5-12-07

33)

$$\sum_{n=1}^{\infty} \lambda(\lambda-1)\cdots(\lambda-n+1)\frac{r^n\cos(n\theta)}{n!}, \quad \lambda\in\mathbb{R},\ r\in(0,1) \quad r=\frac{1}{\sqrt{3}},\ \theta=\frac{\pi}{2},\ \lambda=\frac{1}{2}$$

Θέτουμε $z_0 = re^{i\theta}$

$$|1+z_0| = \sqrt{1+r^2+2r\cos\theta}$$

$$\Sigma = \mathrm{Re}\left((1+z_0)^{\lambda}\right) - 1$$

$$(1+z_0)^{\lambda} = e^{\lambda\log(1+z_0)} = e^{\lambda\log|1+z_0| + i\arg(1+z_0)} = e^{\lambda\log|1+z_0|}\cdot(\cos\lambda\arg(1+z_0) + i\sin\lambda\arg(1\ldots$$

$$\mathrm{Re}\, e^{x+yi} = e^x\cos y \,/\, e^{x+yi} = e^x(\cos y + i\sin y)$$

$$\mathrm{Re}\,(1+z_0)^{\lambda} = e^{\lambda\log|1+z_0|}\cos(\lambda\arg|1+z_0|)$$

$$e^{\lambda\log|1+z_0|} = \left(e^{\log|1+z_0|}\right)^{\lambda} = |1+z_0|^{\lambda}$$

$1+z_0 = \underset{\text{αρνητικός}}{a} \Rightarrow z_0 = 1-a \Rightarrow |z_0| = 1+|a| \geq 1$ άτοπο γιατί $|z_0| = r$

Άρα $1+z_0 \neq a$ Άρα έχει έννοια ο log που γράψαμε πιο πάνω.

$$\mathrm{Re}\,(1+z_0) = |1+z_0|^{\lambda}\cos(\lambda\arg(1+z_0))$$

$z\neq 0$

$$z = x+yi = \sqrt{x^2+y^2}\left(\frac{x}{\sqrt{x^2+y^2}} + i\frac{y}{\sqrt{x^2+y^2}}\right)$$

$$\mathbb{R}\ni\theta \Rightarrow \cos\theta^2 + \sin^2\theta = 1$$

Λήμμα

$(x,y)\in\mathbb{R}^2 : x^2+y^2=1$

Τότε $\exists$ μοναδικό $\theta\in(-\pi,\pi]$ ώστε : $x=\cos\theta$

$y=\sin\theta$

Άρα για κάθε $\exists\,\theta\in(-\pi,\pi]$ ώστε:

$$\cos\theta=\frac{x}{\sqrt{x^2+y^2}} \quad \text{και} \quad \sin\theta=\frac{y}{\sqrt{x^2+y^2}}$$

Άρα $z=\sqrt{x^2+y^2}\,(\cos\theta+i\sin\theta)$

$+yi=1+z_0=|1+z_0|(\cos\varphi+i\sin\varphi)$

$\arg(1+z_0)=\varphi$

$$\cos\varphi=\frac{1+r\cos\theta}{\sqrt{1+r^2+2r\cos\theta}} \qquad \sin\varphi=\frac{r\sin\theta}{\sqrt{1+r^2+2r\cos\theta}}$$

$1+z_0=(1+r\cos\theta)+ir\sin\theta$

$$\Sigma=\left(\sqrt{1+r^2+2r\cos\theta}\right)^{\lambda}\cos(\lambda\varphi)-1$$

$1+z_0\to\varphi$ και $\varphi\neq\pi,-\pi$ άρα $1+z_0$ μη αρνητικός

$$1+z_0=1+\frac{1}{\sqrt3}e^{i\frac{\pi}{2}}=1+\frac{1}{\sqrt3}\cos\frac{\pi}{2}+i\frac{1}{\sqrt3}\sin\frac{\pi}{2}=1+i\frac{1}{\sqrt3}$$

$$\sqrt{1^2+\left(\frac{1}{\sqrt3}\right)^2}=\sqrt{\frac{4}{3}}$$

$$\varphi=\arg(1+z_0)=\arg\left(1+i\frac{1}{\sqrt3}\right)=\pi/6$$

$$\cos\varphi=\frac{1}{\sqrt{\frac{4}{3}}}=\frac{\sqrt3}{2} \qquad \sin\varphi=\frac{1/\sqrt3}{\sqrt{4/3}}=\frac{1}{2}$$

$\longrightarrow$

Άρα $S = \sqrt[4]{\frac{4}{3}} \cos\left(\frac{\pi}{12}\right) - 1$

$$\mathrm{Re}\left(\sum_{i=0}^{\infty} a_i\right) = \sum_{i=0}^{\infty} \mathrm{Re}(a_i)$$

Απόδειξη

$a_j = x_j + y_j \cdot i$

$$\sum_{j=0}^{\infty} a_j = \sum_{j=0}^{\infty}(x_j + y_j i) = \sum_{j=0}^{\infty} x_j + i \sum_{j=0}^{\infty} y_j$$

$$\mathrm{Re}\left(\sum_{i=0}^{\infty} a_i\right) = \sum_{i=0}^{\infty} x_i = \sum_{i=0}^{\infty} \mathrm{Re}(a_i)$$

36)

$(1+z)^{2n} + (1-z)^{2n} = 0$, $n \in \mathbb{N}$ (1)

$z_k = i \tan\left(\frac{2k+1}{4n}\pi\right)$ $k = 0, 1, \ldots, 2n-1$

Δεν είναι ρίζες τα $1, -1$

$$(1+z_0)^{2n}\left(1 + \left(\frac{1-z_0}{1+z_0}\right)^{2n}\right) = 0 \quad (2)$$

Από (2) $\Rightarrow \left(\frac{1-z_0}{1+z_0}\right)^{2n} = -1$ (3)

Θέτω $w_0 = \frac{1-z_0}{1+z_0}$ (4)

Από (3) και (4) $w_0^{2n} = -1$ (5)

$z^n = a$

$$w_0 = \sqrt[2n]{|-1|}\left(\cos\left(\frac{(2k+1)\pi}{2n}\right) + i\sin\left(\frac{(2k+1)\pi}{2n}\right)\right) \quad k = 0,1,\dots,2n-$$

$(-1 = \cos\pi + i\sin\pi)$

$$= e^{i\frac{(2k+1)\pi}{2n}} \quad k = 0,1,\dots,2n-1$$

$$w_0 = e^{i\frac{(2k+1)\pi}{2n}} \Rightarrow |w_0| = 1$$

$$w_0 = \frac{1-z_0}{1+z_0} \quad (4)$$

$$w_0 = \frac{1-z_0}{1+z_0} \Rightarrow (1+z_0)w_0 = 1-z_0 \Rightarrow w_0 + z_0 w_0 = 1 - z_0 \Rightarrow$$

$$\Rightarrow z_0(1+w_0) = 1 - w_0 \Rightarrow z_0 = \frac{1-w_0}{1+w_0}$$

Av $(w_0)^{2n} = -1$

$(-1)^{2n} = 1 \neq -2$

$$z_0 = \frac{1-w_0}{1+w_0} = \frac{(1-w_0)(1+\overline{w}_0)}{|1+w_0|^2}$$

$$|1+w_0|^2 = \left|1 + e^{i\frac{(2k+1)\pi}{2n}}\right|^2 = \left(1 + \cos\left(\frac{(2k+1)\pi}{2n}\right)\right)^2 + \left(\sin\left(\frac{(2k+1)\pi}{2n}\right)\right)$$

$$= 1 + \cos^2\left(\frac{(2k+1)\pi}{2n}\right) + \sin^2\left(\frac{(2k+1)}{2n}\pi\right) + 2\cos\frac{(2k+1)\pi}{2n}$$

$$= 2\left(1 + \cos\left(\frac{(2k+1)\pi}{2n}\right)\right)$$

$\cos 2a = 2\cos^2 a - 1 \Rightarrow 1 + \cos 2a = 2\cos^2 a \rightarrow$

$$a = \frac{(2k+1)\pi}{4n}$$

$\rightarrow$

(55)

$$= 2\cdot 2\cos^2\left(\frac{(2k+1)}{4n}\pi\right) = 4\cos^2\left(\frac{(2k+1)\pi}{4n}\right)$$

Άρα $|1+w_0|^2 = 4\cos^2\left(\frac{(2k+1)}{4n}\pi\right)$

$$(1-w_0)(1+\overline{w_0}) = 1+\overline{w_0} - w_0 - |w_0|^2 = \overline{w_0} - w_0 = e^{-i\frac{(2k+1)\pi}{2n}} - e^{i\frac{(2k+1)\pi}{2n}}$$

$$\left(\overline{w_0} = e^{-i\left(\frac{(2k+1)\pi}{2n}\right)}\right)$$

$$= \cos\left(-\frac{(2k+1)\pi}{2n}\right) + i\sin\left(-\frac{(2k+1)\pi}{2n}\right) - \left(\cos\left(\frac{(2k+1)\pi}{2n}\right) + i\sin\left(\frac{(2k+1)\pi}{2n}\right)\right)$$

$$= -2i\sin\frac{(2k+1)\pi}{2n}$$

$$z_0 = \frac{-2i\sin\left(\frac{(2k+1)}{2n}\pi\right)}{4\cos^2\left(\frac{(2k+1)}{4n}\pi\right)} = \frac{-2i\,2\sin\frac{(2k+1)\pi}{4n}\cos\frac{(2k+1)}{4n}\pi}{4\cos^2\frac{(2k+1)\pi}{4n}}$$

$$\left(\begin{array}{l}\sin 2a = 2\sin a\cos a\\ a = \frac{(2k+1)\pi}{2n}\end{array}\right) = -i\tan\frac{(2k+1)\pi}{4n} \qquad k=0,1,\dots,2n-1$$

ίδιες: $a+bi$, $a-bi$

Βρήκαμε τις $-bi$ ρίζες θα έχει και τις συζυγείς!

$$z_k = -i\tan\frac{(2k+1)\pi}{4n} \qquad k=0,1,\dots,2n-1$$

$$0 \le k \le 2n-1 \Longrightarrow \frac{(2k+1)\pi}{4n} < \pi$$

→

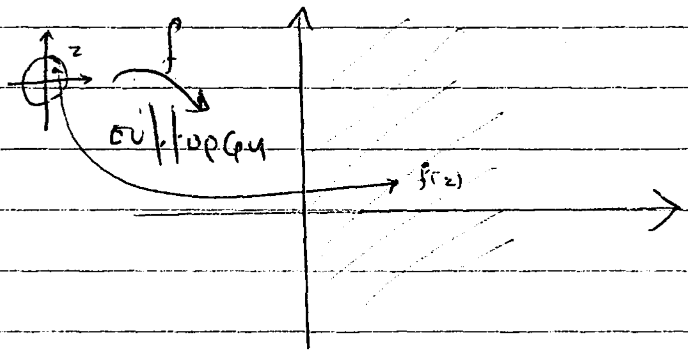
Είναι $\frac{(2k+1)\pi}{4n} = \frac{\pi}{2}$; (ΑΝ ΝΑΙ, ΔΕΝ ΘΑ ΟΡΙΖΕΤΑΙ Η ΤΑΝ)

$\frac{(2k+1)\pi}{4n} = \frac{\pi}{2} \to 2k+1 = 2n$ ΑΤΟΠΟ

(35)

$w = f(z) = \frac{1+z}{1-z}$

$D(0,1) \xrightarrow{f} I = \{z \in \mathbb{C} / Re\, z > 0\}$



Από το θεώρημα
αν $f'(z) \neq 0 \ \forall z \in D(0,1)$ τότε f σύμμορφη

$$f'(z) = \left(\frac{1+z}{1-z}\right)' = \frac{(1+z)'(1-z) - (1+z)(1-z)'}{(1-z)^2} = \frac{1-z+(1+z)}{(1-z)^2}$$

$$= \frac{2}{(1-z)^2} \neq 0$$

$f'(z) \neq 0 \ \forall z \in D(0,1) \to f$ είναι σύμμορφη

$\forall z \in D(0,1)$ $f(z) \in I$
Πρέπει: $Re(f(z)) > 0$

$$f(z) = \frac{1+z}{1-z} = \frac{(1+z)(1-\bar{z})}{|1-z|^2} = \frac{1-\bar{z}+z-|z|^2}{|1-z|^2} = \cancel{\frac{1}{|1-z|^2}(1-|z|}$$

(Ξέρω $z - \bar{z} = \frac{Im(z)}{2i}$ $\forall z \in \mathbb{C}$) $z - \bar{z} = 2i\, Im z$

$$= \frac{1}{|1-z|^2}\left(1 - |z|^2 - \frac{1}{2} i\, Im(z)\right)$$

$\rightarrow$

$\mathrm{Re}\, f(z) = \frac{1-|z|^2}{|1-z|^2} > 0 \qquad |z|<1$

Έστω $w_0 \in I$ $\exists ?$ $z_0 \in \Delta(0,1)$: $f(z_0) = w_0$

$$w_0 = \frac{1+z_0}{1-z_0} \Leftrightarrow w_0(1-z_0) = 1+z_0 \Rightarrow w_0 - w_0 z_0 = 1+z_0$$

$$\Rightarrow z_0(1+w_0) = w_0 - 1 \Rightarrow z_0 = \frac{w_0-1}{1+w_0}$$

$(|z|^2 = z\cdot\bar{z})$

$$|z_0|^2 = \left|\frac{w_0-1}{1+w_0}\right|^2 = \frac{(w_0-1)(\overline{w_0}-1)}{(1+w_0)(1+\overline{w_0})} = \frac{|w_0|^2 - w_0 - \overline{w_0} + 1}{1+\overline{w_0}+w_0+|w_0|}$$

$z+\bar{z} = 2\,\mathrm{Re}(z))$

$$= \frac{1+|w_0|^2 - 2\,\mathrm{Re}(w_0)}{1+|w_0|^2 + 2\,\mathrm{Re}(w_0)} \overset{;}{<} 1$$

Πράγματι γιατί,

$$1+|w_0|^2 - 2\,\mathrm{Re}(w_0) < 1+|w_0|^2 + 2\,\mathrm{Re}(w_0) \Rightarrow$$

$\mathrm{Re}(w_0) > 0$ ισχύει

Άρα η f είναι επί

(37)

(c_n) φραγμ. $\sum_{n=0}^{\infty} c_n z^n$

(α) $R \geq 1$

(β) $\sum_{n=0}^{\infty} \frac{c_n}{n!} z^n / \rho = +\infty$

$|c_n| < M$

Άρα $\sqrt[n]{|c_n|} < \sqrt[n]{M} \Rightarrow \limsup_{n\to+\infty} \sqrt[n]{|c_n|} \leq \limsup_{n\to+\infty} \sqrt[n]{M} = \lim_{n\to+\infty} \sqrt[n]{M} = 1$

(Cauchy-Hadamard) $\frac{1}{R}$

$\rightarrow$

$$\frac{1}{R} \leq 1 \Rightarrow R \geq 1$$

β) $$\sqrt[n]{\left|\frac{c_n}{n!}\right|} = \frac{\sqrt[n]{|c_n|}}{\sqrt[n]{|n!|}} < \frac{\sqrt[n]{M}}{\sqrt[n]{n!}} \Rightarrow \limsup_{n\to\infty} \sqrt[n]{\left|\frac{c_n}{n!}\right|} \leq \limsup_{n\to\infty} \frac{\sqrt[n]{M}}{\sqrt[n]{n!}}$$

$$= \lim_{n\to+\infty} \frac{1}{\sqrt[n]{n!}} = 0$$

Είναι $0 \leq \underbrace{\limsup \sqrt[n]{\frac{|c_n|}{n!}}}_{= \frac{1}{R}} \leq 0$

$$R = +\infty$$

22) $$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin z \quad \frac{e^{iz} - e^{-iz}}{2i}$$

23) $$\cosh z = \frac{e^{z} + e^{-z}}{2}$$

$$\sinh z = \frac{e^{z} - e^{-z}}{2}$$

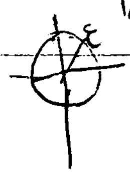
24)

$$e^{iz} = y$$

$$\cos z_0 = w_0$$

25) $$f(z) = e^{1/z}$$

$\forall \varepsilon$ και $\forall\, w \neq 0 \quad \exists\, z \in \mathbb{C} \setminus \{0\} \ : \ f(z) = w$



$W_0 \ : \ e^{1/z_0} = W_0$

→

$\exists\, a_0 \in \mathbb{C} : e^{a_0} = w_0$ (από το επί της e)

$$e^{a_0} = e^{a_0 + 2\kappa\pi i} \quad \forall \kappa \in \mathbb{Z}$$

$$e^{a_0} = e^{a_0 + 2n\pi i} \quad \forall n \in \mathbb{N}$$

$$b_n = a_0 + 2n\pi i \to +\infty$$

$$\frac{1}{b_n} = \frac{1}{a_0 + 2n\pi i} \to 0$$

$$e^{a_0} = w_0 \Rightarrow e^{b_n} = w_0 \Longrightarrow e^{\frac{1}{\left(\frac{1}{b_n}\right)}} = w_0 \Longrightarrow e^{\frac{1}{\gamma_n}} = w_0$$

$\gamma_n \to 0$

$f(\gamma_n) = W_0$

$$\frac{1}{b_n} = \gamma_n$$