Example

Let $X_1, \dots, X_n$ r.s of $U(0,\theta)$. Find a sufficient and complete statistic for θ.

Solution

a) Factorization criterion: $f(\underline{x};\theta) = \prod_{i=1}^{n} f(x_i;\theta) =$

$$= \prod_{i=1}^{n} \frac{1}{\theta} I(0<x_i<\theta) = \theta^{-n} \prod_{i=1}^{n} I(0<x_i<\theta) \quad (1)$$

We demand $I(0<x_i<\theta)=1 \ \forall i \Leftrightarrow x_i \in (0,\theta) \ \forall i \Leftrightarrow$

$$\Leftrightarrow 0 < x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)} < \theta \Leftrightarrow$$

$$\Leftrightarrow I(0<x_{(1)}<\infty) \cdot I(0<x_{(n)}<\theta) = 1$$

So $(1) \Rightarrow f(\underline{x};\theta) = \theta^{-n} I(0<x_{(1)}<\infty) \cdot I(0<x_{(n)}<\theta) =$

$= g(T(\underline{x});\theta) \cdot h(\underline{x})$, where $h(\underline{x}) = I(0<x_{(1)}<\infty)$ and $g(T(\underline{x});\theta) = \theta^{-n} I(0<x_{(n)}<\theta)$. Hence, $T(\underline{x}) = x_{(n)} = \max\{x_i\}$ sufficient for θ.

b) To prove that $T(\underline{x})$ is complete we need the PDF of $T(X) = X_{(n)} = T$. We will work with the cumulative distribution function (CDF) of T.

$$F_T(t) = P(T \leq t) = P(X_{(n)} \leq t) = P(X_1, \dots, X_n \leq t) =$$

$$= P(X_1 \leq t) \cdot \dots \cdot P(X_n \leq t) = F_{X_1}(t) \cdot \dots \cdot F_{X_n}(t) = [F_X(t)]^n$$

We know $X \sim U(0,\theta)$, hence: $F_X(t) = \int_0^t f(x)\,dt =$

$= \int_0^t \frac{1}{\theta}\,dx = \frac{t}{\theta}$, so $F_T(t) = \left(\frac{t}{\theta}\right)^n = t^n/\theta^n$

Finally, $f_T(t) = \frac{\partial}{\partial t} F_T(t) = \frac{n t^{n-1}}{\theta^n}$

c) Completeness: Let $g(T)$ function of T

We have: $E[g(T)] = \int_0^{\theta} g(t) \frac{\nu t^{\nu-1}}{\theta^{\nu}} dt = 0 \quad \forall \theta > 0 \Rightarrow$

$\Rightarrow \int_0^{\theta} g(t) t^{\nu-1} dt = 0 \quad \forall \theta > 0 \xrightarrow[\partial/\partial\theta]{} g(\theta)\theta^{\nu-1} = 0 \quad \forall \theta > 0$

$\Rightarrow g(\theta) = 0 \quad \forall \theta > 0 \quad \Rightarrow$

$\Rightarrow g(t) = 0 \quad \forall t \in (0, \theta)$. Hence T is also complete.

Example

Let $X_1, \dots, X_\nu$ r.s of PDF $f(x; \theta) = \frac{\theta}{x^2}$, $\theta > 0$, $x > \theta$

Prove that $T(\underline{x}) = X_{(1)} = \min(X_i)$ is a sufficent and complete statistic for θ.

Solution

a) Factorization criterion: $f(\underline{x}; \theta) = \prod_{i=1}^{\nu} \frac{\theta}{x_i^2} I(\theta < x_i < \infty) =$

$= \theta^{\nu} \cdot \frac{1}{\prod_{i=1}^{\nu} x_i^2} \prod_{i=1}^{\nu} I(\theta < x_i < \infty)$

We demand $\prod_{i=1}^{\nu} I(\theta < x_i < \infty) = 1 \Leftrightarrow I(\theta < x_i < \infty) = 1 \quad \forall i \Leftrightarrow$

$\Leftrightarrow x_i \in (\theta, +\infty) \quad \forall i \Leftrightarrow \theta < x_{(1)} \le \dots \le x_{(\nu)} < \infty$

Hence $f(\underline{x}; \theta) = \theta^{\nu} \frac{1}{\prod_{i=1}^{\nu} x_i^2} I(\theta < x_{(1)} < \infty) \cdot I(0 < x_{(\nu)} < \infty) =$

$= g(T(\underline{x}); \theta) \cdot h(\underline{x})$ where $h(\underline{x}) = \frac{1}{\prod x_i^2} \cdot I(0 < x_{(\nu)} < \infty)$,

$g(\underline{x}) = \theta^{\nu} \cdot I(\theta < x_{(1)} < \infty)$, so $T(\underline{x}) = X_{(1)}$ is sufficient for θ.

b) We find the distribution of $T = X_{(1)}$:

$F_T(t) = P(T \le t) = P(X_{(1)} \le t) = 1 - P(X_{(1)} > t) =$

$= 1 - P(X_1, \dots, X_\nu > t) = 1 - P(X_1 > t) \cdot P(X_2 > t) \cdots P(X_\nu > t) =$

$= 1 - \prod_{i=1}^{\nu} P(X_i > t) = 1 - \prod_{i=1}^{\nu} (1 - P(X_i \le t)) = 1 - \prod_{i=1}^{\nu} (1 - F_{X_i}(t)) =$

$= 1 - (1 - F_X(t))^{\nu}$.

We know: $F_X(t) = \int_{\theta}^{t} \frac{\theta}{x^2} dx = \theta \cdot \left(-\frac{1}{t} + \frac{1}{\theta}\right) = 1 - \frac{\theta}{t}$

Hence: $F_T(t) = 1 - \left[1 - \left(1 - \frac{\theta}{t}\right)\right]^{\nu} = 1 - \left(\frac{\theta}{t}\right)^{\nu}$

and $f_T(t) = \frac{\partial}{\partial t} F_T(t) = \frac{\nu \theta^{\nu}}{t^{\nu+1}}$

c) Completeness: Let $g(T)$ function of T. Then:

$$E[g(T)] = 0 = \int_{\theta}^{\infty} g(t) \frac{v\theta^{v}}{t^{v+1}} dt = 0 \quad \forall \theta > 0 \Rightarrow$$

$$\Rightarrow \int_{\theta}^{\infty} g(t) \cdot t^{-v-1} = 0 \quad \forall \theta > 0 \overset{\partial/\partial\theta}{\Rightarrow} -g(\theta) t^{-v-1} = 0 \quad \forall \theta > 0$$

$$\Rightarrow g(\theta) = 0 \quad \forall \theta > 0 \Rightarrow g(t) = 0 \quad \forall t > \theta$$

Hence T is complete.

Reminders

1) $X_1, \dots, X_v \sim Bernoulli(p) \Rightarrow T = \sum_{i=1}^{v} X_i \sim Bin(v, p)$
2) $X_1, \dots, X_v \sim Bin(N, p) \Rightarrow T \sim Bin(vB, p)$
3) $X_1, \dots, X_v \sim Poisson(\lambda) \Rightarrow T \sim Poisson(v\lambda)$
4) $X_1, \dots, X_v \sim N(\mu, \sigma^2) \Rightarrow T \sim (v\mu, v\sigma^2)$
5) $X_1, \dots, X_v \sim Exp(\theta) \Rightarrow T \sim Gamma(v, \theta)$

↳ Forms of $Gamma(a, b)$:

(I) $f(x) = \frac{1}{\Gamma(a)\beta^a} x^{a-1} \cdot e^{-x/\beta} = \frac{(1/\beta)^a}{\Gamma(a)} \cdot x^{a-1} \cdot e^{-x/\beta}$

(II) $f(x) = \frac{\beta^a}{\Gamma(a)} \cdot x^{a-1} e^{-\beta x}$

6) $Gamma(1, \theta)$ is $Exp(\theta)$
7) $Gamma(\frac{r}{2}, 1)$ is χ^2_r (form I)
   or $Gamma(\frac{r}{2}, \frac{1}{2})$ is χ^2_r (form II)

8) χ^2_2 is $Gamma(1, 2) \rightarrow Exp(2)$ (form I)
   $Gamma(1, 1/2) \rightarrow Exp(\frac{1}{2})$ (form II)
9) $X_1, \dots, X_v \sim Exp(2)$ (form I) $\Rightarrow T \sim Gamma(v, 2)$
   $Exp(1/2)$ (form II) $\Rightarrow T \sim Gamma(v, 1/2)$