$\Rightarrow \int_\lambda^\infty g(t)\cdot 2\nu\lambda^{2\nu}\cdot t^{-2\nu-1}\,dt=0\ \forall\lambda \Rightarrow$$
$$\Rightarrow \int_\lambda^\infty g(t)\, t^{-2\nu-1}\,dt=0\ \forall\lambda \xRightarrow{\partial/\partial\lambda} -g(\lambda)\lambda^{-2\nu-1}=0\ \forall\lambda\Rightarrow$$
$$\Rightarrow g(\lambda)=0\ \forall\lambda>0 \Rightarrow g(t)=0\ \forall t>\lambda$$

So T is also complete

ii) We are looking for a $\varphi(T)$ such that $E[\varphi(T)]=\lambda^{-k}$.

We have: $E[\varphi(T)]=\lambda^{-k} \Rightarrow \int_\lambda^\infty \varphi(t) f_T(t)\,dt = \lambda^{-k} \Rightarrow$

$$\Rightarrow \int_\lambda^\infty \varphi(t)\frac{2\nu\lambda^{2\nu}}{t^{2\nu+1}}\,dt = \lambda^{-k} \Rightarrow \int_\lambda^\infty \varphi(t)\cdot t^{-2\nu-1}\,dt = \frac{\lambda^{-k-2\nu}}{2\nu} \xRightarrow{\partial/\partial\lambda}$$
$$\Rightarrow -\varphi(\lambda)\lambda^{-2\nu-1}\,dt = (-k-2\nu)\frac{\lambda^{-k-2\nu-1}}{2\nu} \Rightarrow \varphi(\lambda)=\frac{k+2\nu}{2\nu}\cdot\lambda^{-k}$$

Finally, the statistic $\varphi(T)=\frac{k+2\nu}{2\nu}T^{-k}=\frac{k+2\nu}{2\nu}X_{(1)}^{-k}$ is m.v.u.e of λ^{-k}.

Example

Let $X_1,\dots,X_\nu$ r.s of Poisson(λ). Find m.v.u.e of $g(\lambda)=\lambda^k$.

Solution

We have previously shown that $T=T(\underline{X})=\sum_{i=1}^{\nu}X_i$ is sufficient and complete for λ.

We are looking for a function $\varphi(T)$ such that $E[\varphi(T)]=\lambda^k$.

$E[\varphi(T)]=\sum_{t=0}^{\infty}\varphi(t)\cdot P(T=t)$ and $T\sim$ Poisson$(\nu\lambda)$

Hence, $E[\varphi(T)]=\lambda^t \Rightarrow \sum_{t=0}^{\infty}\varphi(T)\cdot e^{-\nu\lambda}\frac{(\nu\lambda)^t}{t!}=\lambda^t \Rightarrow$

$$\Rightarrow \sum_{t=0}^{\infty}\varphi(t)\frac{(\nu\lambda)^t}{t!}=\lambda^t\cdot e^{\nu\lambda}=\sum_{t=0}^{\infty}\varphi(t)\frac{(\nu\lambda)^t}{t!}=\lambda^t\cdot\sum_{t=0}^{\infty}\frac{(\nu\lambda)^t}{t!}\ \forall\lambda \Rightarrow$$

$$\Rightarrow \sum_{t=0}^{\infty}\varphi(t)\frac{(\nu\lambda)^t}{t!}=\frac{1}{\nu^k}\sum_{t=0}^{\infty}\frac{(\nu\lambda)^{t+k}}{t!}\ \forall\lambda \Rightarrow \sum_{t=0}^{\infty}\frac{\varphi(t)}{t!}(\nu\lambda)^t=\sum_{t=k}^{\infty}\frac{1}{\nu^k(t-k)!}(\nu\lambda)^t\ \forall\lambda$$

Since the equality is true $\forall\lambda$ we have:

For $t \geq k$: $\frac{\psi(t)}{t!} = \frac{1}{\nu^k (t-k)!} \Rightarrow \psi(t) = \frac{1}{\nu^k} \cdot \frac{t!}{(t-k)!}$

For $t < k$: $\psi(t) = 0$

So $\psi(T) = \begin{cases} 0 & t < k \\ \frac{1}{\nu^k} \cdot \frac{t!}{(t-k)!}, & t \geq k \end{cases}$

For example, for $k=1$ $\psi(T) = \frac{1}{\nu} \cdot \frac{T!}{(T-1)!} = \frac{1}{\nu} \cdot T = \frac{\sum X_i}{\nu}$

Example

Let $X_1, \dots, X_\nu$ r.s of $U(-\theta, \theta)$.

a) Prove that $T = \max\{|X_1|, \dots, |X_\nu|\}$ is sufficient and complete for θ.

b) Find m.v.u.e for 2θ and for $\frac{\theta^2}{3}$

Solution

- $X_i \sim U(-\theta, \theta) \Rightarrow f(x_i; \theta) = \frac{1}{2\theta} \cdot I(-\theta < x_i < \theta)$

$f(x; \theta) = \prod_{i=1}^{\nu} f(x_i, \theta) = \prod_{i=1}^{\nu} \frac{1}{2\theta} \cdot I(-\theta < x_i < \theta) = (2\theta)^{-\nu} \cdot \prod_{i=1}^{\nu} I(-\theta < x_i < \theta)$

It is: $\prod_{i=1}^{\nu} I(-\theta < x_i < \theta) = 1 \Leftrightarrow I(-\theta < x_i < \theta) = 1 \ \forall i \in \{1, \dots, \nu\} \Leftrightarrow$

$\Leftrightarrow I(0 < |x_i| < \theta) = 1 \ \forall i \in \{1, \dots, \nu\} \Leftrightarrow 0 < \max\{|x_1|, \dots, |x_\nu|\} < \theta \Leftrightarrow$

$\Leftrightarrow I(0 < \max\{|x_1|, \dots, |x_\nu|\} < \theta) = 1$

Hence, $f(x; \theta) = (2\theta)^{-\nu} I(0 < \max\{|x_1|, \dots, |x_\nu|\} < \theta)$

and $T = T(X) = \max\{|X_1|, \dots, |X_\nu|\}$ is sufficient for θ.

- Distribution of T: $F_T(t) = P(T \leq t) = P(\max\{X_i\} \leq t) =$

$= P(|X_1|, \dots, |X_\nu| \leq t) = \prod_{i=1}^{\nu} P(|X_i| \leq t) = \prod_{i=1}^{\nu} P(-t \leq X_i \leq t) =$

$= \prod_{i=1}^{\nu} (F_X(t) - F_X(-t)) = (F_X(t) - F_X(-t))^\nu = \left(\frac{t+\theta}{2\theta} - \frac{-t+\theta}{2\theta}\right)^\nu = \left(\frac{t}{\theta}\right)^\nu$

So $f_T(t) = \frac{d}{dt} F_T(t) = \frac{\nu t^{\nu-1}}{\theta^\nu}$

- Let $g(T)$ function, then: $E[g(T)] = 0 \Rightarrow \int_0^\theta g(t) f_T(t) dt = 0 \Rightarrow$

$\Rightarrow \int_0^\theta g(t) \frac{\nu t^{\nu-1}}{\theta^\nu} dt = 0 \Rightarrow \int_0^\theta g(t) t^{\nu-1} dt = 0 \ \forall \theta \xRightarrow{d/d\theta}$

$\Rightarrow g(\theta)\cdot\theta^{v-1}=0 \ \forall\theta \Rightarrow g(\theta)=0 \ \forall\theta>0 \Rightarrow g(t)=0 \ \forall t\in(0,\theta)$

Finally, T is complete.

Alternatively: We set $Y=|X|$ and it is:

$$F_Y(y)=P(Y\le t)=P(|X|<y)=P(-y<x<y)=$$
$$=F_X(y)-F_X(-y)=\frac{y+\theta}{2\theta}-\frac{-y+\theta}{2\theta}=\frac{y}{\theta}=\frac{y-0}{\theta-0} \quad \text{so} \quad Y\sim U(0,\theta)$$

b) We have: $E[\psi_1(t)]=2\theta \Rightarrow \int_0^\theta \psi_1(t) f_T(t)\,dt=2\theta \Rightarrow$

$$\Rightarrow \int_0^\theta \psi_1(t)\frac{vt^{v-1}}{\theta^v}\,dt=2\theta \Rightarrow \int_0^\theta \psi_1(t)t^{v-1}\,dt=\frac{2\theta^{v+1}}{v} \xRightarrow{\partial/\partial\theta}$$
$$\Rightarrow \psi_1(\theta)\theta^{v-1}=\frac{2(v+1)}{v}\cdot\theta^v \Rightarrow \psi_1(\theta)=\frac{2(v+1)}{v}\cdot\theta$$

So the m.v.u.e of 2θ is $\psi_1(T)=\frac{2(v+1)}{v}T=\frac{2(v+1)}{v}\cdot\max\{|X_i|\}$

We have: $E[\psi_2(t)]=\frac{\theta^2}{3} \Rightarrow \int_0^\theta \psi_2(t) f_T(t)\,dt=\frac{\theta^2}{3} \Rightarrow$

$$\Rightarrow \int_0^\theta \psi_2(t)\frac{vt^{v-1}}{\theta^v}\,dt=\frac{\theta^2}{3} \Rightarrow \int_0^\theta \psi_2(t)t^{v-1}\,dt=\frac{\theta^{v+2}}{3v} \xRightarrow{\partial/\partial\theta}$$
$$\Rightarrow \psi_2(\theta)\theta^{v-1}=\frac{v+2}{3v}\cdot\theta^{v+1} \Rightarrow \psi_2(\theta)=\frac{v+2}{3v}\cdot\theta^2$$

So the m.v.u.e of $\frac{\theta^2}{3}$ is $\psi_2(T)=\frac{v+2}{3v}\cdot T=\frac{v+2}{3v}\cdot\max\{X_i\}$