Υπολογισμός αθροισμάτων (21/04)

① τύπος Cauchy

$$\binom{x+y}{\nu} = \sum_{k=0}^{\nu} \binom{x}{y} \binom{y}{\nu-k}, \quad x,y \in \mathbb{R}, \ \nu = 0,1,2,\dots$$

Απόδειξη

$$(1+t)^{x+y} = (1+t)^x \cdot (1+t)^y$$

$$\sum_{k=0}^{\infty} \underbrace{\binom{x+y}{k}}_{\gamma_k} t^k = \left(\sum_{k=0}^{\infty} \underbrace{\binom{x}{k}}_{a_k} t^k\right) \left(\sum_{k=0}^{\infty} \underbrace{\binom{y}{k}}_{b_k} t^k\right)$$

$$\gamma_\nu = \sum_{k=0}^{\nu} a_k b_{\nu-k} \Rightarrow \binom{x+y}{\nu} = \sum_{k=0}^{\nu} \binom{x}{k} \binom{y}{\nu-k}$$

② Παράδειγμα

$$\sum_{k=\cancel{0}1}^{\nu} k \binom{r}{k} \binom{s}{\nu-k} \qquad \binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1}$$

$$\sum_{k=1}^{\nu} k \, \frac{r}{k} \binom{r-1}{k-1} \binom{s}{\nu-k} =$$

$$\overset{j=k-1}{=} r \sum_{j=0}^{\nu-1} \binom{r-1}{j} \binom{s}{\nu-j-1}$$

$$= r \binom{r+s-1}{\nu-1}$$

③ Παράδειγμα

$$\sum_{\kappa=0}^{\nu} \kappa^2 \binom{r}{\kappa}\binom{s}{\nu-\kappa}$$

$$= \sum_{\kappa=0}^{\nu} [\kappa(\kappa-1)+\kappa] \binom{r}{\kappa}\binom{s}{\nu-\kappa}$$

$$= \underbrace{\sum_{\kappa=0}^{\nu} \kappa(\kappa-1)\binom{r}{\kappa}\binom{s}{\nu-\kappa}}_{S_2} + \underbrace{\sum_{\kappa=0}^{\nu} \kappa \binom{r}{\kappa}\cdot\binom{s}{\nu-\kappa}}_{\text{Παράδειγμα (2)}\ r\binom{r+s-1}{r-1}}$$

$$S_2 = \sum_{\kappa=\cancel{0}\,2}^{\nu} \kappa(\kappa-1)\frac{r}{\kappa}\cdot\frac{r-1}{\kappa-1}\binom{r-2}{\kappa-2}\binom{s}{\kappa}$$

$$= r(r-1)\sum_{\kappa=2}^{\nu}\binom{r-2}{\kappa-2}\binom{s}{\nu-\kappa}$$

$$\overset{j=\kappa-2}{=} r\cdot(r-1)\sum_{j=0}^{\nu-2}\binom{r-2}{j}\binom{s}{\nu-2-j} \overset{\text{Cauchy}}{=} r(r-1)\binom{r+s-2}{r-2}$$

④ Βασικές σχέσεις

$$(x)_\nu = x(x-1)(x-2)\dots(x-\nu+1) = [x-\nu+1]_\nu$$

$$= (-1)^\nu(-x)(-x+1)(-x+2)\dots(-x+\nu-1) = (-1)^\nu[-x]_\nu$$

$$\Rightarrow \binom{x}{\nu} = \begin{bmatrix} x-\nu+1 \\ \nu \end{bmatrix} = (-1)^\nu \begin{bmatrix} -x \\ \nu \end{bmatrix}$$

ομοίως $\begin{bmatrix} x \\ \nu \end{bmatrix} = \binom{x+\nu-1}{\nu} = (-1)^\nu \binom{-x}{\nu}$

⑤ Παράδειγμα

$$\sum_{k=0}^{\nu}\binom{r+k}{k}\binom{s-k}{\nu-k} = \sum_{k=0}^{\nu}\underbrace{\binom{r+1+k-1}{k}}_{=\left[{r \atop k}\right]}\underbrace{\binom{(s-\nu+1)+\nu-k-1}{\nu-k}}_{=\left[{s-\nu+1 \atop \nu-k}\right]}$$

$$= \sum_{k=0}^{\nu}(-1)^{k}\binom{-1-r}{k}(-1)^{\nu-k}\binom{-s+\nu-1}{r-k} =$$

$$= (-1)^{\nu}\sum_{k=0}^{\nu}\binom{-(r+1)}{k}\binom{-s+\nu-2}{\nu-k} =$$

$$= (-1)^{\nu}\binom{-r-s+\nu-2}{\nu} =$$

$$= \left[{r+s-\nu+2 \atop \nu}\right] =$$

$$= \binom{r+s-\cancel{\nu}+2+\cancel{\nu}-1}{\nu} = \binom{r+s+1}{\nu}$$

⑥ Παραδείγματα

$$\sum_{k=0}^{\nu}\frac{1}{(k+1)(k+2)}\binom{r}{k}\binom{s}{\nu-k}$$

$$= \frac{1}{(r+1)\cdot(r+2)}\sum_{k=0}^{\nu}\underbrace{\frac{(r+1)\cdot(r+2)}{(k+1)(k+2)}\binom{r}{k}}_{\binom{r+2}{k+2}}\binom{s}{\nu-k}$$

$$\overset{j=k+2}{=} \frac{1}{(r+1)\cdot(r+2)}\sum_{j=2}^{\nu+2}\binom{r+2}{j}\binom{s}{\nu+2-j}$$

$$= \frac{1}{(r+1)(r+2)} \left[\binom{r+s+2}{v+2} - \binom{r+2}{0}\binom{s}{v+2} - \binom{r+2}{1}\binom{s}{v+1} \right]$$

⑦ Παράδειγμα SOS για εξετάσεις

$$\sum_{k=0}^{v-1} \frac{v-k}{k+1} \binom{r}{k}\binom{s}{v-k} =$$

$$= \frac{1}{r+1} \sum_{k=0}^{v-1} \underbrace{\frac{r+1}{k+1}\binom{r}{k}}_{\binom{r+1}{k+1}} (v-k)\frac{s}{v-k}\binom{s-1}{v-k-1}$$

$$\overset{j=k+1}{=} \frac{s}{r+1} \sum_{j=1}^{v} \binom{r+1}{j}\binom{s-1}{v-j}$$

$$= \frac{s}{r+1}\left[\binom{r+s}{v} - \binom{r+1}{0}\binom{s-1}{v}\right]$$

$$= \frac{s}{r+1}\left[\binom{r+s}{v} - \binom{s-1}{v}\right]$$

⑧ Αθροίσματα πάνω σε άρτιους και σε περιττούς δείκτες

$$\sum_{\substack{k=0 \\ k\ \text{άρτιος}}}^{v} a_k = a_0 + a_2 + a_4 + \dots + a_{2\lfloor \frac{v}{2} \rfloor}$$

τελευταίος άρτιος $\leq v$

$$S_n = \sum_{\substack{k=0 \\ k\ \text{περιττός}}}^{v} a_k = a_1 + a_3 + a_5 + \dots + a_{2\lfloor \frac{v-1}{2} \rfloor + 1}$$

τελευταίος περιττός $\leq v$

= τελευταίος άρτιος $(v-1)+1$

πχ. $\binom{\nu}{0}+\binom{\nu}{2}+\binom{\nu}{4}+\dots+\binom{\nu}{2[\frac{\nu}{2}]}=;$

Βασική τεχνική:

Είτε ζητείται το S_a, S_π βρίσκω και τα δύο λύνοντας σύστημα:

$$S_a+S_\pi=\sum_{\kappa=0}^{\nu} a_\kappa$$

$$S_a-S_\pi=\sum_{\kappa=0}^{\nu}(-1)^\kappa a_\kappa$$

⑨ Παράδειγμα

$$S_\pi=\frac{1}{2}\binom{\nu}{1}+\frac{1}{4}\binom{\nu}{3}+\frac{1}{6}\binom{\nu}{5}+\dots+\binom{\nu}{2\lfloor\frac{\nu-1}{2}\rfloor+1}\frac{1}{2[\frac{\nu-1}{2}]+2}$$

↑ τελευταίος περιττός $\leq \nu$

$$S_\pi=\sum_{\substack{\kappa=0\\ \kappa\ \pi\varepsilon\rho}}^{\nu}\frac{1}{\kappa+1}\binom{\nu}{\kappa} \quad \text{θέλω } S_a=\sum_{\substack{\kappa=0\\ \kappa\ \alpha\rho\tau\iota\alpha}}^{\nu}\frac{1}{\kappa+1}\binom{\nu}{\kappa}$$

$$S_\pi+S_a=\sum_{\kappa=0}^{\nu}\frac{1}{\kappa+1}\binom{\nu}{\kappa}$$

$$-S_\pi+S_a=\sum_{\kappa=0}^{\nu}(-1)^\kappa\frac{1}{\kappa+1}\binom{\nu}{\kappa}$$

θα υπολογίσω το

$$\sum_{\kappa=0}^{\nu}\frac{1}{\kappa+1}\binom{\nu}{\kappa}t^\kappa=\frac{1}{\nu+1}\sum_{\kappa=0}^{\nu}\underbrace{\frac{\nu+1}{\kappa+1}\binom{\nu}{\kappa}}_{(\nu+1)}t^\kappa$$

$$\underline{j=k+1}\quad \frac{1}{\nu+1}\sum_{j=1}^{\nu+1}\binom{\nu+1}{j} t^{j-1}$$

$$= \frac{1}{(\nu+1)t}\underbrace{\sum_{j=1}^{\nu+1}\binom{\nu+1}{j} t^{j}}_{(1+t)^{\nu+1}-\binom{\nu+1}{0}t^{0}} = \frac{(1+t)^{\nu+1}-1}{(\nu+1)t}$$

Διωνυμικό Άθροισμα

$$\boxed{\sum_{k=0}^{\nu}\binom{\nu}{k} t^{k} = (1+t)^{\nu}}$$

Άρα

$$\left.\begin{aligned} S_a + S_\pi &\overset{t=1}{=} \frac{2^{\nu+1}-1}{\nu+1} \\ S_a - S_\pi &\overset{t=-1}{=} \frac{1}{\nu+1} \end{aligned}\right\} \Rightarrow \begin{aligned} 2S_\pi &= \frac{2^{\nu+1}-1-1}{\nu+1} \\ S_\pi &= \frac{2^{\nu}-1}{\nu+1} \end{aligned}$$

(10) Αθροίσματα με συμμετρία

$$\binom{\nu}{k} = \binom{\nu}{\nu-k}$$

(11) Παραδείγματα

$$\binom{2\nu}{0}+\binom{2\nu}{1}+\binom{2\nu}{2}+\dots+\binom{2\nu}{\nu} = \sum_{k=0}^{\nu}\binom{2\nu}{k} \Rightarrow$$

έχω: $\sum_{k=0}^{2\nu}\binom{2\nu}{k} = 2^{2\nu} \Rightarrow$

$$\begin{aligned}&\binom{2\nu}{0}+\binom{2\nu}{1}+\dots+\binom{2\nu}{\nu} \\ &\binom{2\nu}{2\nu}+\binom{2\nu}{2\nu-1}+\dots+\binom{2\nu}{\nu+2}+\binom{2\nu}{\nu+1}\end{aligned} = 2^{2\nu} \Rightarrow$$

$$\binom{2\nu}{0}+\binom{2\nu}{1}+\dots+\overset{=\lambda}{\binom{2\nu}{\nu}} = 2^{2\nu} \Rightarrow 2\lambda-\binom{2\nu}{\nu}=2^{2\nu}$$

$$+\binom{2\nu}{0}+\binom{2\nu}{1}+\dots+\binom{2\nu}{\nu-1} = \lambda-\binom{2\nu}{\nu}$$

⑫ Παράδειγμα

$$\sum_{\kappa=0}^{\nu}\binom{\nu}{\kappa}^2 \underset{\substack{\uparrow\\ \text{συμμετρική}\\ \text{ιδιότητα}}}{=} \sum_{\kappa=0}^{\nu}\binom{\nu}{\kappa}\cdot\binom{\nu}{\nu-\kappa} \underset{\substack{\uparrow\\ \text{Cauchy}\\ x=y=\nu}}{=} \binom{\nu+\nu}{\nu}=\binom{2\nu}{\nu}$$

⑬ Παράδειγμα

$$\sum_{\kappa=0}^{\nu}(\kappa+1)\binom{\nu}{\kappa}^2 = \sum_{\kappa=0}^{\nu}\kappa\underbrace{\binom{\nu}{\kappa}}\binom{\nu}{\kappa} + \underbrace{\sum_{\kappa=0}^{\nu}\binom{\nu}{\kappa}^2}_{\binom{2\nu}{\nu}}$$

$$= \sum_{\kappa=1}^{\nu}\cancel{\kappa}\frac{\nu}{\cancel{\kappa}}\binom{\nu-1}{\kappa-1}\binom{\nu}{\nu-\kappa}+\binom{2\nu}{\nu}$$

$$\overset{j=\kappa-1}{=} \nu\sum_{j=0}^{\nu-1}\binom{\nu-1}{j}\binom{\nu}{\nu-1-j}+\binom{2\nu}{\nu} =$$

$$= \nu\binom{2\nu-1}{\nu-1}+\binom{2\nu}{\nu}$$

⑭ Παράδειγμα

$$S=\sum_{s=\kappa}^{\nu}\binom{s}{\kappa}\binom{\nu}{s}=;$$

Λύση: $S=\sum_{s=\kappa}^{\nu}\frac{\cancel{s!}}{\kappa!\,(s-\kappa)!}\frac{\nu!}{\cancel{s!}\,(\nu-s)!} = \frac{\nu!}{\kappa!\,(\nu-\kappa)!}\sum_{s=\kappa}^{\nu}\underbrace{\frac{(\nu-\kappa)!}{(s-\kappa)!\,(\nu-s)!}}_{\binom{\nu-\kappa}{s-\kappa}}$

$$= \binom{v}{k} \sum_{s=k}^{v} \binom{v-k}{s-k} =$$

$$= \binom{v}{k} \sum_{j=0}^{v-k} \binom{v-k}{j} = \binom{v}{k} 2^{v-k}$$