Preperation Problems

Analysis

Problem 1

Let x_1 be a real number in the interval $(0, 1)$ and define a sequence by $x_{n+1} = x_n - x_n^{n+1}$. Show that the sequence converges and $\lim_{n\to\infty} x_n > 0$.

Problem 2

Prove that:

$$\int_0^\infty \frac{|\sin x|}{x^{1/2}} dx = \infty$$

Problem 3

Let a, x_0 be positive numbers, and define the sequence (x_n) recursively by:

$$x_n = \frac{1}{2}\left(x_{n-1} + \frac{a}{x_{n-1}}\right)$$

Prove that this sequence converges, and find its limit.

Problem 4

a) Let $f : [a, b] \to [0, \infty)$ be a continuous function which is not identically zero. Then prove that:

$$\lim_{n\to\infty} \frac{\int_a^b f^{n+1}(x)dx}{\int_a^b f^n(x)dx} = \max\{f(x) : x \in [a, b]\}$$

b) Find a real number c and a positive number L such that:

$$\lim_{r\to\infty} \frac{r^c \int_0^{\pi/2} x^r \sin x dx}{\int_0^{\pi/2} x^r \cos x dx} = L$$

Problem 5

For an arbitrary number $x_0 \in (0, \pi)$ define recursively the sequence (x_n) by $x_{n+1} = \sin x_n, n \geq 0$. Compute $\lim_{n\to\infty} n^{1/2} x_n$.

Problem 6

Let $a_0 = 5/2$ and $a_k = a_{k-1}^2 - 2, k \geq 1$. Evaluate the product:

$$\prod_{k=0}^{\infty} \left(1 - a_k^{-1}\right)$$

Problem 7

Let f be a continuous function defined on the unit square. Prove that:

$$\int_0^1 \left(\int_0^1 f(x, y)dy\right)^2 dx + \int_0^1 \left(\int_0^1 f(x, y)dx\right)^2 dy \leq \left(\int_0^1 \int_0^1 f(x, y)dxdy\right)^2 + \int_0^1 \int_0^1 f^2(x, y)dxdy$$

(Hint: Weierstass Theorem)

Problem 8

Let S be an uncountable set of disks in the plane. Prove that there is an uncountable subset S' such that all the disks in S' have a common interior point.

Problem 9

Evaluate the limit:

$$\lim_{x\to 1^-}\prod_{n=0}^{\infty}\left(\frac{1+x^{n+1}}{1+x^n}\right)^{x^n}$$

Linear Algebra

Problem 1

Let A,B be 2×2 real matrices satisfying $(AB-BA)^n=\mathcal{I}_2$ for some positive integer n. Prove that n is even and $(AB-BA)^4=\mathcal{I}_2$.

Problem 2

Let A be a real $n\times n$ matrix such that $A+A^t=\mathcal{O}_n$. Prove that:

$$\det\left(\mathcal{I}_n+sA^2\right)\geq 0$$

for every real number s.

Problem 3

Compute the determinant of the matrix $A=\left(\frac{1}{\min\{i,j\}}\right)_{i,j=1,\ldots,n}$.

Problem 4

Let A,B be $n\times n$ matrices that commute. Prove that if $\det(A+B)\geq 0$ then $\det\left(A^k+B^k\right)\geq 0$ for every natural number k.

Problem 5

Let A be an $n\times n$ matrix with complex entries whose minimal polynomial has degree k. Let $a_1,...,a_k$ be distinct complex numbers which are not eigenvalues of the matrix. Prove that there exists complex numbers $b_1,...,b_k$ such that:

$$\sum_{i=1}^{k}b_i\left(A-a_i\mathcal{I}_n\right)^{-1}=\mathcal{I}_n$$

Problem 6

Let A be an $n\times n$ matrix. Prove that there exists an $n\times n$ matrix B with the property:

$$ABA=A$$

Problem 7

Let $A_1,...A_n$ be n points in the plane. We consider the following set:

$$\mathcal{C}_n:=\{(i,j):d\left(A_iA_j\right)=1\}$$

Prove that: $|\mathcal{C}_n|\leq 2n^{3/2}$.