Example

Let $X_1, \dots, X_\nu$ r.s of $N(\mu, \sigma^2)$, where μ, σ^2 are unknown. Let $S^2 = \frac{1}{\nu-1}\sum_{i=1}^{\nu}(X_i - \bar{X})^2$ u.e. of σ^2. Find the value of the constant c so that $c \cdot S^2$ is a minimal-MSE estimator of θ.

Solution

The MSE of an estimator Y of θ is:

$$MSE = E[(Y-\theta)^2] = V(Y) + b^2(\theta), \quad \text{where } b(\theta) = E(Y) - \theta$$

We are looking for c so that $E[(cS^2 - \sigma^2)^2]$ is minimal.

It is: $E[(cS^2-\sigma^2)^2] = V(cS^2)^2 + E[cS^2 - \sigma^2]^2 =$

$$= c^2 V(S^2) + (c\sigma^2 - \sigma^2)^2 = c^2 V(S^2) + \sigma^4 (c-1)^2 \qquad (1)$$

We know that $\frac{(\nu-1)S^2}{\sigma^2} \sim \chi^2_{\nu-1}$

So $V\left[\frac{(\nu-1)S^2}{\sigma^2}\right] = 2(\nu-1) \Rightarrow \frac{(\nu-1)^2}{\sigma^4} V(S^2) = 2(\nu-1) \Rightarrow$

$\Rightarrow V(S^2) = \frac{2\sigma^4}{\nu-1}$

Hence, $(1) \Rightarrow E[(cS^2-\sigma^2)^2] = c^2 \cdot \frac{2\sigma^4}{\nu-1} + \sigma^4(c-1)^2$.

To find the minimum we set the derivative (with respect to c) equal to zero: $2c\frac{2\sigma^4}{\nu-1} + 2\sigma^4(c-1) = 0 \Rightarrow$

$\Rightarrow 2c + (c-1)(\nu-1) = 0 \Rightarrow c = \frac{\nu-1}{\nu+1}$

The 2nd derivative is >0, hence the statistic $\frac{\nu-1}{\nu+1} \cdot S^2 =$
$= \frac{1}{\nu+1}\sum_{i=1}^{\nu}(X_i - \bar{X})^2$ is the minimal-MSE estimator of σ^2.

Reminder

R-B Theorem: Let $U = U(\underline{X})$ u.e. of θ and $T = T(\underline{X})$ sufficient, then $W = E[U|T]$ u.e of θ and $V(W) \leq V(U)$

Generalization of R-B

Let $X_1, \dots, X_n$ r.s. of $f(x;\theta)$. If $U = U(\underline{X})$ u.e of $g(\theta)$ and $T(\underline{X})$ sufficient statistic for θ, then the statistic $\delta = \delta(\underline{X}) = \varphi(T(\underline{X})) = E[U \mid T]$ is an u.e. of $g(\theta)$ and $V(\delta) \leq V(U)$

Theorem (Lehmann - Scheffé)

With the same assumptions as the R-B Theorem and the assumption that $T = T(\underline{X})$ is also complete, we have that $\delta(\underline{X}) = \varphi(T(\underline{X})) = E(U \mid T)$ is m.v.u.e. of $g(\theta)$

Proof

Let $\varphi_0(t)$ and $\varphi_1(t)$ functions of the sufficient and complete statistic T and $\varphi_0(T), \varphi_1(T)$ u.e. of $g(\theta)$. Then:

$$E[\varphi_0(T)] = g(\theta) = E[\varphi(T)] \Rightarrow E[\varphi_0(T) - \varphi_1(T)] = 0 \quad \forall \theta \in \Theta$$

$$\xrightarrow[T \text{ complete}]{} \varphi_0(t) - \varphi_1(t) = 0 \quad \forall t \Rightarrow \varphi_0(t) = \varphi_1(t) \quad \forall t$$

Proposition

If $T = T(X)$ sufficient and complete statistic for θ and $\varphi_1(T)$ u.e. of $g(\theta)$, then $\varphi_1(T)$ is m.v.u.e of θ

Proof

From L-S Theorem we get a function $\varphi(T) = E[U \mid T]$ that is an u.e. of $g(\theta)$ and has minimal variance among the estimators of $g(\theta)$. Then:

$E[\varphi[T] - \varphi_1(T)] = 0$ and since T is complete $\varphi(T) = \varphi_1(T)$

hence, $\varphi_1(T)$ is m.v.u.e. of $g(\theta)$.

Example

Let $X_1, \dots, X_n$ r.s of $N(\theta, 1)$.

a) Find m.v.u.e of θ

b) Prove that $\bar{X}^2 - \frac{1}{n}$ is m.v.u.e. of θ^2

Solution

a) We are looking for a sufficient and complete statistic for θ. We have previously proven that $T = T(\underline{X}) = \sum_{i=1}^{n} X_i$ is sufficient and complete for θ. We have also proven that $E[\bar{X}] = E[X] = \theta$. Hence, $E[\bar{X}] = E\left[\frac{\sum X_i}{n}\right] = E\left[\frac{T}{n}\right] = \theta$, so from L-S theorem $\varphi(T) = \frac{T}{n}$ is m.v.u.e. of θ (as it is an u.e. of θ and a function of the sufficient and complete T.

b) $T = \sum_{i=1}^{n} X_i$ is sufficient and complete for θ so, we just have to prove that $\bar{X}^2 - \frac{1}{n}$ is an u.e. of $g(\theta) = \theta^2$ (because it is a function of T)

$$E\left[\bar{X}^2 - \frac{1}{n}\right] = E\left[\bar{X}^2\right] - \frac{1}{n} = V(\bar{X}) + E[\bar{X}]^2 - \frac{1}{n} = \frac{1}{n} \cdot V(X) + \theta^2 - \frac{1}{n} =$$
$$= \frac{1}{n} \cdot 1 + \theta^2 - \frac{1}{n} = \theta^2.$$

Finally, $\bar{X}^2 - \frac{1}{n}$ is m.v.u.e. of θ^2.

Alternatively: $X_i \sim N(\theta, 1) \Rightarrow \bar{X} \sim N(\theta, 1/n) \Rightarrow$

$$\Rightarrow \frac{\bar{X} - \theta}{\sqrt{1/n}} \sim N(0, 1) \Rightarrow n(\bar{X} - \theta)^2 \sim \chi_1^2$$

$$E\left[n(\bar{X} - \theta)^2\right] = 1 \Rightarrow E\left[(\bar{X} - \theta)^2\right] = \frac{1}{n} \Rightarrow$$

$$\Rightarrow E\left[\bar{X}^2 + \theta^2 - 2\theta\bar{X}\right] = \frac{1}{n} \Rightarrow E\left[\bar{X}^2\right] + \theta^2 - 2\theta E[\bar{X}] = 1/n$$

$$\Rightarrow E\left[\bar{X}^2\right] + \theta^2 - 2\theta \cdot \theta = 1/n \Rightarrow E\left[\bar{X}^2 - \frac{1}{n}\right] = \theta^2$$

Example

Let $X_1, \dots, X_\nu$ r.s. of $U(0,\theta)$.

a) Find m.v.u.e of $g(\theta)=\theta^\kappa$

b) Find m.v.u.e of $E[X]$ and $V(X)$

Solution

a) $f(x;\theta) = \frac{1}{\theta} \cdot I(0<x<\theta)$, where $I(0<x<\theta) = \begin{cases} 1, & 0<x<\theta \\ 0, & \text{otherwise} \end{cases}$

We have proven that $T = X_{(\nu)} = \max\{X_i\}$ is a sufficient statistic for θ. For completeness we need $E[g(T)]=0 \ \forall \theta>0 \Rightarrow g(t)=0 \ \forall t$ for all functions $g(T)$.

$F_T(t) = P(T \le t) = P(X_{(\nu)} \le t) = P(X_1, \dots, X_\nu \le t) = \prod_{i=1}^{\nu} P(X_i \le t) = [F_X(t)]^\nu$

but $X \sim U(0,\theta)$ so $F_X(t) = \int_0^t \frac{1}{\theta} dx = \frac{t}{\theta}$.

Hence, $F_T(t) = \left(\frac{t}{\theta}\right)^\nu \xRightarrow{\partial/\partial t} f_T(t) = \frac{\nu t^{\nu-1}}{\theta^\nu}$, $t \in (0,\theta)$

Then: $E[g(T)]=0 \ \forall \theta>0 \Rightarrow \int_0^\theta g(t) f_T(t) dt = 0 \ \forall \theta>0 \Rightarrow$

$\Rightarrow \int_0^\theta g(t) \frac{\nu t^{\nu-1}}{\theta^\nu} dt = 0 \ \forall \theta>0 \Rightarrow \int_0^\theta g(t) t^{\nu-1} = 0 \ \forall \theta>0 \Rightarrow$

$\xRightarrow{\partial/\partial\theta} g(\theta)\cdot\theta^{\nu-1} = 0 \ \forall \theta>0 \Rightarrow g(\theta)=0 \ \forall \theta>0 \Rightarrow g(t)=0 \ \forall t \in (0,\theta)$

Finally, $T = X_{(\nu)}$ is complete.

To find a m.v.u.e of $g(\theta)=\theta^\kappa$ we just need to find a function $\varphi(T)$: $E[\varphi(T)] = \theta^\kappa \ \forall \theta>0$.

$E[\varphi(T)] = \theta^\kappa \Rightarrow \int_0^\theta \varphi(t) f_T(t) dt = \theta^\kappa \Rightarrow \int_0^\theta \varphi(t) \frac{\nu t^{\nu-1}}{\theta^\nu} dt = \theta^\kappa$

$\Rightarrow \int_0^\theta \varphi(t) t^{\nu-1} = \frac{\theta^{\kappa+\nu}}{\nu} \xRightarrow{\partial/\partial\theta} \varphi(\theta)\cdot\theta^{\nu-1} = \frac{\nu+\kappa}{\nu}\cdot\theta^{\nu+\kappa-1} \ \forall \theta>0 \Rightarrow$

$\Rightarrow \varphi(\theta) = \frac{\nu+\kappa}{\nu}\cdot\theta^\kappa$

Hence, the m.v.u.e of θ^κ is $\varphi(T) = \frac{\nu+\kappa}{\nu}\cdot\theta^\kappa$

b) Since $X \sim U(0,\theta)$, it is $E[X] = \frac{\theta}{2}$ and $V(X) = \frac{\theta^2}{12}$

We just proved that $\varphi(T) = \frac{\nu+\kappa}{\nu}\cdot\theta^\kappa$ m.v.u.e of θ^κ

For $k=1$: $E\left[\frac{v+1}{v}\cdot T\right]=\theta \Rightarrow E\left[\frac{v+1}{2v}\cdot T\right]=\frac{\theta}{2}$ so $\psi_2(T)=\frac{v+1}{2v}\cdot T$

m.v.u.e of $E[X]$

For $k=2$: $E\left[\frac{v+2}{v}T^2\right]=\theta^2 \Rightarrow E\left[\frac{v+2}{12v}T^2\right]=\frac{\theta^2}{12}$

so $\psi_3(T)=\frac{v+2}{12v}T^2$ m.v.u.e of $V(X)$