M.V.U.E (Minimum Variance Unbiased Estimator)

Reminder: Let $X_1, \dots, X_n$ r.s. with PDFs $f(x;\theta)$ and we want to estimate θ (or $g(\theta)$). An estimator $\delta = \delta(\underline{X})$ is called unbiased estimator of θ (or $g(\theta)$) if $E[\delta] = \theta$ (or $E[\delta] = g(\theta)$)

⇝ Among all unbiased estimators we seek for the one with the minimum variance.

Definition
An estimator $\delta = \delta(\underline{X})$ is called m.v.u.e. of θ (or $g(\theta)$) $V(\delta) \le V(\delta_1)$ for all unbiased estimators δ_1 of θ (or $g(\theta)$)

Reminder: MSE: $E[(\delta-\theta)^2] = V(\delta) + \cancel{b_\theta^2(\delta)}^{\,0}$

Unbias - Sufficiency Relation
Let $T_1 = T_1(\underline{X})$ and $T_2 = T_2(\underline{X})$ unbiased estimators of θ such that $E[T_1] = E[T_2] = \theta$. Let $T = T(\underline{X})$ sufficient statistic for θ. We suppose that $T_1 = f(T)$ for some function $f(\cdot)$. If sufficiency has any value then we expect $V(T_1) \le V(T_2)$

Lemma
Let $U = U(\underline{X})$ and $Y = Y(\underline{X})$ such that $E[U] = \theta$ and $0 \le V(U) < \infty$

Let $E[U \mid Y=y] = W(y) = W$, then $E[W] = \theta$ and $V(W) \leq V(U)$

<u>Proof</u>

Suppose that U, Y are continuous random variables (with PDFs $g_2(u)$, $g_1(y)$), $g(u,v)$ is their joint PDF and $h(u \mid y)$ the ∂ PDF of $U \mid Y=y$

We have $E[U \mid Y=y] = \int_{-\infty}^{+\infty} u h(u \mid y) \partial u$, so:

$$E[W] = \int_{-\infty}^{+\infty} w(y) g_1(y) \partial y = \int_{-\infty}^{+\infty} E[U \mid Y=y] g_1(y) \partial y =$$

$$= \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} u h(u \mid y) \partial u \right] g_1(y) \partial y = \iint u \frac{g(u,y)}{g_1(y)} \cdot g_1(y) \partial y =$$

$$= \int u \left[\int g(u,y) \partial y \right] \partial u = \int u g_2(u) \partial u = E[U] = \theta$$

So $E[U] = E[W] = \theta$.

We have: $V(U) = E[(U - \overset{\theta}{E(U)})^2] = E[(U - W + W - \theta)^2] =$

$$= E[(U-W)^2] + E[(W-\theta)^2] + 2E[(U-W)(W-\theta)]$$

and it is $E[(U-W)(W-\theta)] = \iint (u-w)(w-\theta) g(u,y) \partial u \partial y =$

$$= \iint (u-w)(w-\theta) h(u \mid y) \partial u \partial y = \int (w-\theta) g(y) \left[\int (u-w) h(u \mid y) \partial u \right] \partial y$$

but $\int (u - w(y)) h(u \mid y) \partial u = \int u h(u \mid y) \partial u - \int w(y) h(u \mid y) \partial u =$

$$= E[U \mid Y=y] - W(y) \cdot \int h(u \mid y) \partial u = E[U \mid Y=y] - W(y) = 0$$

So $V(U) = E[(U-W)^2] + E[(W-\theta)^2]$

and $E[(W-\theta)^2] = V(W)$ since $E[W-\theta]^2 = 0$

Finally $V(U) = E[(U-W)^2] + V(W) \geq V(W)$, since $E[(U-W)^2] \geq 0$

(Alternatively: $V(U) = V(E[U \mid Y]) + E[V(U \mid Y)] = V(W) + \underbrace{E[V(U \mid Y)]}_{\geq 0}$

Note: $W = W(y) = E[U \mid Y=y]$ is a statistic because U is unbiased.

Rao-Blackwell Theorem

Let $X_1, \dots, X_v$ r.s with PDF $f(x;\theta)$ (discrete or continuous).
Let $Y = Y(\underline{X})$ sufficient statistic for θ and $U = U(\underline{X})$ unbiased estimator of θ. Then, the statistic $W = W(Y) = E[U \mid Y=y]$ is an unbiased estimator of θ (so $E[W] = \theta$) and $V(W) \le V(U)$

Proof

The work above

Example

Let $X_1, \dots, X_v$ r.s. of Bernoulli(θ) with PDF $f(x;\theta) = \theta^x (1-\theta)^{1-x}$, $x \in \{0,1\}$, $\theta \in (0,1)$

We know that $E[X_i] = \theta$ and that $T = T(\underline{X}) = \sum_{i=1}^{v} X_i$ is sufficient for θ. According to the R-B theorem, the r.v. $W = E[X_1 \mid T=t]$ is an unbiased estimator of θ and $V(W) \le V(U)$.

For $1 \le t \le v$: $E[X_1 \mid T=t] = \sum_{x_1=0}^{1} x_1 \cdot P(X_1 = x_1 \mid T=t) =$

$= 0 \cdot P(X_1 = 0 \mid T=t) + 1 \cdot P(X_1 = 1 \mid T=t) = P(X_1 = 1 \mid T=t) =$

$= \dfrac{P(X_1=1, T=t)}{P(T=t)} = P(X_1 = 1, \sum_{i=2}^{v} X_i = t-1) / P(T=t) =$

$= P(X_1=1) \cdot P(\sum_{i=2}^{v} X_i = t-1) / P(\sum_{i=1}^{v} X_i = t) = \dfrac{\theta \binom{v-1}{t-1} \theta^{t-1} (1-\theta)^{v-t}}{\binom{v}{t} \theta^t (1-\theta)^{v-t}} = \dfrac{\binom{v-1}{t-1}}{\binom{v}{t}} = \dfrac{t}{v}$

For $t=0$: $E[X_1 \mid T=0] = \sum_{x_1=0}^{1} P(X_1 = x_1 \mid T=0) = 0$

Hence, $E[X_1 \mid T=t] = \frac{t}{v} = \sum_{i=1}^{v} x_i / v = \bar{X} \quad \forall t \in \{0, \dots, v\}$

Obviously, $E[W] = E[\bar{X}] = \theta$ and $V(W) = V\left(\frac{\sum X_i}{v}\right) = \frac{\theta(1-\theta)}{v} < \theta(1-\theta) = V(X_1)$

The statistic $T_1(X_v, \sum_{i=1}^{v-1} X_i)$ is also sufficient for θ.

$$f[X_1 \mid T_1 = t_1] = P(X_1 = x_1 \mid T_1 = t_1) = \frac{P(X_1 = x_1, X_v = x_v, \sum_{i=1}^{v-1} X_i = t_1)}{P(X_v = x_v, \sum_{i=1}^{v-1} X_i = t_1)} =$$

$$= \frac{P(X_1 = x_1) \cdot P(X_v = x_v) \cdot P(\sum_{i=2}^{v-1} X_i = t_1 - x_1)}{P(X_v = x_v) \cdot P(\sum_{i=1}^{v-1} X_i = t_1)} = \frac{\theta \binom{v-2}{t_1-1} \theta^{t_1-1} (1-\theta)^{v-t_1-1}}{\binom{v-1}{t_1} \theta^{t_1} (1-\theta)^{v-t_1-1}} =$$

$$= \frac{\binom{v-2}{t_1-1}}{\binom{v-1}{t_1}} = \frac{t_1}{v-1}$$, so $W_1 = f[X_1 \mid T_1]$ u.e. of θ

an $$V(W) = V\left(\sum_{i=1}^{v-1} X_i / v-1\right) = \frac{1}{(v-1)^2} \cdot (v-1) \cdot \theta(1-\theta) =$$

$$= \frac{\theta(1-\theta)}{v-1} > \frac{\theta(1-\theta)}{v} = V(W)$$