PL

Παρασκευή, 1 Οκτωβρίου 2021 12:36 μμ

Δειγμ. διασπορά: $S^2 = \frac{1}{\nu-1} \sum_{i=1}^{\nu} (x_i - \bar{x})^2$

επειδή: $\hat{\mu} = \bar{x} = \frac{\sum_{i=1}^{\nu} x_i}{\nu}$.

$\hat{\sigma}^2 = \frac{1}{\nu} \cdot \sum_{i=1}^{\nu} (x_i - \mu)^2$ όταν μ - γνωστό!

ΕΚΘΕΤΙΚΗ ΟΙΚΟΓΕΝΕΙΑ ΚΑΤΑΝΟΜΩΝ

Θα έχει:

$$f(x;\underset{\sim}{\theta}) = \exp\left\{ \sum_{i=1}^{s} q_i(\underset{\sim}{\theta}) T_i(x) - B(\underset{\sim}{\theta}) + H(x) \right\}$$

όπου: $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_s)$.

$$= b(\underset{\sim}{\theta}) \cdot \exp\left\{ \sum_{i=1}^{s} q_i(\underset{\sim}{\theta}) T_i(x) \right\} \cdot h(x)$$

όπου: $T_i(\cdot), B(\cdot), q_i(\cdot), H(\cdot), b(\cdot), h(\cdot)$

είναι πραγματικές συναρτήσεις. και ισχύει

$h(x) > 0 \quad \forall x \in S_f$ και $\beta(\underline{\theta}) > 0 \quad \forall \underline{\theta} \in \mathbb{R}^s$

Το στήριγμα (support) της τ.μ. X δεν εξαρτάται από το $\underset{\sim}{\theta}$

↳ $S_f = \{ x \in \mathbb{R} ; f(x;\underline{\theta}) > 0 \}$.

P2

Παρασκευή, 1 Οκτωβρίου 2021 1:48 μμ

Πχ] $X \sim Bin(\nu, p)$ ν - γνωστό, p - άγνωστο $\boxed{s=1}$

$$f(x;p) = \binom{\nu}{x} p^x \cdot (1-p)^{\nu - x} =$$

$$= \binom{\nu}{x} \left(\frac{p}{1-p}\right)^x (1-p)^{\nu} =$$

$$= \binom{\nu}{x} \cdot \exp\left\{x \log \frac{p}{1-p}\right\} \cdot (1-p)^{\nu} =$$

$$= b(p) \cdot \exp\{\eta(p) T(x)\} \cdot h(x)$$

<u>όπου</u>: $b(p) = (1-p)^{\nu}$ || $h(x) = \binom{\nu}{x}$

$\eta(p) = \log \frac{p}{1-p}$ || $T(x) = x$

$\rightarrow \beta(p) = -\nu \log(1-p)$

Το στήριγμα <u>δεν</u> εξαρτάται από το p.

Άρα $\rightarrow$ <u>ΕΟΚ</u>

Με κατάλληλη αναπαραμετροποίηση

$\underline{\eta} = (\eta_1(\underline{\theta}), \eta_2(\underline{\theta}), \dots, \eta_s(\underline{\theta}))$

παίρνουμε την κανονική μορφή της ΕΟΚ

$$f(x;\underline{\eta}) = \exp\left\{\sum_{i=1}^{s} \eta_i T_i(x) - A(\underline{\eta})\right\} h(x).$$

Πχ) Διωνυμική

Θέτω: $\eta(p) = \log\frac{p}{1-p} = \eta \Rightarrow$

$\Rightarrow \frac{p}{1-p} = e^{\eta} \Rightarrow p = e^{\eta} - e^{\eta}\cdot p \Rightarrow$

$$\Rightarrow \boxed{p = \frac{e^{\eta}}{1+e^{\eta}}}$$

Συνεπώς, η κανονική μορφή είναι:

$$f(x;\underline{\eta}) = \exp\left\{\eta\cdot x + \nu\log\left(1 - \frac{e^{\eta}}{1+e^{\eta}}\right)\right\}\binom{\nu}{x}$$

$$= \exp\{\eta \cdot T(x) - A(\eta)\} \cdot h(x)$$

כאשר: $T(x) = x \quad , \quad h(x) = \binom{\nu}{x}$

וכן: $A(\eta) = -\nu \log\left(1 - \frac{e^{\eta}}{1+e^{\eta}}\right).$

πχ) $X \sim N(\mu,\sigma^2)$ με σ^2-γνωστό

$$f(x;\mu) = \frac{1}{\sqrt{2\pi}\,\sigma}\cdot \exp\left\{-\frac{1}{2\sigma^2}\cdot(x-\mu)^2\right\} =$$

$$= \frac{1}{\sqrt{2\pi}\cdot\sigma}\exp\left\{-\frac{x^2}{2\sigma^2}+\frac{x\mu}{\sigma^2}-\frac{\mu^2}{2\sigma^2}\right\} =$$

$$= b(\mu)\cdot\exp\{\eta(\mu)\cdot T(x)\}\cdot h(x)$$

όπου: $b(\mu)=\exp\left\{-\frac{\mu^2}{2\sigma^2}\right\}$ || $T(x)=X$

$\eta(\mu)=\frac{\mu}{\sigma^2}$ || $h(x)=\frac{1}{\sqrt{2\pi}\,\sigma}\cdot\exp\left\{-\frac{x^2}{2\sigma^2}\right\}$

$\to B(\mu)=\mu^2/2\sigma^2$

Έστω: $\eta(\mu)=\frac{\mu}{\sigma^2}=\eta \Rightarrow \mu=\eta\sigma^2$

Άρα: $A(\eta)=\frac{\mu^2}{2\sigma^2}=\frac{\eta^2\sigma^4}{2\sigma^2}=\frac{\eta^2\sigma^2}{2}$

Κανονική Μορφή

$$f(x;\eta)=\frac{1}{\sqrt{2\pi}\,\sigma}\cdot\exp\left\{x\cdot\eta-A(\eta)-\frac{x^2}{2\sigma^2}\right\}$$

$h(x)=\frac{1}{\ldots}\exp\left\{-\frac{x^2}{\ldots}\right\}$ $T(x)=X$

$\sqrt{2\pi}\,\sigma \qquad 2\sigma^2) \; , \qquad$

Sol: $A(\mu) = \dfrac{\mu^2 \sigma^2}{2}$

P5

Παρασκευή, 1 Οκτωβρίου 2021 2:28 μμ

<u>πχ)</u> $X \sim N(\mu, \sigma^2)$ με μ-γνωστό

$$f(x;\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \cdot \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

$$= b(\sigma^2) \cdot \exp\{\eta(\sigma^2) \cdot T(x)\}\, h(x)$$

<u>όπου:</u> $b(\sigma^2) = \frac{1}{\sqrt{\sigma^2}}$, $\eta(\sigma^2) = \frac{1}{\sigma^2}$

$T(x) = -\frac{(x-\mu)^2}{2}$, $h(x) = \frac{1}{\sqrt{2\pi}}$

<u>και:</u> $B(\sigma^2) = \frac{1}{2}\log\sigma^2$

Έστω: $\eta(\sigma^2) = \frac{1}{\sigma^2} = \eta \Rightarrow \sigma^2 = \frac{1}{\eta}$

<u>Οπότε:</u> $A(\eta) = \frac{1}{2}\log\sigma^2 = -\frac{1}{2}\log\eta$

<u>κανονική μορφή</u>

$$f(x;\eta) = \exp\left\{\frac{1}{2}\log\eta\right\} \cdot \exp\left\{-\eta\frac{(x-\mu)^2}{2}\right\} \cdot \frac{1}{\sqrt{2\pi}}$$

$$f(x,\eta) = \exp\left\{\frac{1}{2}\log\eta\right\}\cdot\exp\left\{-\eta\frac{(x-\mu)}{2}\right\}\cdot\frac{1}{\sqrt{2\pi}}$$

Παρ/ $X \sim N(\mu,\sigma^2)$ με μ,σ^2-άγνωστα ⊛

$$f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma}\cdot\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} =$$

— —

$s=2$

$$\boxed{⊛\ f(x;\mu,\sigma^2) = \exp\left\{\eta_1(\mu,\sigma^2)T_1(x)+\eta_2(\mu,\sigma^2)T_2(x)-B(\mu,\sigma^2)\right\}\cdot h(x)}$$

$$= \frac{1}{\sqrt{2\pi}}\cdot\exp\left\{-\frac{1}{2}\log\sigma^2-\frac{x^2}{2\sigma^2}+\frac{x\mu}{\sigma^2}-\frac{\mu^2}{2\sigma^2}\right\}$$

άρα: $h(x) = \frac{1}{\sqrt{2\pi}}$

$\eta_1(\mu,\sigma^2) = -\frac{1}{2\sigma^2}$, $T_1(x) = x^2$

$$\eta_2(\mu,\sigma^2)=\frac{\mu}{\sigma^2}\quad,\quad T_2(x)=x$$

$$B(\mu,\sigma^2)=\frac{1}{2}\log\sigma^2+\frac{\mu^2}{2\sigma^2}$$

<u>Κανονική Μορφή</u>

$$\eta_1(\mu,\sigma^2)=-\frac{1}{2\sigma^2}=\eta_1 \Rightarrow \sigma^2=-\frac{1}{2\eta_1}$$

$$\eta_2(\mu,\sigma^2)=\frac{\mu}{\sigma^2}=\eta_2 \Rightarrow \mu=\eta_2\sigma^2$$

$$\Rightarrow \mu=-\frac{\eta_2}{2\eta_1}$$

<u>Άρα</u>: $A(\eta_1,\eta_2)=\frac{1}{2}\log\sigma^2+\frac{\mu^2}{2\sigma^2}$

$$=-\frac{1}{2}\log(-2\eta_1)+\frac{\eta_2^2}{4\eta_1^2}\cdot(-\eta_1)=$$

$$=-\frac{1}{2}\log(-2\eta_1)-\frac{\eta_2^2}{4\eta_1}$$

<u>Οπότε</u>:

$$f(x;\eta_1,\eta_2)=\frac{1}{\sqrt{2\pi}}\exp\left\{\eta_1x^2+\eta_2x+\frac{1}{2}\log(-2\eta_1)+\frac{\eta_2^2}{4\eta_1}\right\}$$

<u>Now</u>: $T_1(x) = x^2$, $T_2(x) = x$

$$A(\eta_1, \eta_2) = -\frac{1}{2}\log(-2\eta_1) - \frac{\eta_2^2}{4\eta_1}$$

$$h(x) = \frac{1}{\sqrt{2\pi}}$$