Υπολογισμοί Αθροισμάτων: 6/5/25

Αθροίσματα σε περιττούς / άρτιους δείκτες με συμμετρία

① Παράδειγμα

$$S_a = \sum_{\substack{k=0 \\ k \text{ άρτιος}}}^{v} k\binom{v}{k} = 0\binom{v}{0} + 2\binom{v}{2} + 4\binom{v}{4} + \dots + 2\left\lfloor \frac{v}{2} \right\rfloor \binom{v}{2\lfloor \frac{v}{2} \rfloor}$$

τελευταίος άρτιος $\leq$ του v $= 2\left\lfloor \frac{v}{2} \right\rfloor$

$$= \sum_{i=0}^{\lfloor \frac{v}{2} \rfloor} 2i\binom{v}{2i}$$

$$S_n = \sum_{\substack{k=0 \\ k \text{ περιττός}}}^{v} k\binom{v}{k} = 1\binom{v}{1} + 3\binom{v}{3} + \dots + \left(2\left\lfloor \frac{v-1}{2} \right\rfloor + 1\right)\binom{v}{2\lfloor \frac{v-1}{2} \rfloor + 1}$$

τελευταίος περιττός $\leq$ του v $=$ (τελευταίος άρτιος $\leq v-1$) $+1$ $= 2\left\lfloor \frac{v-1}{2} \right\rfloor + 1$

$$= \sum_{i=0}^{\lfloor \frac{v-1}{2} \rfloor} (2i+1)\binom{v}{2i+1}$$

{ * Για να υπολογ. το 1 από τα 2 πρέπει να υπολογίσουμε και τα 2 }

Ιδέα:

$$S_a + S_n = \sum_{k=0}^{v} k\binom{v}{k} = \sum_{k=1}^{v} k \cdot \frac{v}{k}\binom{v-1}{k-1} = v\sum_{k=1}^{v}\binom{v-1}{\underbrace{k-1}_{j}}$$

$$= v\sum_{j=0}^{v-1}\binom{v-1}{j} = v \cdot 2^{v-1}$$

$$S_a - S_n = \sum_{k=0}^{v} (-1)^k k\binom{v}{k} = \sum_{k=1}^{v} (-1)^k k \frac{v}{k}\binom{v-1}{k-1} = v\sum_{k=1}^{v}\binom{v-1}{\underbrace{k-1}_{j}}(-1)^k$$

$j \Longleftrightarrow k = j+1$

$$= -v\underbrace{\sum_{j=0}^{v-1}\binom{v-1}{j}(-1)^j}_{(1+(-1))^{v-1}} = 0$$

$\to$ κρατάω μόνο το j

Ἀρα $\left.\begin{aligned} S_a + S_n &= \nu 2^{\nu-1} \\ S_a - S_n &= 0 \end{aligned}\right\} \Rightarrow \boxed{S_a = S_n = \nu 2^{\nu-2}}$

$$S_a - S_n = 0\binom{\nu}{0} - 1\binom{\nu}{1} + 2\binom{\nu}{2} - 3\binom{\nu}{3} + 4\binom{\nu}{4} - 5\binom{\nu}{5} + \dots$$

$$+ 2\left\lfloor \frac{\nu}{2} \right\rfloor \binom{\nu}{2\left\lfloor \frac{\nu}{2} \right\rfloor} - \left(2\left\lfloor \frac{\nu-1}{2} \right\rfloor + 1\right)\binom{\nu}{2\left\lfloor \frac{\nu-1}{2} \right\rfloor + 1}$$

② Παράδειγμα

$$S_n = \frac{1}{2}\binom{\nu}{1} + \frac{1}{4}\binom{\nu}{3} + \frac{1}{6}\binom{\nu}{5} + \dots\dots = ?$$

$$= \sum_{i=0}^{\infty} \frac{1}{i+1}\binom{\nu}{i} = \sum_{j=0}^{\infty} \frac{1}{2j+2}\binom{\nu}{2j+1}$$

S_n $\quad$ i περιττός $\longrightarrow$ Αυτό με συμφέρει

Έστω $S_a = \sum_{i=0}^{\infty} \frac{1}{i+1}\binom{\nu}{i}$

i άρτιος

$$\left.\begin{aligned} S_a + S_n &= \sum_{i=0}^{\infty} \frac{1}{i+1}\binom{\nu}{i} \\ S_a - S_n &= \sum_{i=0}^{\infty} (-1)^i \frac{1}{i+1}\binom{\nu}{i} \end{aligned}\right\}$$

Έχω: $\sum_{i=0}^{\infty} \frac{1}{i+1}\binom{\nu}{i} t^i = \frac{1}{\nu+1}\sum_{i=0}^{\infty} \underbrace{\frac{\nu+1}{i+1}\binom{\nu}{i}}_{} t^i \to \binom{\nu+1}{i+1} \to k$

$$= \frac{1}{\nu+1}\sum_{k=1}^{\infty}\binom{\nu+1}{k} t^{k-1} = \frac{1}{(\nu+1)t}\underbrace{\sum_{k=1}^{\infty}\binom{\nu+1}{k} t^k}_{(1+t)^{\nu+1} - \underline{1}}$$

$\underline{1} \to 0$ $1^{\text{ος}}$ όρος

Άρα : $S_a + S_n \overset{t=1}{=} \frac{1}{\nu+1}(2^{\nu+1}-1)$

$S_a - S_n = \frac{1}{(\nu+1)(-1)}\left((1+\cancel{(-1)})^{\nu+1}-1\right) = \frac{1}{\nu+1}$

Άρα $S_n = \dfrac{\frac{2^{\nu+1}-1}{\nu+1} - \frac{1}{\nu+1}}{2} = \dfrac{2^{\nu}-1}{\nu+1}$

③ Αθροίσματα με συμμετρία

Βασική σχέση $\boxed{\binom{\nu}{k} = \binom{\nu}{\nu-k}}$

④ Παράδειγμα

$S = \binom{2\nu}{0} + \binom{2\nu}{1} + \binom{2\nu}{2} + \dots + \binom{2\nu}{\nu} = \sum_{k=0}^{\nu}\binom{2\nu}{k} = ?$

$S = \binom{2\nu}{2\nu} \quad \binom{2\nu}{2\nu-1} \quad \binom{2\nu}{2\nu-2} + \dots + \binom{2\nu}{\nu}$

$= \sum_{k=0}^{2\nu}\binom{2\nu}{k}$

$\Rightarrow 2S = \underbrace{\sum_{k=0}^{2\nu}\binom{2\nu}{k}}_{2^{2\nu}} + \binom{2\nu}{\nu} \quad \Rightarrow S = \dfrac{2^{2\nu} + \binom{2\nu}{\nu}}{2}$

⑤ Άσκηση

$$S = \sum_{k=0}^{\nu} \binom{\nu}{k}^2$$

$$= \sum_{k=0}^{\nu} \binom{\nu}{k}\binom{\nu}{k} = \sum_{k=0}^{\nu} \binom{\nu}{k}\binom{\nu}{\nu-k}$$ → Συμμετρική ιδιότητα

$$= \binom{\nu+\nu}{\nu} = \binom{2\nu}{\nu}$$ → Cauchy

⑥ Φεβρ. 2011 / Θέμα 2β

$$S = \sum_{k=0}^{\nu} \binom{\nu}{k}\left(\nu k + \binom{\nu}{k}\right)$$

$$= \nu \sum_{k=0}^{\nu} \binom{\nu}{k}k + \sum_{k=0}^{\nu} \binom{\nu}{k}^2$$

* $\binom{\nu}{k} = \binom{\nu}{\nu-k}$

$\binom{\nu}{k} = \frac{\nu}{k}\binom{\nu-1}{k-1}$

$$= \nu \sum_{k=1}^{\nu} \frac{\nu}{k}\binom{\nu-1}{k-1}k + \sum_{k=0}^{\nu} \binom{\nu}{k}\binom{\nu}{\nu-k}$$

$$= \nu^2 \underbrace{\sum_{k=1}^{\nu} \binom{\nu-1}{k-1}}_{j} + \binom{\nu+\nu}{\nu}$$

$$= \nu^2 \sum_{j=0}^{\nu-1} \binom{\nu-1}{j} + \binom{\nu+\nu}{\nu}$$

διων. θεωρ.

$$= \nu^2 \cdot (1+1)^{\nu-1} + \binom{2\nu}{\nu}$$

$$= \nu^2 2^{\nu-1} + \binom{2\nu}{\nu}$$

7 Σεπτ. 2010 Ομ. Β / Θέμα 3α

$$S = \sum_{j=m}^{\infty} \binom{j}{m} 3^{m-j}$$

$k = j-m$
$$= \sum_{k=0}^{\infty} \binom{k+m}{m} 3^{-k}$$

συμμετρ. ιδ.
$$= \sum_{k=0}^{\infty} \binom{k+m}{k} 3^{-k}$$

$$\left[{x \atop k} \right] = \binom{x+k-1}{k}$$

$$= \sum_{k=0}^{\infty} \binom{(m+1)+k-1}{k} 3^{-k}$$

$$= \sum_{k=0}^{\infty} \left[{m+1 \atop k} \right] \left(\frac{1}{3}\right)^{k}$$

$$* \sum_{k=0}^{\infty} \left[{v \atop k} \right] t^{k} = (1-t)^{-v}$$

$v = m+1$

$t = \frac{1}{3}$

$$= \left(1-\frac{1}{3}\right)^{-(m+1)} = \left(\frac{2}{3}\right)^{-(m+1)}$$

$$= \left(\frac{3}{2}\right)^{m+1}$$

8 Άσκηση

SOS

$$S = \sum_{s=k}^{\nu} \binom{s}{k}\binom{\nu}{s}$$

$$= \sum_{s=k}^{\nu} \frac{s!}{k!(s-k)!} \cdot \frac{\nu!}{s!(\nu-s)!} = \frac{\nu!}{k!} \sum_{s=k}^{\nu} \frac{1}{(s-k)!(\nu-s)!}$$

$j = s-k$

$$= \frac{\nu!}{k!} \sum_{j=0}^{\nu-k} \frac{1}{j!(\nu-k-j)!}$$

$$= \frac{\nu!}{k!} \cdot \frac{1}{(\nu-k)!} \sum_{j=0}^{\nu-k} \frac{(\nu-k)!}{j!(\nu-k-j)!}$$

$$= \binom{\nu}{k} \sum_{j=0}^{\nu-k} \binom{\nu-k}{j} = \binom{\nu}{k} 2^{\nu-k}$$

Διων.
Νεύτωνα

9 Άσκηση

$$S = \sum_{k=0}^{\nu} \underbrace{(k+1)^2}_{k^2+2k+1} \binom{\nu}{k}^2 = ?$$

$k^2+2k+1 \to k(k-1) + 3k + 1$

$$= \overbrace{\sum_{k=0}^{\nu} k(k-1)\binom{\nu}{k}^2}^{S_1} + 3 \overbrace{\sum_{k=0}^{\nu} k \binom{\nu}{k}^2}^{S_2} + \overbrace{\sum_{k=0}^{\nu} \binom{\nu}{k}^2}^{S_3}$$

$$S_3 = \sum_{k=0}^{\nu} \binom{\nu}{k}^2 = \sum_{k=0}^{\nu} \binom{\nu}{k}\binom{\nu}{\nu-k} = \binom{\nu+\nu}{\nu} = \binom{2\nu}{\nu}$$

$$S_2 = \sum_{k=0}^{\nu} k \binom{\nu}{k}^2 = \sum_{k=1}^{\nu} k \cdot \frac{\nu}{k} \binom{\nu-1}{k-1}\binom{\nu}{\nu-k}$$

$$= \nu \sum_{k=1}^{\nu} \binom{\nu-1}{k-1}\binom{\nu}{\nu-k}$$

$$= \nu \sum_{j=0}^{\nu-1} \binom{\nu-1}{j}\binom{\nu}{\nu-1-j}$$

$$= \nu \binom{2\nu-1}{\nu-1}$$

$$S_1 = \sum_{k=\cancel{0}\,2}^{\nu} k(k-1)\binom{\nu}{k}^2 = \sum_{k=2}^{\nu} k(k-1)\frac{\nu}{k}\cdot\frac{\nu-1}{k-1}\binom{\nu-2}{k-2}\binom{\nu}{\nu-k}$$

$$= \nu(\nu-1)\sum_{k=2}^{\nu}\binom{\nu-2}{\underbrace{k-2}_{=j}}\binom{\nu}{\nu-k}$$

$$= \nu(\nu-1)\sum_{j=0}^{\nu-2}\binom{\nu-2}{j}\binom{\nu}{\nu-2-j}$$

$$= \nu(\nu-1)\binom{2\nu-2}{\nu-2}$$

Άρα $S = \nu(\nu-1)\binom{2\nu-2}{\nu-2} + 3\nu\binom{2\nu-1}{\nu-1} + \binom{2\nu}{\nu}$