

Υπολογισμοί Αθροισμάτων:

Αθροίσματα σε περιττούς/άρτιους δείκτες με συμμετρία

① Παράδειγμα

$$S_a = \sum_{k=0}^v k \binom{v}{k} = 0 \binom{v}{0} + 2 \binom{v}{2} + 4 \binom{v}{4} + \dots + 2 \left\lfloor \frac{v}{2} \right\rfloor \binom{v}{2 \left\lfloor \frac{v}{2} \right\rfloor}$$

k άρτιος $\lfloor \frac{v}{2} \rfloor$ ζεγγυραίος άρτιος $\leq$ του v

$$= \sum_{i=0}^{\lfloor \frac{v}{2} \rfloor} 2i \binom{v}{2i}$$

$$S_n = \sum_{k=0}^v k \binom{v}{k} = 1 \binom{v}{1} + 3 \binom{v}{3} + \dots + (2 \left\lfloor \frac{v-1}{2} \right\rfloor + 1) \binom{v}{2 \left\lfloor \frac{v-1}{2} \right\rfloor + 1}$$

k περιττός ζεγγυραίος περιττός $\leq$ του v

$$= \sum_{i=0}^{\lfloor \frac{v-1}{2} \rfloor} (2i+1) \binom{v}{2i+1}$$

(ζεγγυραίος άρτιος $\leq v-1$) + 1

$$2 \left\lfloor \frac{v-1}{2} \right\rfloor + 1$$

{* Για να υπολογ. το 1 από τα 2 ηρèneι
να υπολογίσουμε και τα 2 }
==

Ιδέα: $S_a + S_n = \sum_{k=0}^v k \binom{v}{k} = \sum_{k=1}^v k \cdot \frac{v}{k} \binom{v-1}{k-1} = v \sum_{k=1}^v \binom{v-1}{k-1}$

$$= v \sum_{j=0}^{v-1} \binom{v-1}{j} = v \cdot 2^{v-1}$$

$$S_a - S_n = \sum_{k=0}^v (-1)^k k \binom{v}{k} = \sum_{k=1}^v (-1)^k k \frac{v}{k} \binom{v-1}{k-1} = v \sum_{k=1}^v \binom{v-1}{k-1} (-1)^k$$

$\text{bij } \Leftrightarrow k=j+1$

$$= -v \sum_{j=0}^{v-1} \binom{v-1}{j} (-1)^j \rightarrow \text{κρατάω μόνο το } j$$

$(1+(-1))^{v-1}$

$$\text{Άρα } \left. \begin{array}{l} S_a + S_n = v 2^{v-1} \\ S_a - S_n = 0 \end{array} \right\} \Rightarrow \boxed{S_a = S_n = v 2^{v-2}}$$

$$S_a - S_n = 0 \binom{v}{0} - 1 \binom{v}{1} + 2 \binom{v}{2} - 3 \binom{v}{3} + 4 \binom{v}{4} - 5 \binom{v}{5} + \dots \\ + 2 \lfloor \frac{v}{2} \rfloor \binom{v}{\lfloor \frac{v}{2} \rfloor} - (2 \lfloor \frac{v-1}{2} \rfloor + 1) \binom{v}{\lfloor \frac{v-1}{2} \rfloor + 1}$$

② Παράδειγμα

$$S_n = \frac{1}{2} \binom{v}{1} + \frac{1}{4} \binom{v}{3} + \frac{1}{6} \binom{v}{5} + \dots = ?$$

$$= \sum_{i=0}^{\infty} \frac{1}{i+1} \binom{v}{i} = \sum_{j=0}^{\infty} \frac{1}{2j+2} \binom{v}{2j+1}$$

$$S_n \xrightarrow{\text{i περίσσις}} \text{Αυτό με συμπέπει}$$

$$\text{Έστω } S_a = \sum_{i=0}^{\infty} \frac{1}{i+1} \binom{v}{i}$$

i άρτιος

$$S_a + S_n = \sum_{i=0}^{\infty} \frac{1}{i+1} \binom{v}{i}$$

$$S_a - S_n = \sum_{i=0}^{\infty} (-1)^i \frac{1}{i+1} \binom{v}{i}$$

$$\begin{aligned} \text{Έχω : } \sum_{i=0}^{\infty} \frac{1}{i+1} \binom{v}{i} t^i &= \frac{1}{v+1} \sum_{i=0}^{\infty} \frac{v+1}{i+1} \binom{v}{i} t^i \\ &= \frac{1}{v+1} \sum_{k=1}^{\infty} \binom{v+1}{k} t^{k-1} = \frac{1}{(v+1)t} \sum_{k=1}^{\infty} \binom{v+1}{k} t^k \end{aligned}$$

$$(1+t)^{v+1} - 1 \rightarrow 0 \text{ ως } \text{φύος}$$

$$\text{Άρα: } S_0 + S_n = \sum_{t=0}^{t=n} \frac{1}{v+1} (2^{v+1} - 1)$$

$$S_0 - S_n = \frac{1}{(v+1)(-1)} ((1+(-1))^{v+1} - 1) = \frac{1}{v+1}$$

$$\text{Άρα } S_n = \frac{\frac{2^{v+1}-1}{v+1} - \frac{1}{v+1}}{2} = \frac{2^v - 1}{v+1}$$

③ Αθροίσματα με συμμετρία

Βασική σχέση

$$\binom{v}{k} = \binom{v}{v-k}$$

④ Παράδειγμα

$$S = \binom{2v}{0} + \binom{2v}{1} + \binom{2v}{2} + \dots + \binom{2v}{v} = \sum_{k=0}^v \binom{2v}{k} = ?$$

$$S = \binom{2v}{2v} + \binom{2v}{2v-1} + \binom{2v}{2v-2} + \dots + \binom{2v}{v}$$

$$\sum_{k=0}^{2v} \binom{2v}{k}$$

$$\Rightarrow 2S = \underbrace{\sum_{k=0}^{2v} \binom{2v}{k}}_{2^{2v}} + \binom{2v}{v} \Rightarrow S = \frac{2^{2v} + \binom{2v}{v}}{2}$$

5) Άσκηση

$$\begin{aligned}
 S &= \sum_{k=0}^v \binom{v}{k}^2 \\
 &= \sum_{k=0}^v \binom{v}{k} \binom{v}{v-k} = \sum_{k=0}^v \binom{v}{k} \binom{v}{v-k} \quad \begin{array}{l} \nearrow \text{συμμετρική} \\ \nearrow \text{ιδιότητα} \end{array} \\
 &= \binom{v+v}{v} = \binom{2v}{v} \quad \nearrow \text{Cauchy}
 \end{aligned}$$

6) Π.Ε.Ρ. 2011 / Θέμα 2 β

$$\begin{aligned}
 S &= \sum_{k=0}^v \binom{v}{k} (vk + \binom{v}{k}) \\
 &= v \sum_{k=0}^v \binom{v}{k} k + \sum_{k=0}^v \binom{v}{k}^2 \quad * \begin{array}{l} \binom{v}{k} = \binom{v}{v-k} \\ \binom{v}{k} = \frac{v}{k} \binom{v-1}{k-1} \end{array} \\
 &= v \sum_{k=1}^v \frac{v}{k} \binom{v-1}{k-1} k + \sum_{k=0}^v \binom{v}{k} \binom{v}{v-k} \\
 &= v^2 \sum_{k=1}^v \binom{v-1}{k-1} + \binom{v+v}{v} \\
 &= v^2 \sum_{j=0}^{v-1} \binom{v-1}{j} + \binom{v+v}{v} \\
 \text{Συν. Θωρ.} &\quad \uparrow \\
 &= v^2 \cdot (1+1)^{v-1} + \binom{2v}{v} \\
 &= v^2 2^{v-1} + \binom{2v}{v}
 \end{aligned}$$

7) ΞΕΠΤ. 2010 Ομ. Β / Θέμα 3α

$$S = \sum_{j=m}^{\infty} \binom{j}{m}_3^{m-j}$$

$$\kappa = j - m \quad = \sum_{\kappa=0}^{\infty} \binom{\kappa+m}{m}_3^{-\kappa}$$

$$\text{συμμετρ. ιδ.} = \sum_{\kappa=0}^{\infty} \binom{\kappa+m}{\kappa}_3^{-\kappa}$$

$$\parallel$$

$$\left[\begin{smallmatrix} x \\ \kappa \end{smallmatrix} \right] = \left(\frac{x+\kappa-1}{\kappa} \right)$$

$$= \sum_{\kappa=0}^{\infty} \binom{(m+1)+\kappa-1}{\kappa}_3^{-\kappa}$$

$$= \sum_{\kappa=0}^{\infty} \left[\begin{smallmatrix} m+1 \\ \kappa \end{smallmatrix} \right] \left(\frac{1}{3} \right)^{\kappa}$$

$$= \left(1 - \frac{1}{3} \right)^{-(m+1)} = \left(\frac{2}{3} \right)^{-(m+1)}$$

$$= \left(\frac{3}{2} \right)^{m+1}$$

$$* \sum_{\kappa=0}^{\infty} \left[\begin{smallmatrix} v \\ \kappa \end{smallmatrix} \right] t^{\kappa} = (1-t)^{-v}$$

$v = m+1$
 $t = \frac{1}{3}$

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⑧ Ασκηση

SOS

$$S = \sum_{s=k}^v \binom{s}{k} \binom{v}{s}$$

$$= \sum_{s=k}^v \frac{s!}{k!(s-k)!} \cdot \frac{v!}{s!(v-s)!} = \frac{v!}{k!} \sum_{s=k}^v \frac{1}{(s-k)!(v-s)!}$$

$$\begin{aligned} j=s-k \\ &= \frac{v!}{k!} \sum_{j=0}^{v-k} \frac{1}{j!(v-k-j)!} \end{aligned}$$

$$= \frac{v!}{k!} \cdot \frac{1}{(v-k)!} \sum_{j=0}^{v-k} \frac{(v-k)!}{j!(v-k-j)!}$$

$$= \binom{v}{k} \sum_{j=0}^{v-k} \binom{v-k}{j} = \binom{v}{k} 2^{v-k}$$

Διuv.
Νεύκωνα

⑨ Ασκηση

$$S = \sum_{k=0}^v \underbrace{(k+1)^2}_{k^2+2k+1} \binom{v}{k}^2 = ?$$

$\rightarrow k(k-1) + 3k + 1$

$$= \sum_{k=0}^v \overset{S_1}{k(k-1) \binom{v}{k}^2} + 3 \sum_{k=0}^v \overset{S_2}{k \binom{v}{k}^2} + \sum_{k=0}^v \overset{S_3}{\binom{v}{k}^2}$$

$$S_3 = \sum_{k=0}^v \binom{v}{k}^2 = \sum_{k=0}^v \binom{v}{k} \binom{v}{v-k} = \binom{v+v}{v} = \binom{2v}{v}$$

$$\begin{aligned}
 S_2 &= \sum_{k=0}^v k \binom{v}{k}^2 = \sum_{k=1}^v k \cdot \frac{v}{k} \binom{v-1}{k-1} \binom{v}{v-k} \\
 &= v \sum_{k=1}^v \binom{v-1}{k-1} \binom{v}{v-k} \\
 &= v \sum_{j=0}^{v-1} \binom{v-1}{j} \binom{v}{v-1-j} \\
 &= v \binom{2v-1}{v-1}
 \end{aligned}$$

$$\begin{aligned}
 S_1 &= \sum_{\substack{k=0 \\ 2}}^v k(k-1) \binom{v}{k}^2 = \sum_{k=2}^v k(k-1) \frac{v}{k} \cdot \frac{v-1}{k-1} \binom{v-2}{k-2} \binom{v}{v-k} \\
 &= v(v-1) \sum_{k=2}^v \binom{v-2}{\underbrace{k-2}_j} \binom{v}{v-k} \\
 &= v(v-1) \sum_{j=0}^{v-2} \binom{v-2}{j} \binom{v}{v-2-j} \\
 &= v(v-1) \binom{2v-2}{v-2}
 \end{aligned}$$

Άρα $S = v(v-1) \binom{2v-2}{v-2} + 3v \binom{2v-1}{v-1} + \binom{2v}{v}$