Υπολογισμοί Αθροισμάτων $\to$ (Cauchy, Συμμετρία, Γεωμετρικά)

29/4/25

① Παραδείγματα

1. $\displaystyle\sum_{k=0}^{\nu} k\cdot 2^k = 1\cdot 2^1 + 2\cdot 2^2 + 3\cdot 3^3 + \ldots + \nu\cdot 2^\nu$

Λύση : $\displaystyle\sum_{k=0}^{\nu} t^k = 1 + t + \ldots + t^\nu = \frac{1-t^{\nu+1}}{1-t}$

$$\overset{d/dt}{\Rightarrow} \sum_{k=0}^{\nu} k t^{k-1} = \frac{-(\nu+1)t^\nu(1-t) + 1 - t^{\nu+1}}{(1-t)^2} = \frac{\nu t^{\nu+1} - (\nu+1)t^\nu + 1}{(1-t)^2}$$

$$\overset{t}{\Rightarrow} \sum_{k=0}^{\nu} k t^k = \frac{\nu t^{\nu+2} - (\nu+1)t^{\nu+1} + t}{(1-t)^2} \overset{k=2}{\Rightarrow} \sum_{k=0}^{\nu} k\cdot 2^k = \nu 2^{\nu+2} - (\nu+1)2^{\nu+1} + 2$$

2) $\displaystyle\sum_{k=0}^{\infty} k^2 \left(\frac{1}{3}\right)^k$

Λύση: $\displaystyle\sum_{k=0}^{\infty} t^k = (1-t)^{-1}$

$$\overset{d/dt}{\Rightarrow} \sum_{k=0}^{\infty} k t^{k-1} = (1-t)^{-2}$$

$$\overset{d/dt}{\Rightarrow} \sum_{k=0}^{\infty} k(k-1) t^{k-2} = 2(1-t)^{-3} \overset{t^2}{\Rightarrow} \sum_{k=0}^{\infty} k(k-1)t^k = 2(1-t)^{-3}t^2 \quad *$$

Άρα $\displaystyle\sum_{k=0}^{\infty} k^2\left(\frac{1}{3}\right)^k = \sum_{k=0}^{\infty} k(k-1)\left(\frac{1}{3}\right)^k + \sum_{k=0}^{\infty} k\left(\frac{1}{3}\right)^k$ (από το 1° παρ.)

$$* \; t = \frac{1}{3} \Rightarrow \sum_{k=0}^{\infty} k(k-1)\left(\frac{1}{3}\right)^k = \ldots = \frac{3}{2}$$

② Γεωμετρικά Αθροίσματα

Για να υπολογ. $\sum_{k=0}^{\nu} f(k)p^k$ ή $\sum_{k=0}^{\infty} f(k)p^k \rightarrow |p|<1$

1) Εκφράζω $f(k) = A_0 + \dots + A_r (k)_r$ $\quad \hookrightarrow$ βαθμός $f(k)$

2) Υπολογίζω $\sum_{k=0}^{\nu \text{ ή } \infty} (k)_j\, p^k$ παραγωγ. j φορές την αντιστ. γ. σειρά ή άθροισμα

③ Τύπος Cauchy

$$\binom{r+s}{\nu} = \sum_{k=0}^{\nu} \binom{r}{k}\binom{s}{\nu-k} \qquad r,s = 1,2,\dots$$

Γενίκευση :

$$\boxed{\binom{x+y}{\nu} = \sum_{k=0}^{\nu} \binom{x}{k}\binom{y}{\nu-k}} \quad x,y\in\mathbb{R} \quad (\text{Απ. : γενν. διων. σειρά})$$

④ Χρήσιμη ταυτότητα διων. συντελ.

$$\boxed{\binom{\nu}{k} = \frac{\nu}{k}\binom{\nu-1}{k-1}} \qquad k \geq 1$$

π.χ $\quad \sum_{k=0}^{\nu} (k-2)^2 \binom{8}{k}\binom{12}{\nu-k} = \,?$

Λύση: $\quad (k-2)^2 = k^2 - 4k + 4 = A_0 + A_1 (k)_1 + A_2 (k)_2$

$$= A_0 + A_1 k + A_2 k(k-1)$$

$$= A_2 k^2 + (A_1 - A_2)k + A_0$$

Άρα $\quad A_0 = 4$

$A_2 = 1$

$A_1 - A_2 = -4 \Rightarrow A_1 = -3$

$$\Rightarrow \sum_{k=0}^{\nu} (k-2)^2\binom{8}{k}\binom{12}{\nu-k} = 4\sum_{k=0}^{\nu}\binom{8}{k}\binom{12}{\nu-k} - 3\sum_{k=\cancel{0}\,1}^{\nu} \underbrace{\cancel{k}\binom{8}{k}}_{=\frac{8}{\cancel{k}}\binom{7}{k-1}}\binom{12}{\nu-k} + \sum_{k=\cancel{0}\,2}^{\nu} \underbrace{\cancel{k(k-1)}\binom{8}{k}}_{=\frac{8}{\cancel{k}}\cdot\frac{7}{\cancel{k-1}}\binom{6}{k-2}}\binom{12}{\nu-k}$$

$$= 4\sum_{k=0}^{\nu}\binom{8}{k}\binom{12}{\nu-k} - 24\sum_{k=1}^{\nu}\binom{7}{\underset{i}{k-1}}\binom{12}{\nu-k} + 56\sum_{k=2}^{\nu}\binom{6}{\underset{j}{k-2}}\binom{12}{\nu-k}$$

$$= 4\binom{8+12}{\nu} - 24\sum_{i=0}^{\nu-1}\binom{7}{i}\binom{12}{\nu-1-i} + 56\sum_{i=0}^{\nu-2}\binom{6}{i}\binom{12}{\nu-2-i}$$

$$= 4\binom{20}{\nu} - 24\binom{7+12}{\nu-1} + 56\binom{6+12}{\nu-2} = 4\binom{20}{\nu} - 24\binom{19}{\nu-1} + 56\binom{18}{\nu-2}$$

⑤ Μετατροπή Απλών - Επαναλ. Συνδυασμών

$$\genfrac{[}{]}{0pt}{}{\nu}{k} = \binom{\nu+k-1}{k} = (-1)^k\binom{-\nu}{k}$$

$$\binom{\nu}{k} = \genfrac{[}{]}{0pt}{}{\nu-k+1}{k} = (-1)^k\genfrac{[}{]}{0pt}{}{-\nu}{k}$$

π.χ.

1) $\displaystyle\sum_{k=0}^{\nu}\binom{r+k}{k}\binom{s-k}{\nu-k}$

Λύση: $\displaystyle\binom{r+k}{k} = \genfrac{[}{]}{0pt}{}{r+1}{k} = (-1)^k\binom{-r-1}{k}$

$$\binom{s-k}{\nu-k} = \genfrac{[}{]}{0pt}{}{y}{\nu-k} = \binom{y+\nu-k-1}{\nu-k} \overset{y=s+1-\nu}{\leadsto} \binom{s-k}{\nu-k} = \genfrac{[}{]}{0pt}{}{s+1-\nu}{\nu-k} = (-1)^{\nu-k}\binom{-s+1+\nu}{\nu-k}$$

$$\Rightarrow \sum_{k=0}^{\nu}\binom{r+k}{k}\binom{s-k}{\nu-k} = \sum_{k=0}^{\nu}(-1)^k\binom{-r-1}{k}(-1)^{\nu-k}\binom{-s+1+\nu}{\nu-k}$$

$$= (-1)^{\nu}\sum_{k=0}^{\nu}\binom{-\nu-1}{k}\binom{-s-1+k}{\nu-k} = (-1)^{\nu}\binom{-r-1-s-1+\nu}{\nu} = \genfrac{[}{]}{0pt}{}{r+s+2-\nu}{\nu}$$

$$= \genfrac{[}{]}{0pt}{}{r+s+1-\nu+1}{\nu} = \binom{r+s+1}{\nu}$$

2) $\sum_{k=0}^{v} \frac{1}{(k+1)(k+2)}\binom{15}{k}\binom{35}{v-k}$

Λύση: $\frac{1}{17\cdot 16}\sum_{k=0}^{v}\underbrace{\frac{17\cdot 16}{(k+1)(k+2)}\binom{15}{k}}_{=\binom{17}{k+2}}\binom{35}{v-k}$

$$= \frac{1}{17\cdot 16}\sum_{k=0}^{v}\binom{17}{\underset{j}{k+2}}\binom{35}{v-k} = \frac{1}{17\cdot 16}\sum_{j=2}^{v+2}\binom{17}{j}\binom{35}{v+2-j}$$

$$= \frac{1}{17\cdot 16}\left[\binom{17+35}{v+2}-\binom{17}{1}\binom{35}{v+1}-\binom{17}{0}\binom{35}{v+2}\right]$$

$$= \frac{1}{17\cdot 16}\left[\binom{52}{v+2}-\frac{1}{16}\binom{35}{v+1}-\binom{35}{v+2}\right]$$