Πχ) Ομοιόμορφη στο $(0,\theta)$, όπου $\theta$-άγνωστη παράμετρος.

$$f(x;\theta) = \begin{cases} 1/\theta, & 0 < x < \theta \\ 0, & \text{αλλού} \end{cases}$$

ή

$$f(x;\theta) = \frac{1}{\theta} \cdot I_{(0,\theta)}(x), \quad \text{όπου}$$

$$I_{(\alpha,\beta)}(x) = \begin{cases} 1, & \alpha < x < \beta \\ 0, & \text{αλλού} \end{cases}$$

Η κατανομή δεν ανήκει στην ΕΟΚ αφού το στήριγμα εξαρτάται από την παράμετρο $\theta$.

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$\boxed{\Theta}$ Για την ΕΟΚ ισχύουν.

α) $$E(T_i(X)) = \frac{\partial A(\underline{\eta})}{\partial \eta_i}$$

β) $$Cov(T_i(X), T_j(X)) = \frac{\partial^2 A(\underline{\eta})}{\partial \eta_i \partial \eta_j}$$

ΚΑΝΟΝΙΚΗ ΜΟΡΦΗ

$$\boxed{f(x;\underline{\eta}) = \exp\left\{\sum_{i=1}^{s} \eta_i T_i(x) - A(\underline{\eta})\right\} \cdot h(x)}$$

γ) Ροπογεννήτρια του $\underline{T}(x)$

$$M_{\underset{\sim}{T}}(\underset{\sim}{u}) = E\left[\exp\left\{\sum u_i T_i(x)\right\}\right]$$

$$= \exp\left\{ A(\underset{\sim}{u}+\underline{\eta}) - A(\underset{\sim}{\eta}) \right\}.$$

<u>Απόδειξη</u>

Έστω η τ.μ. X ανήκει στην ΕΟΚ έχουμε:

$$f(x;\underline{\eta}) = \exp\left\{\sum_{i=1}^{s} \eta_i T_i(x) - A(\underline{\eta})\right\} \cdot h(x)$$

$$E(T_i(x)) = \int T_i(x) f(x) dx$$

Γνωρίζουμε ότι: $\int f(x;\underline{\eta}) dx = 1 \Rightarrow$

$$\Rightarrow \int \exp\left\{\sum \eta_i T_i(x) - A(\underline{\eta})\right\} \cdot h(x) dx = 1 \Rightarrow$$

$$\Rightarrow \int \exp\left\{\sum \eta_i T_i(x)\right\} \cdot h(x) dx = \exp\left\{A(\underline{\eta})\right\}$$

$$\xRightarrow[\text{ως προς } \eta_i]{\text{παρ.}} \frac{\partial}{\partial \eta_i} \int \exp\left\{\sum \eta_i T_i(x)\right\} \cdot h(x) dx = \frac{\partial}{\partial \eta_i} \exp\left\{A(\underline{\eta})\right\}$$

$$\Rightarrow \int \frac{\partial}{\partial \eta_i} \exp\left\{\sum \eta_i T_i(x)\right\} \cdot h(x) dx = \exp\left\{A(\underline{\eta})\right\} \cdot \frac{\partial A(\underline{\eta})}{\partial \eta_i}$$

$$\Rightarrow \int h(x) \exp\left\{\sum \eta_i T_i(x)\right\} \cdot T_i(x) dx = \exp\left\{A(\underline{\eta})\right\} \cdot \frac{\partial A(\underline{\eta})}{\partial \eta_i} \quad (*)$$

$$\Rightarrow \underbrace{\int T_i(x) \exp\left\{\sum \eta_i T_i(x) - A(\underline{\eta})\right\} \cdot h(x) dx}_{} = \frac{\partial A(\underline{\eta})}{\partial \eta_i}$$

$$\Rightarrow \quad E(T_i(x)) = \frac{\partial A(\underline{\eta})}{\partial \eta_i} \quad \blacksquare$$

β) (*) $\longrightarrow$ παραγωγίζω ως προς $\eta_j$

$$\frac{\partial}{\partial \eta_j}\left[\int h(x)\exp\left\{\sum \eta_i T_i(x)\right\} T_i(x)\,dx\right] = \frac{\partial}{\partial \eta_j}\left[\exp\{A(\underline{\eta})\}\frac{\partial A(\underline{\eta})}{\partial \eta_i}\right]$$

$$\Rightarrow \int h(x)\exp\left\{\sum \eta_i T_i(x)\right\} T_i(x) T_j(x)\,dx =$$

$$= \exp\{A(\underline{\eta})\}\frac{\partial}{\partial \eta_i}A(\underline{\eta})\frac{\partial}{\partial \eta_j}A(\underline{\eta}) + \exp\{A(\underline{\eta})\}\frac{\partial^2 A(\underline{\eta})}{\partial \eta_i \partial \eta_j}$$

$$\Rightarrow \underbrace{\int T_i(x) T_j(x) \exp\left\{\sum \eta_i T_i(x) - A(\underline{\eta})\right\}\cdot h(x)\,dx}_{\longrightarrow E(T_i(x)T_j(x))} =$$

$$= \underbrace{\frac{\partial}{\partial \eta_i}A(\underline{\eta})}_{E(T_i(x))}\underbrace{\frac{\partial}{\partial \eta_j}A(\underline{\eta})}_{E(T_j(x))} + \frac{\partial^2}{\partial \eta_i \partial \eta_j}A(\underline{\eta}) \Rightarrow$$

$$\Rightarrow E(T_i(x)T_j(x)) - E(T_i(x))\cdot E(T_j(x)) = \frac{\partial^2}{\partial \eta_i \partial \eta_j}A(\underline{\eta})$$

$$\Rightarrow Cov(T_i(x), T_j(x)) = \frac{\partial^2}{\partial \eta_i \partial \eta_j}A(\underline{\eta}) \qquad \blacksquare$$

γ) Ροπογεννήτρια της τ.μ.

$$T(x) = (T_1(x), T_2(x), \ldots, T_s(x))$$

$$T(x) = (T_1(x), T_2(x), \ldots, T_s(x))$$

Απόδειξη: $$M_{\underline{T}}(\underline{u}) = E\left\{e^{\sum_{i=1}^{s} u_i T_i(x)}\right\} =$$

$$= \int \exp\left\{\sum u_i T_i(x)\right\} \cdot f(x;\eta)\, dx =$$

$$= \int \exp\left\{\sum u_i T_i(x) + \sum \eta_i T_i(x) - A(\underline{\eta})\right\} \cdot h(x)\, dx =$$

$$= \exp\{-A(\underline{\eta})\} \cdot \int \exp\left\{\sum (u_i+\eta_i) T_i(x)\right\} \cdot h(x)\, dx =$$

$$= \exp\{A(\underline{u}+\underline{\eta}) - A(\underline{\eta})\} \cdot \underbrace{\int \exp\left\{\sum (u_i+\eta_i) T_i(x) - A(\underline{u}+\underline{\eta})\right\} \cdot h(x)\, dx}_{\longrightarrow 1}$$

$$= \exp\{A(\underline{u}+\underline{\eta}) - A(\underline{\eta})\}.$$

Οπότε: $$M_{\underline{T}}(\underline{u}) = \exp\{A(\underline{u}+\underline{\eta}) - A(\underline{\eta})\} \quad \blacksquare$$

# P3

Δευτέρα, 4 Οκτωβρίου 2021 2:07 μμ

**Παρατήρηση:** Στην μονοπαραμετρική ΕΟΚ

$$f(x;\eta) = \exp\{\eta T(x) - A(\eta)\} \cdot h(x)$$

σύμφωνα με το θεώρημα:

$$\to E(T(x)) = \frac{\partial A(\eta)}{\partial \eta}$$

$$\to V(T(x)) = Cov(T(x), T(x)) = \frac{\partial^2 A(\eta)}{\partial \eta^2}$$

$$\to M_T(u) = E\left(e^{uT(x)}\right) = \exp\{A(u+\eta) - A(\eta)\}$$

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**Ροπογεννήτριες**

$$\underline{M_X(u)} = E\left(e^{uX}\right) = \begin{cases} \Delta \to \sum_x e^{ux} P(X=x) \\ \Sigma \to \int e^{ux} f(x)\,dx \end{cases}$$

**Διδιάστατη:** $M_{X_1,X_2}(u_1,u_2) = E\left[e^{u_1 X_1 + u_2 X_2}\right]$

**Πολυδιάστατη:** $M_{\underline{X}}(\underline{u}) = E\left[e^{\sum u_i X_i}\right]$

Έχουμε: $E(X) = \frac{d}{du} M_X(u) \Big|_{u=0}$

Π.χ. $X \sim \text{Poisson}(\lambda)$ με $M_X(u) = e^{\lambda(e^u - 1)}$

Άρα: $\frac{d}{du} M_X(u) = \lambda e^u \cdot e^{\lambda \cdot (e^u - 1)}$

$$\frac{d}{du} M_X(u) \Big|_{u=0} = \lambda$$

Μέσω ΕΟΚ $\quad f(x;\lambda) = e^{-\lambda} \frac{\lambda^x}{x!}$

$$= \exp\{ \underbrace{-\lambda}_{B(\lambda)} + \underbrace{x \log \lambda}_{\eta(\lambda) \cdot T(x)} \} \underbrace{\frac{1}{x!}}_{h(x)}$$

Συνεπώς ανήκει στην ΕΟΚ.

Έχουμε: $\eta(\lambda) = \log \lambda = \eta \Rightarrow \lambda = e^{\eta}$

Άρα: $A(\eta) = e^{\eta}$, $h(x) = 1/x!$, $T(x) = x$

α) $E(T(X)) = E(X) = \frac{d}{d\eta} A(\eta) = \frac{d}{d\eta} e^{\eta} = e^{\eta} = \lambda$

β) $V(T(x)) = V(x) = \frac{d^2}{d\eta^2} A(\eta) = \frac{d}{d\eta} e^{\eta} = e^{\eta} = \lambda$

γ) $M_T(u) = \exp\{A(u+\eta) - A(\eta)\} = \exp\{e^{u+\eta} - e^{\eta}\} =$

$= \exp\{e^{\eta} \cdot (e^{u} - 1)\} = \exp\{\lambda \cdot (e^{u} - 1)\}$ ⊠